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NECESSARY AND SUFFICIENT CONDITIONS FOR THE BOUNDEDNESS OF THE GEGENBAUER–RIESZ POTENTIAL IN MODIFIED MORREY SPACES

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Abstract. In this paper we study the Gegenbauer–Riesz potential I_G^α (G-Riesz potential) generated by Gegenbauer differential operator $G_\lambda = (x^2-1)^{1/2-\lambda}\frac{d}{dx}(x^2-1)^{\lambda+1/2}\frac{d}{dx}$. We prove that the operator I_G^α is bounded from the modified Morrey space $\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+)$ to the weak modified Morrey space $W\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+)$ if and only if $\frac{\alpha}{2\lambda+1} \le 1-\frac{1}{q} \le \frac{\alpha}{2\lambda+1-\gamma}$ for $1<q<\infty$ and from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+)$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+)$ if and only if $\frac{\alpha}{2\lambda+1} \le \frac{1}{p}-\frac{1}{q} \le \frac{\alpha}{2\lambda+1-\gamma}$ for $1<p<q<\infty$. Obtained results are the analogue of the results taken in [6].

1. Definitions and Auxiliary Results

The study of boundedness of the Riesz potential, singular integrals and commutators were studied by lots of researchers in the last decades. Morrey estimates of such kind of operators is a more recent problem and is still very popular. Just as an example we recall the study made in [1,2,8,12]. Our aim is to continue this research focusing in necessary and sufficient conditions in suitable Morrey estimates of some kind of Riesz potential. The Gegenbauer differential operator was introduced in [3]. About properties of Gegenbauer differential operator we reference detail in [7].

In this paper, we consider the following generalized shift operator

$$A_{ch\,t} f(ch\,x) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)}\int_0^\pi f(ch\,x ch\,t - sh\,x sh\,t\cos\varphi)(\sin\varphi)^{2\lambda-1}d\varphi$$

generated by the Gegenbauer differential operator

$$G_\lambda = (x^2-1)^{1/2-\lambda}\frac{d}{dx}(x^2-1)^{\lambda+1/2}\frac{d}{dx},\ \ x\in(1,\infty),\ \ \lambda\in\big(0,1/2\big).$$

Let $H(x,r) = (x-r,x+r)\cap[0,\infty), r\in(0,\infty), x\in[0,\infty)$. For all measurable set $E\subset[0,\infty)$, $\mu E \equiv |E|_\lambda = \int_E sh^{2\lambda}\,t dt$. In [10] the Gegenbaur maximal function (G-maximal function) is defined as follows:

$$M_G f(ch\,x) = \sup_{r>0}\frac{1}{|(0,r)|_\lambda}\int_0^r A_{ch\,t}|f(ch\,x)| sh^{2\lambda} t dt.$$

Also we consider the following maximal function

$$M_\mu f(ch\,x) = \sup_{r>0}\frac{1}{|H(x,r)|_\lambda}\int_{H(x,r)}|f(ch\,t)| sh^{2\lambda} t dt.$$

Symbol $A\lesssim B$ denote that there exists a constant $C>0$ with that $0<A\le CB$ and C can depends of some parameters. If $A\lesssim B$ and $B\lesssim A$ we write $A\approx B$.

Note that, the following inequality is valid (see [10, Theorem 2.1]):

$$M_G f(ch\,x) \lesssim M_\mu f(ch\,x). \tag{1.1}$$

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In what follows we need the following theorems to prove our main results (see [11, Theorem 2.1 and Theorem 2.2]).

Theorem A ([11, Theorem 2.1]). *For all non-negative function $g \in L^{loc}_{1,\lambda}(\mathbb{R}_+)$ and $1 \le p < \infty$ the inequality*

$$\int\limits_{H(x,r)} (M_\mu f(ch\, y))^p g(ch\, y) sh^{2\lambda} y dy \le \int\limits_{H(x,r)} |f(ch\, y)|^p M_\mu(g(ch\, y)) sh^{2\lambda} y dy$$

is valid.

Theorem B ([11, Theorem 2.2]). *For all $\alpha > 0$ the following Chebyshev type inequality*

$$|\{y \in H(x,r) : M_\mu f(ch\, y) > \alpha\}|_\gamma \lesssim \frac{1}{\alpha} \int\limits_{H(x,r)} M_\mu f(ch\, y) sh^{2\lambda} y dy$$

is valid.

For $1 \le p \le \infty$ let $L_p([0,\infty), G) \equiv L_{p,\lambda}[0,\infty)$ be the space of functions measurable on $[0,\infty)$ with the finite norm

$$\|f\|_{L_{p,\lambda}} = \left(\int\limits_0^\infty |f(ch\, t)|^p sh^{2\lambda}\ t dt \right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$$\|f\|_{\infty,\lambda} = \operatorname*{ess\,sup}_{t \in [0,\infty)} |f(ch\, t)|, \quad p = \infty.$$

The following theorem was proved in [10].

Theorem C. a) *If $f \in L_{1,\lambda}[0,\infty)$, then for all $\alpha > 0$ the inequality*

$$|\{x : M_\mu f(ch\, x) > \alpha\}|_\lambda \le \frac{c_\lambda}{\alpha} \|f\|_{L_{1,\lambda}[0,\infty)}$$

holds, where $c_\lambda > 0$ depends only on λ.

b) *If $f \in L_{p,\lambda}[0,\infty)$, $1 < p \le \infty$, then $M_\mu f(ch\, x) \in L_{p,\lambda}[0,\infty)$ and*

$$\|M_\mu f\|_{L_{p,\lambda}[0,\infty)} \le c_{p,\lambda} \|f\|_{L_{p,\lambda}[0,\infty)}.$$

Corollary A. *If $f \in L_{p,\lambda}[0,\infty)$, $1 \le p \le \infty$, then*

$$\lim_{r \to 0} \frac{1}{|(0,r)|_\lambda} \int\limits_{(0,r)} A^\lambda_{ch\, t} f(ch\, x) sh^{2\lambda}\ t dt = f(ch\, x)$$

for a.e., $x \in [0,\infty)$.

2. Some Embeddings into the G-Morrey and Modified G-Morrey Spaces

We introduce the following notation analogously in [4–6].

Definition 2.1. Let $1 \le p < \infty$, $0 < \lambda < \frac{1}{2}$, $0 \le \gamma \le 2\lambda + 1$, $[r]_1 = \min\{1, r\}$. We denote by $L_{p,\lambda,\gamma}(\mathbb{R}_+, G)$, $\mathbb{R}_+ = [0,\infty)$, the G-Morrey space, and by $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ the modified G-Morrey space, as the set of locally integrable functions $f(ch\, x)$, $x \in \mathbb{R}_+ = [0,\infty)$, with the finite norms

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{x \in \mathbb{R}_+, r > 0} \left(r^{-\gamma} \int\limits_{H(x,r)} |f(ch\, t)|^p sh^{2\lambda} t dt \right)^{\frac{1}{p}},$$

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \sup_{x \in \mathbb{R}_+, r > 0} \left([r]_1^{-\gamma} \int\limits_{H(x,r)} |f(ch\, t)|^p sh^{2\lambda} t dt \right)^{\frac{1}{p}},$$

respectively.

Note that $\widetilde{L}_{p,\lambda,0}(\mathbb{R}_+,G) = L_{p,\lambda,0}(\mathbb{R}_+,G) = L_{p,\lambda}(\mathbb{R}_+,G)$. $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+,G) \subset L_{p,\lambda,\gamma}(\mathbb{R}_+,G) \cap L_{p,\lambda}(\mathbb{R}_+,G)$ and $\max\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\lambda}}\} \le \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$ and if $\gamma < 0$ or $\gamma > 2\lambda+1$, then $L_{p,\lambda,\gamma}(\mathbb{R}_+,G) = \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+,G) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}_+$.

Definition 2.2. Let $1 \le p < \infty$, $0 < \lambda < \frac{1}{2}$, $0 \le \gamma \le 1+2\lambda$. We denote by $WL_{p,\lambda,\gamma}(\mathbb{R}_+,G)$ the weak G-Morrey space and by $W\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+,G)$ the modified weak G-Morrey space as the set of locally integrable functions $f(ch\,x)$, $x \in \mathbb{R}_+$ with finite norms

$$\|f\|_{WL_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{t>0,\, x\in\mathbb{R}_+} \left(t^{-\gamma} |\{y \in H(x,t) : |f(ch\,y)| > r\}|_\gamma\right)^{\frac{1}{p}},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{t>0,\, x\in\mathbb{R}_+} \left([t]_1^{-\gamma} |\{y \in H(x,t) : |f(ch\,y)| > r\}|_\gamma\right)^{\frac{1}{p}}$$

respectively.

Note that $WL_{p,\lambda}(\mathbb{R}_+,G) = WL_{p,\lambda,0}(\mathbb{R}_+,G) = W\widetilde{L}_{p,\lambda,0}(\mathbb{R}_+,G)$, $L_{p,\lambda,\gamma}(\mathbb{R}_+,G) \subset WL_{p,\lambda,\gamma}(\mathbb{R}_+,G)$ and $\|f\|_{WL_{p,\lambda,\gamma}} \le \|f\|_{L_{p,\lambda,\gamma}}$, $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+,G) \subset WL_{p,\lambda,\gamma}(\mathbb{R}_+,G)$ and $\|f\|_{W\widetilde{L}_{p,\lambda,\gamma}} \le \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$.

We note that

$$L_{p,\lambda,0}(\mathbb{R}_+,G) = L_{p,\lambda}(\mathbb{R}_+,G),$$

and if $\gamma < 0$ or $\gamma > 1+2\lambda$, then $L_{p,\lambda,\gamma}(\mathbb{R}_+,G) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}_+$.

Lemma 2.1. *Let* $1 \le p < \infty$, $0 < \lambda < \frac{1}{2}$. *Then*

$$L_{p,\lambda,1+2\lambda}(\mathbb{R}_+,G) = L_\infty(\mathbb{R}_+)$$

and

$$c_\lambda^{-1/p} \|f\|_{L_\infty} \le \|f\|_{L_{p,\lambda,1+2\lambda}} \le \|f\|_{L_\infty},$$

where $c_\lambda = \frac{2^{\frac{1}{2}-\lambda}}{(1+2\lambda)(1+ch\,1)^{\frac{1}{2}-\lambda}}$.

Proof. Let $f \in L_\infty(\mathbb{R}_+)$. Then

$$\left(\frac{1}{|(0,r)|_\lambda} \int\limits_{(0,r)} A^\lambda_{ch\,t} f(ch\,x) sh^{2\lambda}\, t dt\right)^{1/p} \le \|f\|_{L_\infty}.$$

Therefore $f \in L_{p,\lambda,1+2\lambda}(\mathbb{R}_+,G)$ and

$$\|f\|_{L_{p,\lambda,1+2\lambda}} \le \|f\|_{L_\infty}.$$

Let $f \in L_{p,\lambda,1+2\lambda}(\mathbb{R}_+,G)$. By the Lebesgue's Theorem we have (see Section 1, Corollary A)

$$\lim_{r\to 0} \frac{1}{|(0,r)|_\lambda} \int\limits_{(0,r)} A^\lambda_{ch\,t} |f(ch\,x)|^p sh^{2\lambda}\, t dt = |f(ch\,x)|^p.$$

Then

$$|f(ch\,x)| = \left(\lim_{r\to 0} \frac{1}{|(0,r)|_\lambda} \int\limits_{(0,r)} A^\lambda_{ch\,t} |f(ch\,x)|^p sh^{2\lambda}\, t dt\right)^{1/p}$$
$$\le \sup_{0<r<1} \left(\frac{r^{1+2\lambda}}{|(0,r)|_\lambda}\right)^{1/p} \|f\|_{L_{p,\lambda,1+2\lambda}}.$$

From the proof of the Lemma 1.1 in [10] for $0 < r < 1$ we have

$$|(0,r)|_\lambda \ge \frac{2^{\lambda+\frac{3}{2}}}{(1+2\lambda)(1+ch\,1)^{\frac{1}{2}-\lambda}} \left(sh\,\frac{r}{2}\right)^{1+2\lambda} \ge \frac{2^{\frac{1}{2}-\lambda}}{(1+2\lambda)(1+ch\,1)^{\frac{1}{2}-\lambda}}\, r^{1+2\lambda}.$$

Therefore $f \in L_\infty(\mathbb{R}_+)$ and

$$\|f\|_{L_\infty} \le c_\lambda^{1/p} \|f\|_{L_{p,\lambda,1+2\lambda}}. \qquad \square$$

Lemma 2.2. *Let $1 \le p < \infty$, $0 < \lambda < \frac{1}{2}$, $0 \le \gamma \le 1 + 2\lambda$. Then*

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G) = L_{p,\lambda,\gamma}(\mathbb{R}_+, G) \cap L_{p,\lambda}(\mathbb{R}_+, G)$$

and

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\lambda}}\right\}.$$

Proof. Let $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$. Then

$$\begin{aligned}\|f\|_{L_{p,\lambda}} &= \sup_{x\in\mathbb{R}_+, r>0} \left(\int\limits_{(0,r)} A^{\lambda}_{ch\,t}|f(ch\,x)|^p sh^{2\lambda}\ tdt\right)^{1/p} \\ &\le \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma}\int\limits_{(0,r)} A^{\lambda}_{ch\,t}|f(ch\,x)|^p sh^{2\lambda}\ tdt\right)^{1/p} \\ &= \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}\end{aligned}$$

and

$$\begin{aligned}\|f\|_{L_{p,\lambda,\gamma}} &= \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma}\int\limits_{(0,r)} A^{\lambda}_{ch\,t}|f(ch\,x)|^p sh^{2\lambda}\ tdt\right)^{1/p} \\ &\le \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma}\int\limits_{(0,r)} A^{\lambda}_{ch\,t}|f(ch\,x)|^p sh^{2\lambda}\ tdt\right)^{1/p} \\ &= \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.\end{aligned}$$

Therefore, $f \in L_{p,\lambda,\gamma}(\mathbb{R}_+, G) \cap L_{p,\lambda}(\mathbb{R}_+, G)$ and the embedding

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G) \subset\!\succ L_{p,\lambda,\gamma}(\mathbb{R}_+, G) \cap L_{p,\lambda}(\mathbb{R}_+, G)$$

is valid.

Let $f \in L_{p,\lambda,\gamma}(\mathbb{R}_+, G) \cap L_{p,\lambda}(\mathbb{R}_+, G)$. Then

$$\begin{aligned}\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} &= \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma}\int\limits_{(0,r)} A^{\lambda}_{ch\,t}|f(ch\,x)|^p sh^{2\lambda}\ tdt\right)^{1/p} \\ &= \max\left\{\sup_{x\in\mathbb{R}_+, 0<r\le 1} \left(r^{-\gamma}\int\limits_{(0,r)} A^{\lambda}_{ch\,t}|f(ch\,x)|^p sh^{2\lambda}\ tdt\right)^{1/p},\right. \\ &\left.\sup_{x\in\mathbb{R}_+, r>1} \left(\int\limits_{(0,r)} A^{\lambda}_{ch\,t}|f(ch\,x)|^p sh^{2\lambda}\ tdt\right)^{1/p}\right\} \le \max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\lambda}}\right\}.\end{aligned}$$

Therefore, $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ and the embedding $L_{p,\lambda,\gamma}(\mathbb{R}_+, G) \cap L_{p,\lambda}(\mathbb{R}_+, G) \subset\!\succ \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ is valid.

Thus $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G) = L_{p,\lambda,\gamma}(\mathbb{R}_+, G) \cap L_{p,\lambda}(\mathbb{R}_+, G)$.

Let now $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$. Then

$$\begin{aligned}\|f\|_{L_{p,\lambda,\gamma}} &= \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{(0,r)} A^{\gamma}_{ch\,t} |f(ch\,x)|^p sh^{2\lambda}\, tdt \right)^{1/p} \\ &= \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-1}[r]_1\right)^{\frac{\gamma}{p}} \left([r]_1^{-\gamma} \int\limits_{(0,r)} A^{\lambda}_{ch\,t} |f(ch\,x)|^p sh^{2\lambda}\, tdt \right)^{1/p} \\ &= \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma} \int\limits_{(0,r)} A^{\lambda}_{ch\,t} |f(ch\,x)|^p sh^{2\lambda}\, tdt \right)^{1/p} \\ &= \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}. \end{aligned}$$

□

3. Hardy-Littlewood-Sobolev Inequality in Modified G-Morrey Spaces

In this section we study the $\widetilde{L}_{p,\lambda,\gamma}$-boundedness of the G-maximal operator M_μ.

Theorem 3.1. 1) *If $f \in \widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+, G)$, $0 \le \gamma < 1 + 2\lambda$, then $M_\mu f \in W\widetilde{L}_{1,\lambda,\gamma}$ and*

$$\|M_\mu f\|_{W\widetilde{L}_{1,\lambda,\gamma}} \lesssim \|f\|_{\widetilde{L}_{1,\lambda,\gamma}}.$$

2) *If $f \in \widetilde{L}_{p,\lambda,\gamma}$, $1 < p < \infty$, $0 \le \gamma < 1 + 2\lambda$, then $M_\mu f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ and*

$$\|M_\mu f\|_{\widetilde{L}_{p,\lambda,\gamma}} \lesssim \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

Proof. 1) From the definition of weak modified Morrey spaces

$$\|M_\mu f\|_{W\widetilde{L}_{1,\lambda,\gamma}} = \sup_{r>0} r \sup_{t>0,\, x\in\mathbb{R}_+} \left([t]_1^{-\gamma} |\{y \in H(x,t) : M_\mu f(ch\,y) > r\}|_\gamma\right)^{\frac{1}{p}}.$$

Using the Theorem B and also Theorem A at $p = 1$ and $g(ch\,y) \equiv 1$ we obtain

$$\|M_\mu f\|_{W\widetilde{L}_{1,\lambda,\gamma}} \lesssim \sup_{t>0,\, x\in\mathbb{R}_+} \left([t]_1^{-\gamma} \int\limits_{H(x,t)} |f(ch\,y)| sh^{2\lambda} y dy \right) = \|f\|_{\widetilde{L}_{1,\lambda,\gamma}}.$$

Assertion 2) follows from Theorem A at $g(ch\,y) \equiv 1$. □

We consider of the Gegenbauer–Riesz potential (G - Riesz potential) (see [10])

$$I^\alpha_G f(ch\,x) = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int\limits_0^\infty \left(\int\limits_0^\infty r^{\frac{\alpha}{2}-1} h_r(ch\,t) dr \right) A_{ch\,t} f(ch\,x) sh^{2\lambda} tdt,$$

where

$$h_r(ch\,t) = \int\limits_1^\infty e^{-\gamma(\gamma+2\lambda)r} P^\lambda_\gamma(ch\,t)(\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma$$

and P^λ_γ is eigenfunction of operator G_λ.

The following Hardy–Littlewood–Sobolev inequality in modified G-Morrey spaces is valid.

Theorem 3.2. *Let $0 \le \alpha < 1 + 2\lambda$, $0 \le \gamma < 2\lambda + 1 - \alpha$ and $1 \le p < \frac{2\lambda+1-\gamma}{\alpha}$.*

1) *If $1 < p < \frac{2\lambda+1-\gamma}{\alpha}$, then the condition $\frac{\alpha}{2\lambda+1} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{2\lambda+1-\gamma}$ is necessary and sufficient for the boundedness of the operator I^α_G from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.*

2) *If $p = 1 < \frac{2\lambda+1-\gamma}{\alpha}$, then the condition $\frac{\alpha}{2\lambda+1} \le 1 - \frac{1}{q} \le \frac{\alpha}{2\lambda+1-\gamma}$ is necessary and sufficient for the boundedness of the operator I^α_G from $\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+, G)$ to $W\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.*

Proof. 1) *Sufficiency.* Let $0 \le \alpha < 1+2\lambda$, $0 \le \gamma < 2\lambda+1-\gamma$, $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ and $1 < p < \frac{2\lambda+1-\gamma}{\alpha}$. For I_G^α take place the following estimate (see [10, the proof of Corollary 3.1])

$$|I_G^\alpha f(ch\,x)| \lesssim \int_0^\infty A_{ch\,t}|f(ch\,x)|\,(sh\,x)^{\alpha-2\lambda-1}\,sh^{2\lambda}tdt$$
$$= \int_0^\infty A_{ch\,t}(sh\,x)^{\alpha-2\lambda-1}|f(ch\,t)|sh^{2\lambda}tdt. \tag{3.1}$$

From (3.1) we have

$$|I_G^\alpha f(ch\,x)| \lesssim \Big(\int_0^r + \int_r^\infty\Big) A_{ch\,t}(sh\,x)^{\alpha-2\lambda-1}|f(ch\,t)|sh^{2\lambda}tdt$$
$$= A_1(x,r) + A_2(x,r).$$

We consider $A_1(x,r)$. Let $0 < r < 2$, then by (1.1) we obtain

$$|A_1(x,r)| \lesssim \int_0^r \frac{A_{ch\,t}|f(ch\,x)|sh^{2\lambda}t}{(sh\,t)^{2\lambda+1-\alpha}}dt \lesssim \sum_{j=0}^\infty \int_{2^{-j-1}r}^{2^{-j}r} \frac{A_{ch\,t}|f(ch\,x)|sh^{2\lambda}t}{(sh\,t)^{2\lambda+1-\alpha}}dt$$
$$\lesssim \sum_{j=0}^\infty \Big(sh\frac{r}{2^{j+1}}\Big)^\alpha \Big(sh\frac{r}{2^{j+1}}\Big)^{-2\lambda-1} \int_0^{2^{-j}r} A_{ch\,t}|f(ch\,x)|sh^{2\lambda}tdt$$
$$\lesssim (sh\,r)^\alpha M_G f(ch\,x)\Big(\sum_{j=0}^\infty 2^{-(j+1)\alpha}\Big) \lesssim (sh\,r)^\alpha M_\mu f(ch\,x). \tag{3.2}$$

Let $2 \le r < \infty$ and $0 < \alpha < 4\lambda$. Then (see [10, the proof of Corollary 3.1])

$$A_1(x,r) \lesssim \int_0^r \frac{A^\lambda_{ch\,t}|f(ch\,x)|sh^{2\lambda}tdt}{(ch\,t)^{2\lambda+1-\alpha}}$$
$$\le \int_0^r \frac{A^\lambda_{ch\,t}|f(ch\,x)|sh^{2\lambda}tdt}{(ch\,t)^{4\lambda-\alpha}} \le \int_0^r \frac{A^\lambda_{ch\,t}|f(ch\,x)|sh^{2\lambda}tdt}{(sh\,t)^{4\lambda-\alpha}}$$
$$\le \sum_{j=0}^\infty \int_{2^{-j-1}r}^{2^{-j}r} \frac{A^\lambda_{ch\,t}|f(ch\,x)|sh^{2\lambda}tdt}{(sh\,t)^{4\lambda-\alpha}}$$
$$\le \sum_{j=0}^\infty \Big(sh\frac{r}{2^{j+1}}\Big)^\alpha \Big(sh\frac{r}{2^{j+1}}\Big)^{-4\lambda} \int_0^{2^{-j}r} A^\lambda_{ch\,t}|f(ch\,x)|sh^{2\lambda}tdt$$
$$\lesssim M_G f(ch\,x)\sum_{j=0}^\infty \Big(sh\frac{r}{2^{j+1}}\Big)^\alpha \le (sh\,r)^\alpha M_\mu f(ch\,x)\sum_{j=0}^\infty 2^{-(j+1)\alpha}$$
$$\lesssim (sh\,r)^\alpha M_\mu f(ch\,x), \quad 0 < \alpha < 4\lambda.$$

Now let $4\lambda \le \alpha < 2\lambda+1$. From the proof of Corollary 3.1 and [10] it follows that $|I_G^\alpha f(ch\, x)| \lesssim 1$, then we have

$$|A_1(x,r)| \lesssim \int_0^r A_{ch\,t}^\lambda |f(ch\,x)| sh^{2\lambda} t dt = \frac{\left(sh\frac{r}{2}\right)^{4\lambda}}{\left(sh\frac{r}{2}\right)^{4\lambda}} \int_0^r A_{ch\,t}^\lambda |f(ch\,x)| sh^{2\lambda} t dt$$
$$\le \left(sh\frac{r}{2}\right)^{4\lambda} M_G f(ch\,x) \lesssim (sh\,r)^\alpha M_\mu f(ch\,x), \quad 4\lambda \le \alpha < 2\lambda+1.$$

Thus for $0<r<\infty$ we have

$$A_1(x,r) \lesssim (sh\,r)^\alpha M_\mu f(ch\,x), \quad 0<\alpha<2\lambda+1. \tag{3.3}$$

We consider $A_2(x,r)$. From (3.1) and Hölder's inequality we get

$$A_2(x,r) \lesssim \Big(\int_r^\infty (A_{ch\,t}|f(ch\,x)|)^p (sh\,t)^{-\beta} sh^{2\lambda} t dt \Big)^{\frac{1}{p}}$$
$$\times \Big(\int_r^\infty (sh\,t)^{\left(\frac{\beta}{p}+\alpha-2\lambda-1\right)p'} sh^{2\lambda} t dt \Big)^{\frac{1}{p'}} = A_{21}\cdot A_{22}. \tag{3.4}$$

Let $\gamma<\beta<2\lambda+1-p\alpha$. Taking into account the inequality (see [9, Lemma 2])

$$\|A_{ch\,t} f\|_{\widetilde{L}_{p,\lambda,\gamma}} \le \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$$

we obtain

$$A_{21} \lesssim \Big(\sum_{j=0}^\infty \int_{2^j r}^{2^{j+1} r} (A_{ch\,t}|f(ch\,x)|)^p (sh\,t)^{-\beta} sh^{2\lambda} t dt \Big)^{\frac{1}{p}}$$
$$\lesssim \|A_{ch\,t} f\|_{\widetilde{L}_{p,\lambda,\gamma}} \Big(\sum_{j=0}^\infty \frac{[2^{j+1} r]_1^\gamma}{(sh\,2^j r)^\beta} \Big)^{\frac{1}{p}}$$
$$\lesssim \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \begin{cases} \Big((2r)^\gamma \sum\limits_{j=0}^{\left[\log_2 \frac{1}{2r}\right]} 2^{(\gamma-\beta)j} + \sum\limits_{j=\left[\log_2 \frac{1}{2r}\right]+1}^\infty 2^{-\beta j} \Big)^{\frac{1}{p}}, & 0<r<\frac{1}{2}, \\ \Big(\sum\limits_{j=0}^\infty 2^{-\beta j} \Big)^{\frac{1}{p}}, & r \ge \frac{1}{2} \end{cases}$$
$$\lesssim (sh\,r)^{-\frac{\beta}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \begin{cases} (r^\gamma + r^\beta), & 0<r<\frac{1}{2}, \\ 1, & r\ge \frac{1}{2} \end{cases}$$
$$\lesssim \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \begin{cases} (r^{\frac{\gamma}{p}} (sh\,r)^{-\frac{\beta}{p}}, & 0<r<\frac{1}{2}, \\ (sh\,r)^{-\beta}, & r\ge \frac{1}{2} \end{cases}$$
$$\lesssim [2r]_1^{\frac{\gamma}{p}} (sh\,r)^{-\frac{\beta}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}. \tag{3.5}$$

For A_{22} we have

$$\begin{aligned} A_{22} &= \Big(\int_r^\infty (sh\,t)^{\left(\frac{\beta}{p}+\alpha-2\lambda-1\right)p'} sh^{2\lambda}tdt\Big)^{\frac{1}{p'}} \\ &\le \Big(\int_r^\infty (sh\,t)^{\left(\frac{\beta}{p}+\alpha-2\lambda-1\right)p'} sh^{2\lambda}td(sh\,t)\Big)^{\frac{1}{p'}} \\ &\lesssim (sh\,r)^{\frac{\beta}{p}+\alpha-2\lambda-1+\frac{2\lambda+1}{p'}} \lesssim (sh\,r)^{\frac{\beta}{p}+\alpha-2\lambda-1+(2\lambda+1)(1-\frac{1}{p})} \\ &\lesssim (sh\,r)^{\frac{\beta}{p}+\alpha-\frac{2\lambda+1}{p}}. \end{aligned} \tag{3.6}$$

Taking into account (3.5) and (3.6) on (3.4) we obtain

$$A_2(x,r) \lesssim [2r]_1^{\frac{\gamma}{p}} (sh\,r)^{\alpha-\frac{2\lambda+1}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}. \tag{3.7}$$

Thus from (3.3) and (3.7) we get

$$\begin{aligned} |I_G^\alpha f(ch\,x)| &\lesssim \Big([r]_1^{\frac{\gamma}{p}} (sh\,r)^{\alpha-\frac{2\lambda+1}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} + (sh\,r)^\alpha M_\mu f(ch\,x)\Big) \\ &\lesssim \min\Big\{(sh\,r)^{\alpha+\frac{\gamma-2\lambda-1}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} + (sh\,r)^\alpha M_\mu f(ch\,x), \\ &\quad (sh\,r)^{\alpha-\frac{2\lambda+1}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} + (sh\,r)^\alpha M_\mu f(ch\,x)\Big\}, \quad r>0. \end{aligned} \tag{3.8}$$

The right-hand side attains its minimum at

$$sh\,r = \Big(\frac{2\lambda+1-p\alpha}{p\alpha} \frac{\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}}{M_\mu f(ch\,x)}\Big)^{\frac{p}{2\lambda+1}} \tag{3.9}$$

and

$$sh\,r = \Big(\frac{2\lambda+1-\gamma-p\alpha}{p\alpha} \frac{\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}}{M_\mu f(ch\,x)}\Big)^{\frac{p}{2\lambda+1-\gamma}}. \tag{3.10}$$

Taking into account (3.9) and (3.10) in (3.8) we obtain

$$|I_G^\alpha f(ch\,x)| \lesssim \min\left\{\Big(\frac{M_\mu f(ch\,x)}{\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}}\Big)^{1-\frac{p\alpha}{2\lambda+1}}, \Big(\frac{M_\mu f(ch\,x)}{\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}}\Big)^{1-\frac{p\alpha}{2\lambda+1-\gamma}}\right\} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

Then

$$|I_G^\alpha f(ch\,x)| \lesssim (M_\mu f(ch\,x))^{\frac{p}{q}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}^{1-\frac{p}{q}}.$$

Hence, by Theorem 3.1, we have

$$\begin{aligned} \int_{H(x,r)} |I_G^\alpha f(ch\,x)|^q sh^{2\lambda}tdt &\lesssim \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}^{q-p} \int_{H(x,r)} (M_\mu f(ch\,t))^p sh^{2\lambda}tdt \\ &\lesssim [r]_1^\gamma \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}^q. \end{aligned}$$

From this it follows that

$$\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}} \lesssim \|f\|_{\widetilde{L}_{p,\lambda,\gamma}},$$

i.e., I_G^α is bounded from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+,G)$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+,G)$.
Necessity. Let $1<p<\frac{2\lambda+1-\gamma}{\alpha}$, $f\in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+,G)$ and I_G^α be bounded from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+,G)$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+,G)$.

Let the function $f(ch\,x)$ be non-negative and monotonically increasing on $\mathbb{R}_+$. The delates function $f_t(ch\,x)$ is defined as follows

$$\begin{cases} f(ch(th\,t)\,x) \le f_t(ch\,x) \le f(ch(cth\,t)\,x), & 0 < t < 1, \\ f(ch(th\,t)\,x) \le f_t(ch\,x) \le f(ch(sh\,t)\,x), & 1 \le t < \infty. \end{cases} \tag{3.11}$$

We suppose $[t]_{1,+} = \max\{1, t\}$.

From (3.11) we have at $0 < t < 1$

$$\begin{aligned}
\|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} &= \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma} \int\limits_{H(x,r)} |f_t(ch\,y)|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \\
&\le \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma} \int\limits_{H(x,r)} |f(ch(cth\,t)\,y)|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \quad [(cth\,t)y = u, \quad dy = (th\,t)du] \\
&= (th\,t)^{\frac{1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma} \int\limits_{H(xcth\,t, rcth\,t)} |f(ch\,u)|^p sh^{2\lambda} (th\,t) u du \right)^{\frac{1}{p}} \\
&\le (th\,t)^{\frac{2\lambda+1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma} \int\limits_{H(xcth\,t, rcth\,t)} |f(ch\,u)|^p sh^{2\lambda} u du \right)^{\frac{1}{p}} \\
&= (sh\,t)^{\frac{2\lambda+1}{p}} \sup_{r>0} \left(\frac{[r\ cth\,t]_1}{[r]_1} \right)^{\frac{\gamma}{p}} \sup_{x\in\mathbb{R}_+, r>0} \left([r\ cth\,t]_1^{-\gamma} \int\limits_{H(xcth\,t, rcth\,t)} |f(ch\,u)|^p sh^{2\lambda} u du \right)^{\frac{1}{p}} \\
&= (th\,t)^{\frac{2\lambda+1}{p}} [cth\,t]_{1,+}^{\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \le (th\,t)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \\
&= \left(\frac{sh\,t}{ch\,t} \right)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \lesssim \frac{1}{(ch\,t)^{\frac{2\lambda+1-\gamma}{p} - \alpha}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \\
&\lesssim (sh\,t)^{\alpha + \frac{\gamma-2\lambda-1}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.
\end{aligned} \tag{3.12}$$

On the other hand

$$\begin{aligned}
\|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} &= \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma} \int\limits_{H(x,r)} |f_t(ch\,y)|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \\
&\ge \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma} \int\limits_{H(x,r)} |f(ch(th\,t)y)|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \quad [(th\,t)y = u, \quad dy = (cth\,t)du] \\
&= (cth\,t)^{\frac{1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma} \int\limits_{H(xth\,t, rth\,t)} |f(ch\,u)|^p sh^{2\lambda} (cth\,t) u du \right)^{\frac{1}{p}} \\
&\ge (cth\,t)^{\frac{2\lambda+1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma} \int\limits_{H(xth\,t, rth\,t)} |f(ch\,u)|^p sh^{2\lambda} u du \right)^{\frac{1}{p}} \\
&= (cth\,t)^{\frac{2\lambda+1}{p}} \left(\sup_{r>0} \frac{[rth\,t]_1}{[r]_1} \right)^{\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \\
&= (cth\,t)^{\frac{2\lambda+1}{p}} [th\,t]_1^{\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \ge (cth\,t)^{\frac{2\lambda+1}{p} - \frac{\gamma}{p} - \alpha} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \\
&= (cth\,t)^{\frac{2\lambda+1-\gamma}{p} - \alpha} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \ge (sh\,t)^{\alpha + \frac{\gamma-2\lambda-1}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.
\end{aligned} \tag{3.13}$$

Now, let $1 \le t < \infty$, then from (3.11) we obtain

$$\|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+, r>0} \Bigg([r]_1^{-\gamma} \int\limits_{H(x,r)} |f_t(ch\, y)|^p sh^{2\lambda} y dy \Bigg)^{\frac{1}{p}}$$
$$\ge \sup_{x\in\mathbb{R}_+, r>0} \Bigg([r]_1^{-\gamma} \int\limits_{H(x,r)} |f(ch(cth\, t)\, y)|^p sh^{2\lambda} y dy \Bigg)^{\frac{1}{p}} \quad [(th\, t)y = u, \quad dy = (cth\, t) du]$$
$$= (cth\, t)^{\frac{1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \Bigg([r]_1^{-\gamma} \int\limits_{H(xth\, t, rth\, t)} |f(ch\, u)|^p sh^{2\lambda} (cth\, t) u du \Bigg)^{\frac{1}{p}}$$
$$\ge (cth\, t)^{\frac{2\lambda+1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \Bigg([r]_1^{-\gamma} \int\limits_{H(xth\, t, rth\, t)} |f(ch\, u)|^p sh^{2\lambda} u du \Bigg)^{\frac{1}{p}}$$
$$= (cth\, t)^{\frac{2\lambda+1}{p}} \sup_{r>0} \left(\frac{[rth\, t]_1}{[r]_1} \right)^{\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$$
$$= (cth\, t)^{\frac{2\lambda+1}{p}} [th\, t]_1^{\gamma} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \ge (cth\, t)^{\frac{2\lambda+1-\gamma}{p} - \alpha} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$$
$$\ge (sh\, t)^{\alpha + \frac{\gamma-2\lambda-1}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}. \tag{3.14}$$

On the other hand

$$\|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+, r>0} \Bigg([r]_1^{-\gamma} \int\limits_{H(x,r)} |f_t(ch\, y)|^p sh^{2\lambda} y dy \Bigg)^{\frac{1}{p}}$$
$$\le \sup_{x\in\mathbb{R}_+, r>0} \Bigg([r]_1^{-\gamma} \int\limits_{H(x,r)} |f(ch(sh\, t) y)|^p sh^{2\lambda} y dy \Bigg)^{\frac{1}{p}} \quad \Big[(sh\, t)y = u, \quad dy = \frac{du}{sh\, t}\Big]$$
$$= (sh\, t)^{-\frac{1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \Bigg([r]_1^{-\gamma} \int\limits_{H(xsh\, t, rsh\, t)} |f(ch\, u)|^p sh^{2\lambda} \frac{u}{sh\, t} du \Bigg)^{\frac{1}{p}}$$
$$\le (sh\, t)^{-\frac{2\lambda+1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \Bigg([r]_1^{-\gamma} \int\limits_{H(xsh\, t, rsh\, t)} |f(ch\, u)|^p sh^{2\lambda} u du \Bigg)^{\frac{1}{p}}$$
$$= (sh\, t)^{-\frac{2\lambda+1}{p}} \left(\sup_{r>0} \frac{[rsh\, t]_1}{[r]_1} \right)^{\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$$
$$= (sh\, t)^{-\frac{2\lambda+1}{p}} [sh\, t]_{1,+}^{\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \le (sh\, t)^{\alpha + \frac{\gamma-2\lambda-1}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}. \tag{3.15}$$

From (3.12)–(3.15) for all $0 < t < \infty$ we obtain

$$\|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} \approx (sh\, t)^{\alpha + \frac{\gamma-2\lambda-1}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}. \tag{3.16}$$

According to the define of G-potential we can write

$$I_G^{\alpha} f_t(ch\, x) = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int\limits_0^{\infty} \Bigg(\int\limits_0^{\infty} u^{\frac{\alpha}{2}-1} h_u(chv) du \Bigg) A_{chv} f_t(ch\, x) sh^{2\lambda} v dv.$$

From this and (3.11) for $0 < t < 1$ we have

$$\|I_G^{\alpha} f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+, r>0} \Bigg([r]_1^{-\gamma} \int\limits_{H(x,r)} |I_G^{\alpha} f_t(ch\, y)|^q sh^{2\lambda} y dy \Bigg)^{\frac{1}{q}}$$

$$\leq \sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(x,r)}|I_G^\alpha f(ch(cth\, t)\, y)|^q sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$[(cth\, t)y = z, \quad dy = (th\, t)dz]$

$$= (th\, t)^{\frac{1}{q}}\sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(xcth\, t, rcth\, t)}|I_G^\alpha f(ch\, z)|^q sh^{2\lambda}(th\, t)zdz\right)^{\frac{1}{q}}$$

$$\leq (th\, t)^{\frac{2\lambda+1}{q}}\sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(xcth\, t, rcth\, t)}|I_G^\alpha f(ch\, z)|^q sh^{2\lambda}zdz\right)^{\frac{1}{q}}$$

$$= (th\, t)^{\frac{2\lambda+1}{q}}\left(\sup_{r>0}\frac{[rcth\, t]_1}{[r]_1}\right)^{\frac{\gamma}{q}}\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$= (th\, t)^{\frac{2\lambda+1}{q}}[cth\, t]_1^{\frac{\gamma}{q}}\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$\leq (cth\, t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}} \leq (sh\, t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.17}$$

On the other hand

$$\|I_G^\alpha f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(x,r)}|I_G^\alpha f_t(ch\, y)|^q sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$$\geq \sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(x,r)}|I_G^\alpha f(ch(th\, t)y)|^q sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$[(th\, t)y = z, \quad dy = (cth\, t)dz]$

$$= (cth\, t)^{\frac{1}{q}}\sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(xth\, t, rth\, t)}|I_G^\alpha f(ch\, z)|^q sh^{2\lambda}(cth\, t)zdz\right)^{\frac{1}{q}}$$

$$\geq (cth\, t)^{\frac{2\lambda+1}{q}}\left(\sup_{r>0}\frac{[rth\, t]_1}{[r]_1}\right)^{\frac{\gamma}{q}}\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$\geq (cth\, t)^{\frac{2\lambda+1}{q}}[th\, t]_1^{\gamma}\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$\geq \left(\frac{ch\, t}{sh\, t}\right)^{\frac{2\lambda+1-\gamma}{q}}\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}} \geq (sh\, t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.18}$$

Combining (3.17) and (3.18) we obtain

$$\|I_G^\alpha f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} \approx (sh\, t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}, \quad 0 < t < 1. \tag{3.19}$$

Now we consider the case, then $1 \leq t < \infty$. From (3.11) we have

$$\|I_G^\alpha f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} \geq \sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(x,r)}|I_G^\alpha f(ch((th\, t)y))|^q sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$[(th\, t)y = z, \quad dy = (cth\, t)dz]$

$$= (cth\, t)^{\frac{1}{q}}\sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(xth\, t, rth\, t)}|I_G^\alpha f(ch\, z)|^q sh^{2\lambda}(cth\, t)zdz\right)^{\frac{1}{q}}$$

$$\geq (cth\, t)^{\frac{2\lambda+1}{q}}\left(\sup_{r>0}\frac{[r\, th\, t]_1}{[r]_1}\right)^{\frac{\gamma}{q}}\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$\geq (cth\, t)^{\frac{2\lambda+1}{q}}[th\, t]_1^{\gamma}\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$\geq \Big(\frac{ch\,t}{sh\,t}\Big)^{\frac{2\lambda+1-\gamma}{q}} \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}} \geq (sh\,t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.20}$$

On the other hand

$$\|I_G^\alpha f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} \leq \sup_{x\in\mathbb{R}_+, r>0} \Bigg([r]_1^{-\gamma} \int\limits_{H(x,r)} |I_G^\alpha f(ch(sh\,t)y)|^q sh^{2\lambda} y dy\Bigg)^{\frac{1}{q}}$$

$$\Big[(sh\,t)y = z, \quad dy = \frac{dz}{sh\,t}\Big]$$

$$= (sh\,t)^{-\frac{1}{q}} \sup_{x\in\mathbb{R}_+, r>0} \Bigg([r]_1^{-\gamma} \int\limits_{H(xsh\,t, rsh\,t)} |I_G^\alpha f(ch\,z)|^q sh^{2\lambda}\Big(\frac{z}{sh\,t}\Big) dz\Bigg)^{\frac{1}{q}}$$

$$= (sh\,t)^{-\frac{2\lambda+1}{q}} \sup_{x\in\mathbb{R}_+, r>0} \Bigg([r]_1^{-\gamma} \int\limits_{H(xsh\,t, r\,sh\,t)} |I_G^\alpha f(ch\,z)|^q sh^{2\lambda} dz\Bigg)^{\frac{1}{q}}$$

$$= (sh\,t)^{-\frac{2\lambda+1}{q}} \Bigg(\sup_{r>0} \frac{[r\,sh\,t]_1}{[r]_1}\Bigg)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$\leq (sh\,t)^{-\frac{2\lambda+1}{q}} [th\,t]_{1,+}^\gamma \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$\leq (sh\,t)^{-\frac{2\lambda+1}{q}} \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}} \leq (sh\,t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.21}$$

From (3.20) and (3.21) it follows that

$$\|I_G^\alpha f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} \approx (sh\,t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}, \quad 1 \leq t < \infty. \tag{3.22}$$

Now from (3.19) and (3.22) we have

$$\|I_G^\alpha f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} \approx (sh\,t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}, \quad 0 < t < \infty. \tag{3.23}$$

Since I_G^α is bounded from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+, G)$, i.e.

$$\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}} \lesssim \|f\|_{\widetilde{L}_{p,\lambda,\gamma}},$$

then taking into account (3.23) and (3.16) we obtain

$$\begin{aligned}\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}} &\approx (sh\,t)^{\frac{2\lambda+1-\gamma}{q}} \|I_G^\alpha f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} \lesssim (sh\,t)^{\frac{2\lambda+1-\gamma}{q}} \|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} \\ &\lesssim (sh\,t)^{\alpha+(\gamma-2\lambda-1)(\frac{1}{p}-\frac{1}{q})} \|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} \\ &\lesssim \begin{cases} (sh\,t)^{\alpha-(2\lambda+1)(\frac{1}{p}-\frac{1}{q})}, & 0 < t < 1, \\ (sh\,t)^{\alpha+(\gamma-2\lambda-1)(\frac{1}{p}-\frac{1}{q})}, & 0 \leq t < \infty. \end{cases}\end{aligned}$$

If $\frac{1}{p} - \frac{1}{q} < \frac{\alpha}{2\lambda+1}$, then in the case $t \to 0$ we have $\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.
As well as if $\frac{1}{p} - \frac{1}{q} > \frac{\alpha}{2\lambda+1-\gamma}$, then $t \to \infty$ we obtain $\|I_G^\alpha f\|_{\widetilde{L}_{p,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$.
Therefore $\frac{\alpha}{2\lambda+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{2\lambda+1-\gamma}$.
2) *Sufficiency.* Let $f \in \widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+, G)$, then

$$\begin{aligned}&|\{y \in H(x,r) : |I_G^\alpha f(ch\,y)| > 2\beta\}|_\gamma \\ &\leq |\{y \in H(x,r) : A_1(y,r) > \beta\}|_\gamma + |\{y \in H(x,r) : A_2(y,r) > \beta\}|_\gamma.\end{aligned}$$

Also

$$A_2(y,r) = \int\limits_r^\infty A_{ch\,t}(sh\,x)^{\alpha-2\lambda-1} |f(ch\,t)| sh^{2\lambda} t dt$$

$$= \sum_{j=0}^{\infty} \int_{2^j r}^{2^{j+1} r} (A_{ch\,t}|f(ch\,x)|)(sh\,t)^{\alpha-2\lambda-1} sh^{2\lambda} t dt$$

$$\leq \|A_{ch\,t} f\|_{\widetilde{L}_{1,\lambda,\gamma}} \sum_{j=0}^{\infty} \frac{[2^{j+1}r]_1^{\gamma}}{(2^j r)^{2\lambda+1-\alpha}}$$

$$= r^{\alpha-2\lambda-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}} \begin{cases} (2r)^{\gamma} \sum\limits_{j=0}^{\left[\log_2 \frac{1}{2r}\right]} 2^{(\alpha+\gamma-2\lambda-1)j} + \sum\limits_{j=\left[\log_2 \frac{1}{2r}\right]+1}^{\infty} 2^{(\alpha-2\lambda-1)j}, & 0 < r < \frac{1}{2}, \\ \sum\limits_{j=0}^{\infty} 2^{(\alpha-2\lambda-1)j}, & r \geq \frac{1}{2} \end{cases}$$

$$\lesssim r^{\alpha-2\lambda-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}} \begin{cases} r^{\gamma} + r^{2\lambda+1-\alpha}, & 0 < r < \frac{1}{2}, \\ 1, & r \geq \frac{1}{2} \end{cases}$$

$$\lesssim \|f\|_{\widetilde{L}_{1,\lambda,\gamma}} \begin{cases} r^{\alpha+\gamma-2\lambda-1}, & 0 < r < \frac{1}{2}, \\ r^{\alpha-2\lambda-1}, & r \geq \frac{1}{2}. \end{cases} \lesssim [2r]^{\gamma}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}. \tag{3.24}$$

Taking into account the inequality (3.2) and Theorem B we obtain at $0 < r < 1$

$$\left|\left\{y \in H(x,r) : A_1(y,r) > \beta\right\}\right|_{\gamma}$$

$$\lesssim \left|\left\{y \in H(x,r) : M_{\mu} f(ch\,y) > \frac{\beta}{C\ sh^{\alpha} r}\right\}\right|_{\gamma} \lesssim \frac{sh^{\alpha} r}{\beta}[r]_1^{\gamma}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}. \tag{3.25}$$

And from (3.3) and Theorem B we have at $1 \leq r < \infty$

$$\left|\left\{y \in H(x,r) : A_1(y,r) > \beta\right\}\right|_{\gamma}$$

$$\lesssim \left|\left\{y \in H(x,r) : M_{\mu} f(ch\,y) > \frac{\beta}{C(sh\,r)^{\alpha}}\right\}\right|_{\gamma} \lesssim \frac{(sh\,r)^{\alpha}}{\beta}[r]_1^{\gamma}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}. \tag{3.26}$$

From (3.25) and (3.26) we obtain, that for all $0 < r < \infty$

$$\left|\left\{y \in H(x,r) : A_1(y,r) > \beta\right\}\right|_{\gamma} \lesssim \frac{(sh\,r)^{\alpha}}{\beta}[r]_1^{\gamma}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}. \tag{3.27}$$

If $[2r]_1^{\gamma}(sh\,r)^{\alpha-2\lambda-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}} = \beta$, then from (3.24) we obtain that $|A_2(y,r)| \lesssim \beta$ and consequently, $|\{y \in H(x,r) : A_2(y,r) > \beta\}|_{\gamma} = 0$. Then by $2r < 1$, $\beta = (sh\,r)^{\gamma+\alpha-2\lambda-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}$ and from (3.27) we have

$$\left|\left\{y \in H(x,r) : |I_G^{\alpha} f(ch\,y)| > 2\beta\right\}\right|_{\gamma} \lesssim \frac{(sh\,r)^{\alpha}}{\beta}[r]_1^{\gamma}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}$$

$$= (sh\,r)^{2\lambda-1-\gamma}[r]_1^{\gamma} = \left(\beta^{-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}\right)^{\frac{2\lambda+1-\gamma}{2\lambda+1-\gamma-\alpha}}[r]_1^{\gamma}. \tag{3.28}$$

And for $2r \geq 1$, $\beta = (sh\,r)^{\alpha-2\lambda-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}$ and from (3.26) we have

$$\left|\left\{y \in H(x,r) : |I_G^{\alpha} f(ch\,y)| > 2\beta\right\}\right|_{\gamma} \lesssim \frac{(sh\,r)^{\alpha}}{\beta}[r]_1^{\gamma}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}$$

$$= [r]_1^{\gamma}(sh\,r)^{2\lambda+1} = \left(\beta^{-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}\right)^{\frac{2\lambda+1}{2\lambda+1-\alpha}}[r]_1^{\gamma}. \tag{3.29}$$

Finally from (3.28) and (3.29) we have

$$\left|\left\{y \in H(x,r) : |I_G^{\alpha} f(ch\,y)| > 2\beta\right\}\right|_{\gamma}$$

$$\lesssim [r]_1^{\gamma} \min\left\{\left(\beta^{-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}\right)^{\frac{2\lambda+1}{2\lambda+1-\alpha}}, \left(\beta^{-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}\right)^{\frac{2\lambda+1-\gamma}{2\lambda+1-\gamma-\alpha}}\right\}$$

$$\lesssim [r]_1^{\gamma}\Big(\beta^{-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}\Big)^q,$$

where by condition of the theorem

$$\frac{2\lambda+1}{2\lambda+1-\alpha}\le q\le \frac{2\lambda+1-\gamma}{2\lambda+1-\gamma-\alpha}\Leftrightarrow \frac{\alpha}{2\lambda+1}\le 1-\frac{1}{q}\le \frac{\alpha}{2\lambda+1-\gamma}.$$

Necessity. Preliminarily we established the estimates for $\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}$. From (3.11) for $0<t<1$ we have

$$\|I_G^{\alpha}f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}}\ge \sup_{r>0} r\sup_{x\in\mathbb{R}_+,u>0}\Big([u]_1^{-\gamma}\int\limits_{\{y\in H(x,u):|I_G^{\alpha}f(ch(th\,t)y)|>r\}} sh^{2\lambda}zdz\Big)^{\frac1q}$$

$$[(th\,t)y=z,\ dy=(cth\,t)dz]$$

$$=(cth\,t)^{\frac1q}\sup_{r>0} r\sup_{x\in\mathbb{R}_+,u>0}\Big([u]_1^{-\gamma}\int\limits_{\{y\in H(xth\,t,uth\,t):|I_G^{\alpha}f(ch\,z)|>rth\,t\}} sh^{2\lambda}(cth\,t)zdz\Big)^{\frac1q}$$

$$=(cth\,t)^{\frac1q}\sup_{u>0}\Big(\frac{[uth\,t]_1}{[u]_1}\Big)^{\frac{\gamma}{q}}\sup_{r>0} rth\,t$$

$$\times\sup_{x\in\mathbb{R}_+,u>0}\Big([uth\,t]_1^{-\gamma}\int\limits_{\{y\in H(xth\,t,uth\,t):|I_G^{\alpha}f(ch\,z)|>rth\,t\}} sh^{2\lambda}(cth\,t)zdz\Big)^{\frac1q}$$

$$\ge (cth\,t)^{\frac{2\lambda+1}{q}}[th\,t]_1^{\frac{\gamma}{q}}$$

$$\times\sup_{r>0} rth\,t\sup_{x\in\mathbb{R}_+,u>0}\Big([uth\,t]_1^{1-\gamma-2\lambda}\int\limits_{\{y\in H(xth\,t,uth\,t):|I_G^{\alpha}f(ch\,z)|>rth\,t\}} sh^{2\lambda}sh^{2\lambda}zdz\Big)^{\frac1q}$$

$$\ge (th\,t)^{-\frac{2\lambda+1}{q}}[th\,t]_1^{\frac{\gamma}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}$$

$$\ge (th\,t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}\ge (sh\,t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}.$$

On the other hand from (3.11) we have

$$\|I_G^{\alpha}f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}}\le \sup_{r>0}\sup_{x\in\mathbb{R}_+,u>0}\Big([u]_1^{-\gamma}\int\limits_{\{y\in H(x,u):|I_G^{\alpha}f(ch(cth\,t)\,y)|>r\}} sh^{2\lambda}ydy\Big)^{\frac1q}$$

$$[(cth\,t)y=z,\ dy=(th\,t)dz]$$

$$=(th\,t)^{\frac1q}\sup_{r>0}\sup_{x\in\mathbb{R}_+,u>0}\Big([u]_1^{-\gamma}\int\limits_{\{y\in H(xcth\,t,u\,cth\,t):|I_G^{\alpha}f(ch\,z)|>r\}} sh^{2\lambda}(th\,t)zdz\Big)^{\frac1q}$$

$$\le (th\,t)^{\frac{2\lambda+1}{q}}\sup_{u>0}\Big(\frac{[u\,cth\,t]_1}{[u]_1}\Big)^{\frac{\gamma}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}$$

$$=(th\,t)^{\frac{2\lambda+1}{q}}[cth\,t]_{1,+}^{\frac{\gamma}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}$$

$$\le (th\,t)^{\frac{2\lambda+1-\gamma}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}\le (sh\,t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.30}$$

From (3.34) and (3.30) it follows that

$$\|I_G^{\alpha}f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}}\lesssim (sh\,t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.31}$$

Now we consider the case then $1\le t<\infty$. From (3.11) we have

$$\|I_G^{\alpha}f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}}\ge \sup_{r>0} r\sup_{x\in\mathbb{R}_+,u>0}\Big([u]_1^{-\gamma}\int\limits_{\{y\in H(x,u):|I_G^{\alpha}f(ch(th\,t)y)|>r\}} sh^{2\lambda}ydy\Big)^{\frac1q}$$

$$[(th\,t)y=z,\ dy=(cth\,t)dz]$$

$$
\begin{aligned}
&= (cth\, t)^{\frac{1}{q}} \sup_{r>0} \sup_{x\in\mathbb{R}_+, u>0} \left([u]_1^{-\gamma} \int\limits_{\{y\in H(x\,th\,t, u\,th\,t): |I_G^\alpha f(ch\,z)|>r\}} sh^{2\lambda}(cth\,t) z dz \right)^{\frac{1}{q}} \\
&= (cth\, t)^{\frac{2\lambda+1}{q}} \sup_{r>0} r\, th\, t \sup_{x\in\mathbb{R}_+, u>0} \left([u]_1^{-\gamma} \int\limits_{\{y\in H(x\,th\,t, u\,th\,t): |I_G^\alpha f(ch\,z)|>r\,th\,t\}} sh^{2\lambda} z dz \right)^{\frac{1}{q}} \\
&= (cth\, t)^{\frac{2\lambda+1}{q}} \sup_{r>0} \left(\frac{[u\,th\,t]_1}{[u]_1} \right)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}} \\
&= (cth\, t)^{\frac{2\lambda+1}{q}} [th\,t]_1^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}} \\
&\geq (th\, t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}} \geq (sh\, t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}}.
\end{aligned}
$$

On the other hand from (3.11) we get

$$
\begin{aligned}
&\|I_G^\alpha f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}} \leq \sup_{r>0} \sup_{x\in\mathbb{R}_+, u>0} \left([u]_1^{-\gamma} \int\limits_{\{y\in H(x,u): |I_G^\alpha f(ch(sh\,t)y)|>r\}} sh^{2\lambda} y dy \right)^{\frac{1}{q}} \\
&\left[(sh\,t)y = z,\ dy = \frac{dz}{sh\,t} \right] \\
&= (sh\,t)^{-\frac{1}{q}} \sup_{r>0} r \sup_{x\in\mathbb{R}_+, u>0} \left([u]_1^{1-\gamma-2\lambda} \int\limits_{\{y\in H(xsh\,t, ush\,t): |I_G^\alpha f(ch\,z)|>r\}} sh^{2\lambda} \frac{z}{sh\,t} dz \right)^{\frac{1}{q}} \\
&\leq (sh\,t)^{-\frac{2\lambda+1}{q}} \sup_{r>0} r\, sh\,t \sup_{x\in\mathbb{R}_+, u>0} \left([u]_1^{1-\gamma-2\lambda} \int\limits_{\{y\in H(xsh\,t, ush\,t): |I_G^\alpha f(ch\,z)|>r\,sh\,t\}} sh^{2\lambda} z dz \right)^{\frac{1}{q}} \\
&= (sh\,t)^{-\frac{2\lambda+1}{q}} \sup_{u>0} \left(\frac{[ush\,t]_1}{[u]_1} \right)^{\frac{\gamma+2\lambda-1}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}} \\
&\leq (sh\,t)^{-\frac{2\lambda+1}{q}} [sh\,t]_{1,+}^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}} \\
&\leq (sh\,t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}}.
\end{aligned} \tag{3.32}
$$

From (3.31) and (3.32) for $1 \leq t < \infty$ we have

$$
\|I_G^\alpha f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}} \lesssim (sh\,t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.33}
$$

Combining (3.31) and (3.33) for all $0 < t < \infty$ we obtain

$$
\|I_G^\alpha f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}} \approx (sh\,t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.34}
$$

From the boundedness I_G^α from $\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+, G)$ to $W\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+, G)$ and from (3.16) and (3.34) we have

$$
\begin{aligned}
\|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}} &\lesssim (sh\,t)^{\frac{2\lambda+1-\gamma}{q}} (sh\,t)^{\alpha+\gamma-2\lambda-1} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}} \\
&\lesssim (sh\,t)^{\alpha+(2\lambda-1)(1-\frac{1}{q})} (sh\,t)^{\gamma(1-\frac{1}{q})} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}} \\
&\lesssim \|f\|_{\widetilde{L}_{1,\lambda,\gamma}} \begin{cases} (sh\,t)^{\alpha-(2\lambda+1)(1-\frac{1}{q})}, & 0 < t < 1, \\ (sh\,t)^{\alpha+(\gamma-2\lambda-1)(1-\frac{1}{q})}, & 1 \leq t < \infty. \end{cases}
\end{aligned}
$$

If $1 - \frac{1}{q} < \frac{\alpha}{2\lambda+1}$, then for $t \to 0$ we have $\|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+, G)$.

Similarly, if $1 - \frac{1}{q} > \frac{\alpha}{2\lambda+1}$, then for $t \to \infty$ we obtain $\|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+, G)$.

Therefore, $\frac{\alpha}{2\lambda+1} \leq 1 - \frac{1}{q} \leq \frac{\alpha}{2\lambda+1-\gamma}$. □

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