Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), 33–35

ON THE DOUBLE LIMIT ASSOCIATED WITH RIEMANN'S SUMMATION METHOD

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Abstract. By Riemann's first theorem the convergence of any series $\sum_{k=0}^{\infty} a_k$ to a finite value s implies the existence of the limit $\lim\limits_{h\to 0}\sum_{k=0}^{\infty} a_k\left(\frac{\sin kh}{kh}\right)^2$, i.e. the existence of the repeated limit $\lim\limits_{h\to 0}\lim\limits_{n\to\infty}\sum_{k=0}^{n} a_k\left(\frac{\sin kh}{kh}\right)^2$ with the value s, but the converse statement does not hold. In the article it is proved the following theorem: *A numerical series* $\sum_{k=0}^{\infty} a_k$ *converges to a finite number* s *if and only if there exists the double limit* $\lim\limits_{\substack{h\to 0\\ n\to\infty}}\sum_{k=0}^{n} a_k\left(\frac{\sin kh}{kh}\right)^2$ *and the limit is equal to* s. The proof is based on Toeplitz's condition on the uniform boundedness for summation (see, relation (12) in the article) and Moore–Osgood's double limit theorem. An application of the theorem to trigonometric Fourier series is given.

Along with an arbitrary series

$$\sum_{k=0}^{\infty} a_k, \tag{1}$$

no matter whether it is converging or not, we will consider the series

$$\sum_{k=0}^{\infty} a_k\Big(\frac{\sin kh}{kh}\Big)^2 \tag{2}$$

which depends on the variable h under the assumption that this series converges for sufficiently small $h \neq 0$ and $\dfrac{\sin 0}{0} = 1$.

In particular, the series (2) will be converging for any $h \neq 0$ if the sequence $|a_k|$, $k = 0, 1, \dots$ is bounded by some number $M > 0$. Indeed, we have

$$\Big|\sum_{k=0}^{\infty} a_k\Big(\frac{\sin kh}{kh}\Big)^2\Big| \le |a_0| + Mh^{-2}\sum_{k=1}^{\infty}\frac{1}{k^2}.$$

If under the above assumption the finite limit

$$\lim_{h\to 0}\sum_{k=0}^{\infty} a_k\Big(\frac{\sin kh}{kh}\Big)^2 = \sigma \tag{3}$$

exists, then the series (1) is called Riemann-summable (or, briefly, R-summable) to the value σ.

It is obvious that the equality (3) can be written in the following form

$$\lim_{h\to 0}\lim_{n\to\infty}\sum_{k=0}^{n} a_k\Big(\frac{\sin kh}{kh}\Big)^2 = \sigma,$$

i.e. in the form of the repeated limit

$$\lim_{h\to 0}\lim_{n\to\infty} A_n(h) = \sigma, \tag{4}$$

2010 *Mathematics Subject Classification.* 26B05.
Key words and phrases. Riemann's summation method; Repeated limit; Double limit.

where it is assumed that

$$A_n(h) = \sum_{k=0}^{n} a_k \Big(\frac{\sin kh}{kh}\Big)^2.$$

Therefore the fulfillment of the equality (4) is equivalent to the R-summability of the series (1) to the value σ.

The existence of another repeated limit with the finite value ω

$$\lim_{n\to\infty} \lim_{h\to 0} A_n(h) = \omega, \tag{5}$$

implies the equality

$$\sum_{k=0}^{\infty} a_k = \omega \tag{6}$$

and vice versa: from the equality (6) there follows the equality (5). Hence we have the following

Proposition. *The convergence of the series* (1) *to the value* ω *is the necessary and sufficient condition for the fulfillment of the equality* (5).

We establish the relationship between the convergence of the series (1) and the existence of the double limit

$$\lim_{\substack{h\to 0\\ n\to\infty}} A_n(h). \tag{7}$$

As to this relationship we have the following statement.

Theorem. *The convergence of the series* (1) *to the finite value* s

$$\sum_{k=0}^{\infty} a_k = s \tag{8}$$

is the necessary and sufficient condition for the fulfillment of the equality

$$\lim_{\substack{h\to 0\\ n\to\infty}} A_n(h) = s. \tag{9}$$

Sufficiency. By virtue of the above Proposition, from the equality (8) we obtain the equality (5) where ω is replaced by s. Therefore the limit

$$\lim_{h\to 0} A_n(h) \tag{10}$$

is finite for any n.

Furthermore, from the equality (8) there follows the equality

$$\lim_{h\to 0} \lim_{n\to\infty} A_n(h) = s \tag{11}$$

by virtue of Riemann's first theorem [5, p. 319].

Along with this, during the proof of this Riemann's first theorem an important fact is established that consists in that the series

$$\sum_{k=0}^{\infty} a_k \Big(\frac{\sin kh}{kh}\Big)^2$$

converges uniformly with respect to h, i.e. the limit

$$\lim_{n\to\infty} A_n(h) \tag{12}$$

exists uniformly with respect to h ([3, Ch. XIII, §13.8.2], [4, Ch. 9, §9.62], [5, Ch. IX, §2, inequality (2.6), which holds uniformly with respect to the family $\{(h_i)\}$ of all sequences (h_i) tending to zero as $i \to \infty$]).

Therefore by virtue of the Moore–Osgood Double Limit Theorem [2, p. 180] modified for the continuous parameter h, the equalities

$$\lim_{n\to\infty} \lim_{h\to 0} A_n(h) = \lim_{h\to 0} \lim_{n\to\infty} A_n(h) = \lim_{\substack{h\to 0\\ n\to\infty}} A_n(h) = s$$

are fulfilled.

Thus the convergence of the series (1) to the value s implies the existence of the limit (7) and the equality (9).

Necessity. If the double limit

$$\lim_{\substack{h\to 0\\ n\to\infty}} A_n(h) = s$$

exists, then there exists the partial limit s equal to $\lim\limits_{n\to\infty} A_n(0)$. But $\lim\limits_{n\to\infty} A_n(0) = \sum\limits_{k=0}^{\infty} a_k$.

Therefore the equality (8) is fulfilled. The theorem is proved.

Finally, we give an application of the above theorem to trigonometric Fourier series. It is well known that there is the summable Kolmogorov function $K(x)$ on $[-\pi,\pi]$, whose Fourier series

$$K \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \tag{13}$$

diverges at every point $x \in [-\pi,\pi]$ [5, p. 310].

However, the series (13) is R-summable at almost all points $x \in [-\pi,\pi]$ to values $K(x)$ [1, Ch. I, paragraph 69]. From the theorem that is proved above it follows.

Corollary. *For the series* (13) *the following statements are true:*

1. *The equality*

$$\lim_{h\to 0}\lim_{n\to\infty}\left[\frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)\Big(\frac{\sin kh}{kh}\Big)^2\right] = K(x)$$

is fulfilled for almost all points $x \in [-\pi,\pi]$;

2. *There is not a point $x \in [-\pi,\pi]$ at which the double limit*

$$\lim_{\substack{h\to 0\\ n\to\infty}}\left[\frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)\Big(\frac{\sin kh}{kh}\Big)^2\right]$$

would exist.

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(Received 11.07.2018)

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