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EXISTENCE RESULTS FOR IMPULSIVE STOCHASTIC NEUTRAL INTEGRODIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY

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Abstract. In this article, we investigate the existence of mild solutions for a class of impulsive neutral stochastic integro-differential equations with state-dependent delay. The results are obtained by using the Krasnoselskii-Schaefer type fixed point theorem combined with theories of resolvent operators. In the end as an application, an example has been presented to illustrate the results obtained.

1. Introduction

The investigation of stochastic differential equations has been picking up much importance and attention of researchers due to its wide applicability in science and engineering. Since arbitrary fluctuations are regular in the real world, scientific (mathematical) models for complex systems are frequently subject to instabilities, for example, indeterminate parameters, fluctuating powers, or random boundary conditions. Also, uncertainties may be created by the absence of knowledge of some chemical, physical or biological systems that are not well known, and in this manner are not suitably represented (or missed totally) in the scientific models. Despite the fact that these fluctuations and unrepresented systems may be extremely little or quick, their long-term effect on the system evolution may be delicate or even meaningful. This kind of delicate effects on the general evolution of dynamical systems has been seen in, for instance, stochastic resonance, stochastic bifurcation and noise-induced pattern development. In this way considering stochastic impacts is of central significance for mathematical modeling of complex systems under uncertainty. Thus, a large number of these systems can be modeled by stochastic differential equations, for example, price processes, exchange rates, and interest rates, among others in finance.

The existence and uniqueness of the mild solutions of stochastic differential equations have been studied by many authors. In [19], author has obtained sufficient conditions for the existence and uniqueness of solution of stochastic differential equations under uniform Lipschitz and the linear growth condition. In [17], author has shown that there exists the unique solution for neutral stochastic functional differential equation under uniform Lipschitz and the linear growth condition. The approximate controllability of nonlocal neutral stochastic fractional differential equations is studied by authors in [7]. In [1], authors have considered an impulsive stochastic semilinear neutral functional differential equations with infinite delays and discussed the existence, uniqueness and stability of mild solutions of considered stochastic differential equations with a Lipschitz condition and without a Lipschitz condition by utilizing the technique of successive approximations. In [35], authors have discussed the existence of solutions to impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay. The asymptotic stability of fractional impulsive neutral stochastic partial integrodifferential equations with state-dependent delay is studied by the authors in [36]. The existence and uniqueness of square-mean almost automorphic solutions for some stochastic differential equations have been studied by authors in [8] in which the asymptotic stability of the unique square-mean almost automorphic solution in the square-mean sense has been discussed. In [11], authors have considered an impulsive neutral stochastic functional integro-differential equation with infinite delays in a separable real Hilbert space and established the existence results. In [32],the existence of the mild

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solution nonlinear fractional stochastic differential equation has been studied by the authors by using fixed point theorems and α-resolvent family. For more study on stochastic differential equation, we refer to papers [5, 8, 11, 14, 20, 22, 23, 25, 29–32, 34–36].

Impulsive effects likewise exist in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time, involving such fields as medicine and biology, economics, mechanics, electronics and telecommunications, etc. Recently, many interesting and important results on impulsive differential equations have been derived in [2, 33] and the references therein. More recently, in [16, 17], Sakthivel and Luo have discussed the asymptotic stability for mild solution of impulsive stochastic partial differential equations by employing the fixed point theorem; by establishing an impulsive-integral inequality, the exponential stability for mild solution of impulsive stochastic partial differential equations with delays was considered in [6]. Besides, there are some results about the existence and uniqueness for mild solution of impulsive stochastic partial functional differential equations, see [10, 15, 24] and references therein.

On the other hand, there has been intense interest in the study of impulsive neutral stochastic partial differential equations with memory (e.g. delay) and integrodifferential equations with resolvent operators. Since many control systems arising for realistic models depends heavily on histories (that is, effect of infinite delay on the state equations), there is real need to discuss the existence results for impulsive partial stochastic neutral integrodifferential equations with sate-dependent delay. Recently, the problem of the existence of solutions for partial impulsive functional differential equations with state-dependent delay has been investigated in many publications such as [18, 26] and the references therein.

As the motivation of above discussed works, we consider the following neutral stochastic impulsive integrodifferential functional equations with state-dependent

$$\begin{cases} d\Big[x(t) - G\Big(t, x_t, \int_0^t g(t,s,x_s)ds\Big)\Big] = A\Big[x(t) - G\Big(t, x_t, \int_0^t g(t,s,x_s)ds\Big)\Big]dt \\ \qquad + \Big(\int_0^t B(t-s)\Big[x(s) - G\Big(s, x_s, \int_0^s g(s,u,x_u)du\Big)\Big]ds\Big)dt + F\left(t, x_{\rho(t,x_t)}\right)dw(t), \\ \qquad t \in J, J = [0,b], \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad k = 1, 2, \ldots, m \ldots \\ x_0(\cdot) = \varphi(\cdot) \in \mathbb{B}, \end{cases} \tag{1}$$

where the state $x(.)$ takes values in a separable real Hilbert space $\mathbb{H}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a the infinitesimal generator of a C_0-semigroup $(T(t))_{t \geq 0}$ on $\mathbb{H}$, for $t \geq 0$, $B(t)$ is a closed linear operator with domain $D(A) \subset D(B(t))$; $0 < t_1 < \cdots < t_m < b$, are prefixed points and the symbol $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Let $\mathbb{K}$ be another separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathbb{K}}$ and norm $\| \cdot \|_K$. Suppose $\{w(t) : t \geq 0\}$ is a given $\mathbb{K}$-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q > 0$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$, which is generated by the Wiener process w. We are also employing the same notation $\| \cdot \|$ for the norm $L(\mathbb{K}; \mathbb{H})$, where $L(\mathbb{K}; \mathbb{H})$ denotes the space of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. The history $x_t : (-\infty, 0] \to \mathbb{H}$, $x_t(\theta) = x(t + \theta)$, belongs to some abstract phase space $\mathbb{B}$ defined axiomatically; the initial data $\{\varphi(t) : -\infty < t \leq 0\}$ is an $\mathcal{F}_0$-adapted, $\mathbb{B}$-valued random variable independent of the Wiener process w with finite second moment, and F, G, g, ρ, I_k; $(k = 1, \ldots, m)$, are given functions to be specified later.

To the best of the authors's knowledge, there is no results about the existence of mild solutions impulsive neutral partial stochastic functional integrodifferential equations with state-dependent, which is expressed in the form (1). The aim of our paper main is to establish some existence results for the system (1). Our main results concerning (1) rely essentially on techniques using strongly continuous family of operators $\{R(t), t \geq 0\}$, defined on the Hilbert space $\mathbb{H}$ and called their resolvent. The resolvent operator is similar to the semigroup operator for abstract differential equations in Banach spaces. There is a rich theory for analytic semigroups and we wish to develop theories for (1) which yield analytic resolvent and fractional Brownian motion. However, the resolvent operator does not satisfy semigroup properties (see, for instance [4, 14]) and our objective in the present paper is to apply the theory developed by Grimmer [5], because it is valid for generators of strongly continuous semigroup, not necessarily analytic. The main contribution of this manuscript is that it proposes a framework for studying the mild solution to stochastic integro-differential equation with state-dependent and impulsive conditions.

The structure of this paper is as follows. In Section 2, we recall some necessary preliminaries on stochastic integral and resolvent operator. In Section 3, we discuss the results on existence and uniqueness of mild solutions. Finally in Section 4, an example is presented which illustrates the main results for equation (1).

2. Preliminaries

2.1. **Wiener process.** Throughout this paper, let $\mathbb{H}$ and $\mathbb{K}$ be tow real separabl Hilbert spaces. We denote by $\langle\cdot,\cdot\rangle_{\mathbb{H}}$, $\langle\cdot,\cdot\rangle_{\mathbb{K}}$ their inner products and by $\|\cdot\|_{\mathbb{H}}$, $\|\cdot\|_{\mathbb{K}}$ their vecteur norms, respectively. $\mathcal{L}(\mathbb{K},\mathbb{H})$ denote the space of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operator norm $\|\cdot\|$ and we abbreviate this notation to $\mathcal{L}(\mathbb{H})$ when $\mathbb{H} = \mathbb{K}$.

In the sequel, we always use the same symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces potentially involved when no confusion possibly arises.

Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition (i.e. it is increasing and right-continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets).

Lets $\{w(t) : t \geq 0\}$ denote a $\mathbb{K}$-valued Wiener process difined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, with covariance operator Q; that is $\mathbb{E}\langle w(t), x\rangle_{\mathbb{K}}\langle w(t), y\rangle_{\mathbb{K}} = (t\wedge s)\langle Qx, y\rangle_{\mathbb{K}}$, for all $x, y \in \mathbb{K}$, where Q is a positive, self-adjoint, trace class operator on $\mathbb{K}$. In particular,we denote W a $\mathbb{K}$-valued Q-wener pocess with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. To define stochastic integrals with respect to the Q-Wiener process with w,we introduce the subspace $\mathbb{K}_0 = Q^{1/2}\mathbb{K}$ of $\mathbb{K}$ endowed with the inner product $\langle u, v\rangle_{\mathbb{K}_0} = \langle Q^{1/2}u, Q^{1/2}v\rangle_{\mathbb{K}}$ as a Hilbert space. We assume that there exists a complete orthonormal system $\{e_i\}$ in $\mathbb{K}$, a bounded sequence of positive real numbers λ_i such that $Qe_i = \lambda_i e_i$, $i = 1, 2, \dots$, and a sequence $\{\beta_i(t)\}_{i\geq 1}$ of independent standard Brownian motions such that $w(t) = \sum_{i=1}^{+\infty} \sqrt{\lambda_i}\beta_i(t)e_i$ for $t \geq 0$ and $\mathcal{F}_t = \mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0 \geq s \geq t\}$. Let $\mathcal{L}_2^0 = \mathcal{L}_2(\mathbb{K}_0, \mathbb{H})$ be the space of all Hilbert-Schmidt operators from $\mathbb{K}_0$ to $\mathbb{H}$. It turns out to be a separable Hilbert space equipped whith the norm $\|v\|^2_{\mathcal{L}_2^0} = tr((vQ^{1/2})(vQ^{1/2})^*)$ for any $v \in \mathcal{L}_2^0$. Obviously, for any bounded operator $v \in \mathcal{L}_2^0$, this norm reduces to $\|v\|^2_{\mathcal{L}_2^0} = tr(vQv^*)$.

2.2. **Deterministic integrodifferential equations.** In the present section, we recall some definitions, notations and properties needed in the sequel.

In what follows, $\mathbb{H}$ will denote a Banach space, A and $B(t)$ are closed linear operators on $\mathbb{H}$. Y represents the Banach space $D(A)$, the domain of operator A, equipped with the graph norm

$$\|y\|_Y = \|Ay\| + \|y\|, \qquad y \in Y.$$

The notation $C([0, +\infty[; Y)$ stands for the space of all continuous functions from $[0, +\infty[$ into Y. We then consider the following Cauchy problem

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds, \text{ for } t \geq 0, \\ v(0) = v_0 \in \mathbb{H}. \end{cases} \tag{2}$$

Definition 1 ([5])**.** A resolvent operator of the Eq. (2) is a bounded linear operator valued function $R(t) \in \mathcal{L}(\mathbb{H})$ for $t \geq 0$, having the following properties:

(1) $R(0) = I$ and $\|R(t)\| \leq \eta e^{\delta t}$ for some constants η and δ.
(2) For each $x \in \mathbb{H}$, $R(t)x$ is strongly continuous for $t \geq 0$.
(3) $R(t) \in \mathcal{L}(\mathbb{H})$. For $x \in Y$, $R(\cdot)x \in C^1([0, +\infty; \mathbb{H}) \cap C([0, +\infty[; Y)$ and

$$R'(t)x = AR(t)x + \int_0^t B(t-s)R(s)xds = R(t)Ax + \int_0^t R(t-s)B(s)xds, \quad \text{for} \quad t \geq 0.$$

For additional detail on resolvent operators, we refer the reader to [5] and [21]. The resolvent operator plays an important role to study the existence of solutions and to establish a variation of constants formula for non-linear systems. For this reason, we need to know when the linear system (2) possesses a resolvent operator. Theorem (1) below provides a satisfactory answer to this problem.

In what follows we suppose the following assumptions:

(H1) A generates a strongly semigroup in Banach space $\mathbb{H}$

(H2) For all $t \geq 0$, $t \mapsto B(t)$ is continuous linear operator from $(Y, \|\cdot\|_Y)$ into $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$. Moreover, there exists an integrable function $c : [0, +\infty[\to \mathbb{R}^+$ such that for any $y \in Y$, $t \mapsto B(t)y$ belongs to $W^{1,1}([0, +\infty[; \mathbb{H})$ and

$$\left\| \frac{d}{dt} B(t)y \right\|_{\mathbb{H}} \leq c(t)\|y\|_Y, \quad \text{for} \quad y \in Y, \quad \text{and} \quad t \geq 0.$$

We recall that $W^{k,p}(\Omega) = \{\tilde{w} \in L^p(\Omega) : D^\alpha \tilde{w} \in L^p(\Omega), \quad \forall |\alpha| \leq k\}$, where $D^\alpha \tilde{w}$ is the weak α-th partial derivative of $\tilde{w}$.

The following theorem gives sufficient conditions of ensuring the existence of resolvent operator for Eq.(2)

Theorem 1 ([5])**.** *Assume that **(H1)** and **(H2)** hold. Then there exists a unique resolvent operator for* (2)*.*

In the sequel, we recall some results on the existence of solutions for the following integrodifferential equation:

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds + q(t), \text{ for } t \geq 0, \\ v(0) = v_0 \in \mathbb{H}. \end{cases} \tag{3}$$

where $q : [0, +\infty[\to \mathbb{H}$ is a continuous function.

Definition 2 ([5])**.** A continuous function $v : [0, +\infty[\to \mathbb{H}$ is said to be a strict solution of equation (3) if

(1) $v \in C^1([0, +\infty[, \mathbb{H}) \cap C([0, +\infty[, Y)$,
(2) v satisfies equation Eq. (3) for $t \geq 0$.

Remark 2.1. From this definition we deduce that $v(t) \in D(A)$, the function $B(t-s)v(s)$ is integrable, for all $t > 0$ and $s \in [0, t[$.

Theorem 2 ([5])**.** *Assume that **(H1)**, **(H2)** hold. If v is a strict solution of the Eq.* (3)*, then the following variation of constants formula holds*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds, \quad \textit{for} \quad t \geq 0. \tag{4}$$

Accordingly, we can establish the following definition.

Definition 3 ([5])**.** A function $v : [0, +\infty[\to \mathbb{H}$ is called a mild solution of equation (3) for $v_0 \in \mathbb{H}$, if v satisfies the variation of constants formula (4).

Theorem 3 ([5]). *Let $q \in C^1([0,+\infty[;\mathbb{H})$ and v be defined by (4). If $v_0 \in D(A)$, then v is a strict solution of the equation (3).*

Lemma 2.1 ([14]). *Assume that **(H1)** and **(H2)** hold. Then, there exists a constant $L = L(T)$ such that $\|R(t+\varepsilon) - R(t)R(\varepsilon)\| \le L\varepsilon$ for $0 < \varepsilon \le t \le T$.*

Theorem 4 ([14]). *Assume that **(H1)** and **(H2)** hold. Let $T(t)$ be a compact for $t > 0$. Then the corresponding resolvent operator $R(t)$ of (2) is also compact for $t > 0$.*

In this work, we will employ an axiomatic definition of the phase space $\mathbb{B}$ introduced by Hale and Kato [9].

Definition 4. The phase space $\mathbb{B}(]-\infty,0],\mathbb{H})$ (denoted by $\mathbb{B}$ simply) is the space of continuous functions from $]-\infty,0]$ to $\mathbb{H}$ endowed with seminorm $\|\cdot\|_{\mathbb{B}}$, and $\mathbb{B}$ satisfies the following axioms:

(A1) If $x :]-\infty,T] \to \mathbb{H}$ is continuous on $[t_0,T]$, $0 \le t_0 \le T$ and $x_{t_0} \in \mathbb{B}$, then, for every $t \in [t_0,T]$, the following conditions hold:
  (1) $x_t \in \mathbb{B}$;
  (2) $\|x_t\|_{\mathbb{B}} \le \tilde{M}(t-t_0) \sup_{0\le s\le t} \|x(s)\|_{\mathbb{H}} + N(t-t_0)\|x_0\|_{\mathbb{B}}$, where $\tilde{M}, N : [0,+\infty[\to [1,+\infty[$, $\tilde{M}$ is continuous and N_1 is locally bounded, $\tilde{M}$, N are independent of $x(\cdot)$.
  (3) $\|x(t)\|_{\mathbb{H}} \le \tilde{H}\|x_t\|_{\mathbb{B}}$, where $\tilde{H} > 0$ such that $\tilde{H}$ are independent of $x(\cdot)$.

(A2) For the fonction $x(\cdot)$ in $(A1)$, the function $t \mapsto x_t$ is continuous from $[t_0,T]$ into $\mathbb{B}$.

(A3) The space $\mathbb{B}$ is complete.

The $\mathbb{B}$-valued stochastic process $x_t : \Omega \to \mathbb{B}$, $t \in J$ is defined by setting $x_t = \{x(t+\theta)(w) : \theta \in]-\infty,0]\}$.

The collection of all strongly measurable, square integrable, $\mathbb{H}$-valued random variables, denoted by $L_2(\Omega,\mathbb{H})$ is a Banach space equipped with norm $\|x(\cdot)\|_{L_2} = (\mathbb{E}\|x(\cdot,w)\|^2)^{\frac{1}{2}}$, where the expectation, $\mathbb{E}$ is defined by $\mathbb{E}x = \int_\Omega x(w)dP$. Let $C(J, L_2(\Omega,\mathbb{H}))$ be the Banach space of all continuous maps from J into $L_2(\Omega,\mathbb{H})$ satisfying the condition $\sup_{0\le t\le T} E\|x(t)\|^2 < \infty$. Let $L_2^0(\Omega,\mathbb{H})$ denote the family of all $\mathcal{F}_0$-measurable, $\mathbb{H}$-valued random variables.

We say that a function $x : [\mu,\tau] \to \mathbb{H}$ is a normalized piecewise continous function on $[\mu,\tau]$, if x is piecewise continuous and left continuous on $(\mu,\tau]$. We denote by $\mathcal{PC}([\mu,\tau],\mathbb{H})$ the space formed by the normalized piecewise continuous, $\mathcal{F}_t$-adapted measurable processes from $[\mu,\tau]$ into $\mathbb{H}$. In particular, we introduce the space $\mathcal{PC}$ formed by all $\mathcal{F}_t$-adapted measurable, $\mathbb{H}$-valued stochastic processes $\{x(t) : t \in [0,T]\}$ such that x is continuous at $t \ne t_k$, $x(t_k) = x(t_k^-)$ and $x(t_k^+)$ exists for $k = 1,2,\dots,m$. In this paper, we always assume that $\mathcal{PC}$ is endowed with the norm

$$\|x\|_{\mathcal{PC}} = \Big(\sup_{0\le t\le T} \mathbb{E}\|x(t)\|^2\Big)^{\frac{1}{2}}.$$

Then $(\mathcal{PC}, \|.\|_{\mathcal{PC}})$ is a Banch space.

To simplify the notations, we put $t_0 = 0$, $t_{m+1} = b$ and for $x \in \mathcal{PC}$, we denote by $\hat{x}_k \in C([t_k,t_{k+1}]; L_2(\Omega,\mathbb{H}))$, $k = 0,1,\dots,m$, the function given by

$$\hat{x}_k(t) := \begin{cases} x(t) & \text{for } t \in (t_k,t_{k+1}], \\ x(t_k^+) & \text{for } t = t_k. \end{cases}$$

Moreover, for $B \subset \mathcal{PC}$ we denote by $\hat{B}_k$, $k = 0,1,\dots,m$, the set $\hat{B}_k = \{\hat{x}_k : x \in B\}$. The notation $B_r(x,\mathbb{H})$ stands for the closed ball with center at x and radius $r > 0$ in $\mathbb{H}$.
Now, we give the definition of mild solution for (1).

Definition 5. An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty,b] \to \mathbb{H}$ is said to be a mild solution of the system (1) if $x_0 = \varphi(t)$, $x_{\rho(s,x_s)} \in \mathbb{B}$ satisfying $x_0 \in L_2^0(\Omega,\mathbb{H})$, $x\mid_J \in \mathcal{PC}$, and $\Delta x(t_k) = I_k(x(t_k))$,

$k = 1, \dots, m$, such that

$$x(t) = R(t)[\varphi(0) - G(0, \varphi, 0)] + G\Bigg(t, x_t, \int_0^t g(t, s, x_s) ds\Bigg)$$

$$+ \int_0^t R(t-s) F\left(s, x_{\rho(s,x_s)}\right) dw(s) + \sum_{0<t_k<t} R(t - t_k) I_k(x(t_k)).$$

Lemma 2.2. *A set $B \subset \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if, and only if, the set $\hat{B}_k$ is relatively compact in $C([t_k, t_{k+1}]; L_2(\Omega, \mathbb{H}))$, for every $k = 0, 1, \dots, m$.*

The next result is a consequence of the phase space axioms.

Lemma 2.3. *Let $x : (-\infty, b] \to \mathbb{H}$ be an $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0 = \varphi(\cdot) \in L_2^0(\Omega, \mathbb{B})$ and $x\mid_J \in \mathcal{PC}(J, \mathbb{H})$, then*

$$\|x_s\|_{\mathbb{B}} \leq M_b \mathbb{E}\|\varphi\|_{\mathbb{B}} + K_b \sup_{0\leq s\leq b} \mathbb{E}\|x(s)\|,$$

where $K_b = \sup\{K(t) : 0 \leq t \leq b\}$, $M_b = \sup\{\tilde{M}(t) : 0 \leq t \leq b\}$.

Finally, we end this section by stating the following Krasnoselskii-Schaefer type fixed point theorem appeared in [3] which is our main tool.

Lemma 2.4 ([3])**.** *Let Φ_1, Φ_2 be two operators such that :*

(a) Φ_1 *is a contraction, and*
(b) Φ_2 *is completely continuous.*

Then either:

(i) *the operator equation $x = \Phi_1 x + \Phi_2 x$ has a solution, or*
(ii) *the set $\Lambda = \{x \in X : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x\}$ is unbounded for $\lambda \in (0, 1)$.*

In the following section, we establish the existence theorem of the mild solution.

3. Main Results

Throughout this paper, for the existence and uniqueness of the mild solution to (1), we shall impose the following assumptions:

(H3) The resolvent operator $R(t)$, $t \geq 0$ is compact and there exists constant M such that $\|R(t)\|^2 \leq M$, $t \in J$.

(H4) The function $t \mapsto \varphi_t$ is continous from $\mathcal{R}(\rho^-) = \{\rho(s, \psi) \leq 0,\ (s, \psi) \in J \times \mathbb{B}\}$ into $\mathbb{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that $\|\phi_t\| \leq J^\varphi(t)\|\phi\|_{\mathbb{B}}$ for each $t \in \mathcal{R}(\rho^-)$.

(H5) The function $F : J \times \mathbb{B} \to \mathcal{L}_2^0(\mathbb{K}, \mathbb{H})$, for each $t \in J$, the function $F(t, .) : \mathbb{B} \to \mathcal{L}_2^0(\mathbb{K}, \mathbb{H})$ is continuous and for each $\psi \in \mathbb{B}$, the function $F(., \psi) : J \to \mathcal{L}_2^0(\mathbb{K}, \mathbb{H})$ is strongly measurable.

(H6) For each positive number $r > 0$, there exists a positive function $l(r)$ dependent on r such that

$$\sup_{\|\psi\|_{\mathbb{B}}^2 \leq r} \mathbb{E}\|F(t, \psi)\|^2 \leq l(r),$$

and there exists a constant d such that

$$0 \leq \limsup_{\|\psi\|_{\mathbb{B}}^2 \to \infty} \left(\sup_{t\in J} \frac{\mathbb{E}\|F(t, \phi)\|^2}{\|\psi\|_{\mathbb{B}}^2}\right) \leq d.$$

(H7) There exists a constant $L_1 > 0$ such that

$$\mathbb{E}\left\|\int_0^t [g(t, s, \psi) - g(t, s, \phi)] ds\right\|^2 \leq L_1 \|\psi - \phi\|_{\mathbb{B}}^2$$

for $t,\ s \in J, \psi, \phi \in \mathbb{B}$.

(H8) The function $G : J\times\mathbb{B}\times\mathbb{H}\to\mathbb{H}$ is continous and satisfies the Lipschitz condition, that is, there exists a constant $L_2>0$ such that

$$\mathbb{E}\|G(t_1,\psi_1,\phi_1)-G(t_2,\psi_2,\phi_2)]\|^2\le L_2[|t_1-t_2|+\|\psi_1-\psi_2\|_{\mathbb{B}}^2+\mathbb{E}\|\phi_1-\phi_2\|^2]$$

for $0\le t_1,\, t_2\le b$, $\psi_i\in\mathbb{B}$, $\phi_i\in\mathbb{H}$, $i=1,2,\dots,m$
with

$$L_0=L_2(1+L_1)K_b^2<1.$$

(H9) $I_k\in C(\mathbb{H},\mathbb{H})$, $k=1,2,\dots,m$, are completely continuous and there exists constants c_k, $k=1,2,\dots,m$, such that

$$0\le\limsup_{\|x\|^2\to\infty}\frac{\|I_k(x)\|^2}{\|x\|^2}\le c_k,\quad x\in\mathbb{H}.$$

Lemma 3.1 ([12])**.** *Let $x:(-\infty,b]\to\mathbb{H}$ such that $x_0=\varphi$. If **(H4)** is satisfied, then $\|x_s\|_{\mathbb{B}}\le(M_b+J_0^{\phi})\|\phi\|_{\mathbb{B}}+K_b\sup\{\|x(\theta)\|;\,\theta\in[0,\max\{0,s\}]\},\quad s\in\mathcal{R}(\rho^-)\cup J$, where $J_0^{\phi}=\sup_{t\in\mathcal{R}(\rho^-)}J^{\phi}(t)$.*

Remark 3.1 ([12,13])**.** Let $\varphi\in\mathbb{B}$ and $t\le0$. The notation φ_t represents the function defined by $\varphi_t=\varphi(t+\theta)$. Consequently, if the function $x(\cdot)$ in axiom $(A1)$ is such that $x_0=\varphi$, then $x_t=\varphi_t$. We observe that φ_t is well-defined for $t<0$ since the domain of φ is $(-\infty,0]$.

Theorem 5. *Let $\varphi\in L_2^0(\Omega,\mathbb{H})$. If the assumptions **(H1)** – **(H8)** hold and $\rho(t,\psi)\le t$, for every $(t,\,\psi)\in J\times\mathbb{B}$, then there exists a mild solution of equation Eq.* (1) *provided that*

$$8\left[L_2(1+L_1)K_b^2+Mm\sum_{k=1}^{m}c_k\right]\le1. \tag{5}$$

Proof. Consider the space $\mathbb{Y}=\{x\in\mathcal{PC}:x(0)=\varphi(0)=0\}$ endowed with the uniform convergence topology $(\|\cdot\|_\infty)$ and define the mapping Φ on $\mathbb{Y}$ by

$$\Phi(x)(t)=\begin{cases}0, & t\in]-\infty,0],\\ R(t)[\varphi(0)-G(0,\varphi,0)]+G\left(t,\bar{x}_t,\int_0^t g(t,s,\bar{x}_s)ds\right)+\int_0^t R(t-s)F\left(s,x_{\rho(s,\bar{x}_s)}\right)dw(s)\\ +\sum_{0<t_k<t}R(t-t_k)I_k(\bar{x}(t_k))\quad \text{for } t\in J,\end{cases}$$

where $\bar{x}:(-\infty,0]\to\mathbb{H}$ is such that $\bar{x}_0=\varphi$ and $\bar{x}=x$ on J.

Then it is clear that to prove the existence of mild solutions of the problem (1) is equivalent to find a fixed point for the operator Φ. First we show that $\Phi(\mathcal{PC})\subset\mathcal{PC}$.

From **(H5)**, **(H7)** and **(H8)**, it follows that the function G,F and I_k, $k=1,2,\dots,m$ are continuous,which enables us to conclude that Φ is well-defined operator from $\mathbb{Y}$ into $\mathbb{Y}$. We show that Φ has a fixed point, which in turn is a mild solution of the problem (1).

Let $\bar{\varphi}:(-\infty,T)\to\mathbb{H}$ be the extension of $(-\infty,0]$ such that $\bar{\varphi}(\theta)=\varphi(0)=0$ on J and $J_0^{\varphi}=\sup\{J^{\varphi}(s)\ :\ \ s\in\mathcal{R}(\rho^-)\}$. Now, we decompose Φ as $\Phi_1+\Phi_2$, where

$$(\Phi_1x)(t)=-R(t)G(0,\varphi,0)+G\left(t,\bar{x}_t,\int_0^t g(t,s,\bar{x}_s)ds\right),\quad t\in J$$

$$(\Phi_2x)(t)=R(t)\varphi(0)+\int_0^t R(t-s)F\left(s,\bar{x}_{\rho(s,\bar{x}_s)}\right)dw(s)+\sum_{0<t_k<t}R(t-t_k)I_k(\bar{x}(t_k)),\quad t\in J.$$

The proof is divided into the following five steps.
Step 1. Φ_1 is a contraction on $\mathbb{Y}$.

Let $t \in J$ and $y^1, y^2 \in \mathbb{Y}$. Then, by using (**H7**) and (**H8**), we have

$$\begin{aligned}\mathbb{E}\|(\Phi_1\overline{y^1})(t) - (\Phi_1\overline{y^2})(t)\|^2 &\leq \mathbb{E}\left\|G\left(t, \overline{y^1}_t, \int_0^t g(t,\tau,\overline{y^1}_\tau)d\tau\right) - G\left(t, \overline{y^2}_t, \int_0^t g(t,\tau,\overline{y^2}_\tau)d\tau\right)\right\|^2 \\ &\leq L_2\left(\|\overline{y^1}_t - \overline{y^2}_t\|_{\mathbb{B}}^2 + L_1\|\overline{y^1}_t - \overline{y^2}_t\|_{\mathbb{B}}^2\right) \\ &\leq L_2(1+L_1)\|\overline{y^1}_t - \overline{y^2}_t\|_{\mathbb{B}}^2 \\ &\leq L_2(1+L_1)K_b^2 \sup_{s\in J} \mathbb{E}\|y^1(s) - y^2(s)\|_{\mathbb{B}}^2\end{aligned}$$

by using $\overline{y} = y$ on J.
Taking supremum over t,

$$\|\Phi_1 y^1 - \Phi_1 y^2\|_{PC}^2 \leq L_0\ \|y^1 - y^2\|_{PC}^2,$$

where $L_0 = L_2(1+L_1)K_b^2 < 1$. Thus Φ_1 is a contraction on $\mathbb{Y}$.

Step 2. Φ_2 maps bounded sets into bounded sets in $\mathbb{Y}$.
For each $r > 0$, let

$$B_r(0,\mathbb{Y}) := \{x \in \mathbb{Y} : \mathbb{E}\|x\|^2 \leq r\}.$$

Then, for each $r, B_r(0,\mathbb{Y})$ is a bounded closed convex subset in $\mathbb{Y}$. Indeed, it is enough to show that there exists a positive constant $\mathcal{L}$ such that for each $x \in B_r(0,\mathbb{Y})$ one has $\mathbb{E}\|\Phi_2 x\|^2 \leq \mathcal{L}$. Now, for $t \in J$ we have

$$(\Phi_2 x)(t) = R(t)\varphi(0) + \int_0^t R(t-s)F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right)dw(s) + \sum_{0<t_k<t} R(t-t_k)I_k(\bar{x}(t_k)), \quad t \in J. \tag{6}$$

In view of **(H6)** and **(H9)**, there exist positive constants ϵ, ϵ_k $(k = 1, \ldots, m)$, γ and $\bar{\gamma}$ such that, for all $\|\psi\|_{\mathcal{B}}^2 > \gamma$, $\|\phi\|^2 > \bar{\gamma}$,

$$\begin{aligned}\|F(t,\psi)\|^2 &\leq (d+\epsilon)\|\psi\|_{\mathbb{B}}^2, \\ \|I_k(\phi)\|^2 &\leq (c_k+\epsilon_k)\|\phi\|^2,\end{aligned}$$

and

$$8[L_2(1+L_1)K_b^2 + Mm\sum_{k=1}^m(\epsilon_k + c_k)] \leq 1. \tag{7}$$

Let

$$\begin{aligned}F_1 &= \{\psi : \|\psi\|_{\mathbb{B}}^2 \leq \gamma\}, \quad F_2 = \{\psi : \|\psi\|_{\mathbb{B}}^2 > \gamma\}, \\ G_1 &= \{\phi : \|\phi\|^2 \leq \bar{\gamma}\}, \quad G_2 = \{\phi : \|\phi\|^2 > \bar{\gamma}\}, \\ C_1 &= \max\{\|I_k(\phi)\|^2, \phi \in G_1\}.\end{aligned}$$

Thus

$$\|F(t,\psi)\|^2 \leq l(\gamma) + (d+\epsilon)\|\psi\|_{\mathbb{B}}^2, \tag{8}$$

$$\|I_k(\phi)\|^2 \leq C_1 + (c_k+\epsilon_k)\|\phi\|^2. \tag{9}$$

If $x \in B_r(0,\mathbb{Y})$, from Lemma 2.3 and 3.1, it follows that

$$\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathcal{B}}^2 \leq 2\left[(M_b + \bar{J}_0^{\varphi})\|\varphi\|_{\mathcal{B}}\right]^2 + 2K_b^2 r := r^*.$$

By (8), (9), from (6) we have for $t \in J$

$$\mathbb{E}\|(\Phi_2 x)(t)\|^2 \leq 3\mathbb{E}\|R(t)\varphi(0)\|^2 + 3\mathbb{E}\left\|\int_0^t R(t-s)F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right)dw(s)\right\|^2$$

$$+ 3\mathbb{E}\Big\| \sum_{0<t_k<t} R(t-t_k)I_k(\bar{x}(t_k))\Big\| \leq 3M\mathbb{E}\|\varphi(0)\|^2 + 3Tr(Q)M$$
$$\times \int_0^t \big[l(\gamma) + (d+\epsilon)\|\bar{x}_{\rho(s,\bar{x}_s)}\|^2_{\mathcal{B}}\big]ds$$
$$+ 3Mm\sum_{k=1}^{m}\big[C_1 + (c_k+\epsilon_k)\mathbb{E}\|\bar{x}(t_k)\|^2\big]$$
$$\leq 3M\tilde{H}^2\|\varphi\|^2_{\mathcal{B}} + 3Tr(Q)MT[l(\gamma) + (d+\epsilon)r^*]$$
$$+ 3Mm\sum_{k=1}^{m}\big[C_1 + (c_k+\epsilon_k)r\big] := \mathcal{L}.$$

Then for each $x \in B_r(0,\mathbb{Y})$, we have $\mathbb{E}\|\Phi_2 x\|^2 \leq \mathcal{L}$.
Step 3. We show that the operator Φ_2 is completely continuous.
For this purpose, we decompose Φ_2 as $\Psi_1 + \Psi_2$, Ψ_1, Ψ_2 are the operators on $B_r(0,\mathbb{Y})$ defined respectively by

$$(\Psi_1 x)(t) = R(t)\varphi(0) + \int_0^t R(t-s)F[s,\bar{x}_{\rho(s,\bar{x}_s)}]dw(s), \quad t\in J,$$
$$(\Psi_2 x)(t) = \sum_{0<t_k<t} R(t-t_k)I_k(\bar{x}(t_k)) \quad t \in J.$$

We first show that Ψ_1 is completely continuous.

(i) $\Psi_1(B_r(0,\mathbb{Y}))$ is equicontinuous.

Let $0<\tau_1<\tau_2\leq T$ and $\varepsilon>0$ be small. For each $x\in B_r(0,\mathbb{Y})$, we have

$$\mathbb{E}\Big\|(\Psi_1x)(\tau_2) - (\Psi_1x)(\tau_1)\Big\|^2 \leq 4\mathbb{E}\Big\|[R(\tau_2)-R(\tau_1)]\varphi(0)\Big\|^2$$
$$+ 4\mathbb{E}\Big\|\int_0^{\tau_1-\epsilon}\big[R(\tau_2-s)-R(\tau_1-s)\big]F\left(s,\bar{x}_{\rho(s,\bar{x}_s)}\right)dw(s)\Big\|^2$$
$$+ 4\mathbb{E}\Big\|\int_{\tau_1-\epsilon}^{\tau_1}\big[R(\tau_2-s)-R(\tau_1-s)\big]F\left(s,\bar{x}_{\rho(s,\bar{x}_s)}\right)dw(s)\Big\|^2$$
$$+ 4\mathbb{E}\Big\|\int_{\tau_1}^{\tau_2}R(\tau_2-s)F\left(s,\bar{x}_{\rho(s,\bar{x}_s)}\right)dw(s)\Big\|^2$$
$$\leq 4\mathbb{E}\Big\|[R(\tau_2)-R(\tau_1)]\varphi(0)\Big\|^2$$
$$+ 4Tr(Q)\big[l(\gamma)+(d+\epsilon)r^*\big]\int_0^{\tau_1-\epsilon}\|R(\tau_2-s)-R(\tau_1-s)\|^2_{\mathcal{L}(\mathbb{H})}ds$$
$$+ 4Tr(Q)\big[l(\gamma)+(d+\epsilon)r^*\big]\int_{\tau_1-\epsilon}^{\tau_1}\|R(\tau_2-s)-R(\tau_1-s)\|^2_{\mathcal{L}(\mathbb{H})}ds$$
$$+ 4Tr(Q)\big[l(\gamma)+(d+\epsilon)r^*\big]\int_{\tau_1}^{\tau_2}\|R(\tau_2-s)\|^2_{\mathcal{L}(\mathbb{H})}ds.$$

From the above inequalities, we see that the right-hand side of $\mathbb{E}\|(\Psi_1 x)(\tau_2) - (\Psi_1 x)(\tau_1)\|^2$ tends to zero independent of $x \in B_r(0, \mathbb{Y})$ as $\tau_2 - \tau_1 \to 0$ with ε sufficiently small, since the compactness of $R(t)$ forb $t > 0$ implies the continuity in the uniform operator topology. Thus the set $\{\Psi_1 x : x \in B_r(0, \mathbb{Y})\}$ is equicontinuous. The equicontinuities for the other cases $\tau_1 < \tau_2 \leq 0$ or $\tau_1 \leq 0 \leq \tau_2 \leq T$ are very simple.

(ii) The set $\Psi_1(B_r(0, \mathbb{Y}))(t)$ is precompact in $\mathbb{H}$ for each $t \in J$.
Let $0 < t \leq s \leq b$ fixed and let ε be a real number satisfying $\varepsilon \in (0, t)$. For $x \in B_r(0, \mathbb{Y})$, we define the operators

$$(\Psi_1^{*\varepsilon} x)(t) = R(t)\varphi(0) + R(\varepsilon) \int_0^{t-\varepsilon} R(t-s-\varepsilon) F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) dw(s),$$

$$(\tilde{\Psi}_1^{\varepsilon} x)(t) = R(t)\varphi(0) + \int_0^{t-\varepsilon} R(t-s) F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) dw(s).$$

By the compactness of the operators $R(t)$, the set $V_\varepsilon^*(t) = \{(\Psi_1^{*\varepsilon} x)(t); x \in B_r(0, \mathbb{Y})\}$ is relatively compact in $\mathbb{H}$, for every ε, $\varepsilon \in (0, t)$.

Moreover, also by Lemma 2.1 and assumption **(H6)** we have

$$\begin{aligned}
\mathbb{E}\|(\Psi_1^{*\varepsilon} x)(t) - (\tilde{\Psi}_1^{\varepsilon} x)(t)\|^2 &\leq \mathbb{E}\Bigg\| \int_0^{t-\varepsilon} R(\varepsilon) R(t-s-\varepsilon) F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) dw(s) \\
&\quad - \int_0^{t-\varepsilon} R(t-s) F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) dw(s) \Bigg\|^2 \\
&\leq \mathbb{E}\| \int_0^{t-\varepsilon} \left[R(\varepsilon)R(t-s-\varepsilon) - R(t-s)\right] F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) dw(s)\|^2 \\
&\leq Tr(Q) M \left[l(\gamma) + (d+\epsilon) r^*\right] \int_0^{t-\varepsilon} \|R(\varepsilon)R(t-s-\varepsilon) - R(t-s)\|^2 ds.
\end{aligned}$$

The right-hand side of the above inequality tends to zero as $\varepsilon \to 0$. So the set $\tilde{V}_\varepsilon(t) = \{(\tilde{\Psi}_1^{\varepsilon} x)(t);$ $x \in B_r(0, \mathbb{Y})\}$ is precompact in $\mathbb{H}$ by using the total Boundedness.

Applying the idea again, we obtain

$$\begin{aligned}
\mathbb{E}\|(\Psi_1 x)(t) - (\tilde{\Psi}_1^{\varepsilon} x)(t)\|^2 &\leq \mathbb{E}\Bigg\| \int_{t-\varepsilon}^{t} R(t-s) F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) dw(s) \Bigg\|^2 \\
&\leq Tr(Q) M \int_{t-\varepsilon}^{t} \left[l(\gamma) + (d+\epsilon) r^*\right] ds \\
&\leq Tr(Q) M \left[l(\gamma) + (d+\epsilon) r^*\right] \varepsilon.
\end{aligned}$$

The right hand side of the above inequality tends to zero as $\varepsilon \to 0$. Since there are precompact sets arbitrarily close to the set $U(t) = \{(\Psi_1 x)(t) : x \in B_r(0, \mathbb{Y})\}$. Hence the set U(t) is precompact in $\mathbb{H}$. By Arzelá-Ascoli theorem, we conclude that Ψ_1 maps $B_r(0, \mathbb{Y})$ into a precompact set in $\mathbb{H}$.

Next, it remains to verify that $\Psi_2(B_r(0, \mathbb{Y}))$ is also completely continuous.

We begin by showing $\Psi_2(B_r(0, \mathbb{Y}))$ is equicontinuous. For any $\varepsilon > 0$ and $0 < t < b$. Since the functions $I_k, k = 1, 2, \ldots, m$, are completely continuous in $\mathbb{H}$, we can choose $\xi > 0$ such that

$$\mathbb{E}\|[R(t+h) - R(t)] I_k(x)\|^2 < \frac{\varepsilon}{Mm}; \quad \mathbb{E}\|x\|^2 \leq r$$

when $|h| < \xi$. For each $x \in B_r(0, \mathbb{Y}), t \in (0, T]$ be fixed, $t \in [t_i, t_{i+1}]$, and $t + \xi \in [t_i, t_{i+1}]$, such that

$$\big[(\widehat{\Psi_2 x})\big]_i(t) = \sum_{0<t_k<t} R(t-t_k)I_k(\bar{x}(t_k)),$$

then we have

$$\begin{aligned}\mathbb{E}\|\big[(\widehat{\Psi_2 x})\big]_i(t+h) - \big[(\widehat{\Psi_2 x})\big]_i(t)\|^2 &\leq \mathbb{E}\|\sum_{0<t_k<t}\big[R(t+h-t_k) - R(t-t_k)\big]I_k(\bar{x}(t_k))\|^2 \\ &\leq m\sum_{k=1}^{m}\mathbb{E}\|\big[R(t+h-t_k) - R(t-t_k)\big]I_k(\bar{x}(t_k))\|^2.\end{aligned}$$

As $h \to 0$ and ε sufficiently small, the right-hand side of the above inequality tends to zero independently of x, so $[\Psi_2(\widehat{B_r(0, \mathbb{Y})}]_i \; i = 1, 2, \ldots, m$, are equicontinuous.

Now we prove that $[\Psi_2(\widehat{B_r(0, \mathbb{Y})})]_i(t) \; i = 1, 2, \ldots, m$, is precompact for every $t \in J$.

From the following relations

$$\big[(\widehat{\Psi_2 x})\big]_i(t) = \sum_{0<t_k<t} R(t-t_k)I_k(\bar{x}(t_k)) \in \sum_{k=1}^{m} R(t-t_k)I_k(B_r(0, \mathbb{H})).$$

We conclude that $[\Psi_2(\widehat{B_r(0, \mathbb{Y})})]_i(t) \; i = 1, 2, \ldots, m$, is precompact for every $t \in [t_i; t_{i+1}]$. By Lemma 2.2, we infer that $\Psi_2(B_r(0, \mathbb{Y}))$ is precompact. As an application of the Arzelá-Ascoli theorem, Ψ_2 is completely continuous.

Step 4. $\Phi_2 : \mathbb{Y} \to \mathbb{Y}$ is continuous.

Let $\{x^n\} \subseteq B_r(0, \mathbb{Y})$ with $x^n \to x (n \to \infty)$ in $\mathbb{Y}$. From Axiom (A1), $(A2)$ and $(A3)$, it is easy to see that $(\overline{x^n}_s) \to (\bar{x}_s)$ uniformly for $s \in (-\infty, b]$ as $n \to \infty$. By assumption (H4)–(H5), we have

$$F\left(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}\right) \to F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) \text{ as } \quad n \to \infty$$

for each $s \in [0, t]$, and since

$$\|F\left(s, (\overline{x^n})_{\rho(s,(\overline{x^n})_s)}\right) - F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right)\| \leq 2[l(\gamma) + (d+\epsilon)r^*].$$

Then by the continuity of I_k $(k = 1, 2, \ldots, m)$ and the dominated convergence theorem we have

$$\begin{aligned}\|\Phi_2 x^n - \Phi_2 x\|^2_{\mathcal{PC}} \leq \sup_{t\in[0,T]}\Bigg\| &\int_0^t R(t-s)\left[F\left(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}\right) - F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right)\right]ds \\ &+ \sum_{0<t_k<t} R(t-t_k)\big[I_k(\overline{x^n}(t_k)) - I_k(\bar{x}(t_k))\big]\Bigg\|^2 \\ &\leq 2Tr(Q)M\int_0^t \mathbb{E}\|F\left(s, (\overline{x^n})_{\rho(s,(\overline{x^n})_s)}\right) - F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right)\|^2 ds \\ &+ 2Mm\sum_{0<t_k<t}\|I_k(\overline{x^n}(t_k)) - I_k(\bar{x}(t_k))\|^2 \to 0 \quad as\, n \to \infty.\end{aligned}$$

Therefore, Φ_2 is continuous.

Step 5. We shall show the set $\Lambda = \{x \in \mathbb{Y} : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2(x) = x$, for some $\lambda \in (0, 1)\}$ is bounded on J.

To do this, we consider the following nonlinear operator equation

$$x(t) = \lambda\Phi x(t),\, 0 < \lambda < 1, \tag{10}$$

where Φ is already defined. Next we gives a priori estimate for the solution of the above equation. Indeed, let $x \in \mathbb{Y}$ be a possible solution of $x = \lambda\Phi(x)$ for some $0 < \lambda < 1$. This implies by (10) that for each $t \in J$ we have

$$x(t) = \lambda R(t)\big[\varphi(0) - G(0,\varphi,0)\big] + \lambda G\Big(t, \bar{x}_t, \int_0^t g(t,s,\bar{x}_s)ds\Big)$$
$$+ \lambda \int_0^t R(t-s)F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) ds + \lambda \sum_{0<t_k<t} R(t-t_k)I_k(\bar{x}(t_k)), \quad t \in J. \tag{11}$$

By **(H6)**, (8), (9), from (11) we have for $t \in J$

$$\mathbb{E}\|x(t)\|^2 \le 4\mathbb{E}\|R(t)\big[\varphi(0) - G(0,\varphi,0)\big]\|^2 + 4\mathbb{E}\Big\|G\Big(t, \bar{x}_t, \int_0^t g(t,s,\bar{x}_s)ds\Big)\Big\|^2$$
$$+ 4\mathbb{E}\Big\|\int_0^t R(t-s)F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) dw(s)\Big\|^2 + 4\mathbb{E}\Big\|\sum_{0<t_k<t} R(t-t_k)I_k(\bar{x}(t_k))\Big\|^2$$
$$\le 4M\big[\tilde{H}^2\|\varphi\|_{\mathbb{B}}^2 + (L_2(M_T + J_0^{\varphi})^2\|\varphi\|_{\mathbb{B}}^2 + l_2)\big] + 4M\big[L_2(\|\bar{x}_t\|_{\mathbb{B}}^2 + L_1\|\bar{x}_t\|_{\mathbb{B}}^2 + l_1) + l_2\big]$$
$$+ 4Tr(Q)M\int_0^t \big[l(\lambda) + (d+\epsilon)\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathbb{B}}^2\big]ds + Mm\sum_{k=1}^m \big[C_1 + (c_k + \epsilon_k)\mathbb{E}\|\bar{x}(t_k)\|^2\big],$$

where $l_1 = \sup\limits_{t_0 \le t \le T} \|G(t,0,0)\|^2$ and $l_2 = \sup\limits_{t_0 \le s \le t \le b} \|g(t,s,0)\|^2$. By Lemmas 2.3 and 2.4, it follows that $\rho(s,\bar{x}_s) \le s$, $s \in [0,t]$, $t \in J$ and

$$\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathbb{B}}^2 \le 2\big[(M_b + J_0^{\varphi})\|\varphi\|_{\mathbb{B}}\big]^2 + 2K_b^2 \sup_{0 \le s \le T} \mathbb{E}\|x(s)\|^2.$$

For each $t \in J$, we have

$$\mathbb{E}\|x(t)\|^2 \le M_* + 8L_2(1+L_1)K_b^2 \sup_{t \in J} \mathbb{E}\|x(t)\|^2$$
$$+ 8Tr(Q)M(d+\epsilon)K_b^2 \int_0^t \sup_{\tau \in [0,s]} \mathbb{E}\|x(\tau)\|^2 ds + 8Mm\sum_{k=1}^m (c_k + \epsilon_k) \sup_{t \in J} \mathbb{E}\|x(t)\|^2,$$

where

$$M_* = 8M[\tilde{H}^2\|\varphi\|_{\mathbb{B}}^2 + (L_2c_1^* + l_2)] + 8 \times \{L_2[(1+L_1)c_1^* + l_1] + l_2\}$$
$$+ 8Tr(Q)MT[l(\lambda) + (d+\epsilon)c_1^*] + 10Mm^2C_1,$$

$$c_1^* = \big[(M_b + J_0^{\varphi})\|\varphi\|_{\mathbb{B}}\big]^2.$$

Since $L_* = 8L_2(1+L_1)K_T^2 + 8Mm\sum_{k=1}^m (c_k + \epsilon_k) < 1$, we have

$$\sup_{t \in [0,T]} \mathbb{E}\|x(t)\|^2 \le \frac{M_*}{1 - L_*} + P_2 \int_0^b \sup_{\tau \in [0,s]} \mathbb{E}\|x(\tau)\|^2 ds,$$

where $P_2 = \frac{1}{1-L_*} 8Tr(Q)M(d+\epsilon)K_T^2$.
Applying Gronwall's inequality in the above expression, we obtain

$$\sup_{t \in [0,T]} \mathbb{E}\|x(t)\|^2 \le \frac{M_*}{1-L_*} \exp\{P_2 T\} := \bar{K}.$$

Then for any $x \in \Lambda(\Phi)$, we get that $\|x\|_{\mathcal{PC}}^2 \le \bar{K}$. This implies that Λ is bounded on J. Consequently, by Lemma 2.4, we deduce that Λ has a fixed point $x \in \mathbb{Y}$, which is a mild solution of problem (1). The proof is complete. □

4. Application

Consider the following impulsive neutral stochastic partial integrodifferential equations of the form

$$\begin{cases} \frac{\partial}{\partial t}\left[z(t,x)-\mu_1\left(t,z(t-\tau,x),\int_0^s \mu_2(t,s,z(s-\tau,x))ds\right)\right]=-\frac{\partial^2}{\partial x^2}\left[z(t,x)\right. \\ \left.\mu_1\left(t,z(t-\tau,x),\int_0^s \mu_2(t,s,z(s-\tau,x))ds\right)\right] \\ +\int_0^t b(t-s)\frac{\partial^2}{\partial \xi^2}\left[z(s,x)-\mu_1\left(s,z(s-\tau,x),\int_0^s \mu_2(s,u,z(u-\tau,x))du\right)\right]ds \\ +\mu_3[t,z(s-\rho_1(\tau)\rho_2(\|z(\tau)\|),x)]dw(t), \quad 0\le t\le b; \quad \tau>0, \quad 0\le x\le\pi \\ z(t,0)=z(t,\pi)=0, \quad 0\le t\le T, \\ z(t_k^+,x)-z(t_k^-,x)=I_k(z(t_k,x)), \quad k=1,\dots,m, \\ z(t,x)=\varphi(t,x), \quad -\infty\le t\le 0, \quad 0\le x\le\pi, \end{cases} \tag{12}$$

where φ is continuous and $I_k \in C(\mathbb{R},\mathbb{R}), w(t)$ denotes a standard cylindrical Wiener process in $\mathbb{H}$ defined on a stochastic space $(\Omega,\mathcal{F},P)$ and $\mathbb{H}=L^2([0,\pi])$ with the norm $\|.\|$ and define the operators $A:\mathbb{H}\to\mathbb{H}$ by $Aw=w^{''}$ with the domain
$D(A):=\{w\in\mathbb{H}: w,w^{'}$ are absolutely continuous, $w^{''}\in\mathbb{H}, w(0)=w(\pi)=0\}$.
Then

$$Aw=\sum_{n=1}^{\infty} n^2\langle w,w_n\rangle w_n, \quad w\in D(A),$$

where $w_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$, $n=1,2,\dots$ is the orthogonal set eigenvectors of A. It is well known that A is the infintesimal generator of an analytic semigroup $T(t)$, $t\ge 0$ in H and is given by

$$T(t)w=\sum_{n=1}^{\infty}\exp(-n^2t)(w,w_n)w_n, \quad w\in\mathbb{H}.$$

Let $B: D(A)\subset\mathbb{H}\to\mathbb{H}$ be the operator defined by

$$B(t)(y)=b(t)Ay \quad t\ge 0 \quad y\in D(A).$$

Let $\sigma>0$, define the phase space

$$\mathbb{B}=\{\phi\in C((-\infty,0],\mathbb{H}): \lim_{\theta\to\infty} e^{\sigma\theta}\phi(\theta) \quad \text{exists in } \mathbb{H}\},$$

and let $\|\phi\|_{\mathbb{B}}=\sup_{-\infty<\theta<0}\{e^{\sigma\theta}\|\phi(\theta)\|\}$. Then $(\mathbb{B},\|\phi\|_{\mathbb{B}})$ is a Banach space which satisfies (A1)–(A3) with $\tilde{H}=1$, $K(t)=\max\{1,e^{-\sigma t}\}$, $M(t)=e^{-\sigma t}$. Hence for $(t,\phi)\in[0,T]\times\mathbb{B}$, where $\phi(\theta)(x)=\phi(\theta,x)$, $(\theta,x)\in(-\infty,0]\times[0,\pi]$, let $z(t)(x)=z(t,x)$,

$$G\left(t,\phi,\int_0^s \mu_1(t,s,\phi)ds\right)(x)=\mu_2\left(t,\phi(\theta,x),\int_0^s \mu_2(t,s,\phi)ds\right),$$
$$g(t,s,\phi)(x)=\mu_2-t,s,\phi(\theta,x)),$$
$$F(t,\phi)(x)=\mu_3(t,\phi(\theta,x)),$$
$$\rho(t,\phi)=\rho_1(t)\rho_2(\|\phi(0)\|).$$

Then the problem (12) can be writen as (1). Moreover, if b is bounded and C^1 function such that b' is bounded and uniformly continuous, then **(H1)** and **(H2)** are satisfied, and hence, by Theorem 1, 2.1 has a resolvent operator $(R(t))_{t\ge 0}$ on $\mathbb{H}$. Thus, under appropriate conditions on the functions G,g,F, and I_k as those in **(H1)**–**(H7)**, the problem (12) has a mild solution on J.

5. Conclusion

In this paper, we have studied the existence results for impulsive stochastic neutral integrodifferential systems with state-dependent delay conditions in a Hilbert space by utilizing the stochastic analysis theory, resolvent operator, and the Krasnoselskii fixed point theorem. To validate the obtained theoretical results, we analyze one example. The impulsive stochastic neutral integrodifferential systems with state-dependent delay are very efficient to describe the real-life phenomena; thus, it is essential to extend the present study to establish the other qualitative and quantitative properties such as stability and controllability. There are two direct issues that require further study. First, we will investigate the controllability of neutral stochastic integrodifferential systems with state-dependent delay in the case of nonlocal conditions. Second, we will study the approximate controllability of a new class of impulsive stochastic integrodifferential equations with state-dependent delay and noninstantaneous impulses.

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