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RINGS WHOSE ELEMENTS ARE LINEAR EXPRESSIONS OF THREE COMMUTING IDEMPOTENTS

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Abstract. We classify up to isomorphism those rings in which all elements are linear expressions over the ring of integers $\mathbb{Z}$ of at most three commuting idempotents. Our results substantially improve on recent publications by the author in Albanian J. Math. (2018), Gulf J. Math. (2018), Mat. Stud. (2018), Bull. Iran. Math. Soc. (2018) and Lobachev. J. Math. (2019) as well as on publications due to Hirano-Tominaga in Bull. Austral. Math. Soc. (1988), Ying et al. in Can. Math. Bull. (2016) and Tang et al. in Lin. & Multilin. Algebra (2019).

1. Introduction and Background

Throughout the text of the paper, all rings R are assumed to be associative, containing the identity element 1 which differs from the zero element 0 of R. The standard terminology and notations are mainly in close agreement with [8]. For instance, $U(R)$ denotes the group of units in R, Id(R) the set of idempotents in R, $Nil(R)$ the set of nilpotents in R and $J(R)$ the Jacobson radical of R. As usual, $\mathbb{Z}$ stands for the ring of integers, and $\mathbb{Z}_k \cong \mathbb{Z}/k\mathbb{Z}$ is its quotient modulo the principal ideal $(k) = k\mathbb{Z}$, where $k \in \mathbb{N}$ is the set of naturals.

About the specific notions, they will be explained below in detail.

The aim of the present work is to describe the isomorphic structure of the following class of rings.

Definition 1.1. We shall say that the ring R is from the class $\mathcal{R}_3$ if, for any $r \in R$, there exist commuting each to other $e_1, e_2, e_3 \in$ Id(R) such that $r = e_1 + e_2 - e_3$ or $r = e_1 - e_2 - e_3$.

It is worthwhile to mention that by substituting $r \to -r$ and an eventual re-numeration of the idempotents, the first equality will yield the second equality, and reversible.

Obvious examples of such rings are the rings $\mathbb{Z}_k$, where $k = 2, 3, 4, 5, 6$. Contrasting with that, the ring $\mathbb{Z}_7$ need not be so.

The most important principally known achievements concerning the subject are as follows: Classically, a ring is said to be *boolean* if each its element is an idempotent – such a ring is known to be a subdirect product of a family of copies of the two element field $\mathbb{F}_2$. A very successful attempt to generalize that concept was made in [7] to the rings whose elements are the sum of two commuting idempotents – in fact, these rings are known to be commutative being a subdirect product of a family of copies of the two and three element fields $\mathbb{F}_2$ and $\mathbb{F}_3$, respectively. In particular, if every element of a ring is an idempotent or minus an idempotent, then this ring is either boolean, or $\mathbb{F}_3$, or the direct product of two such rings.

Further expansions of these notions, in terms of linear expressions over $\mathbb{Z}$ of at most three commuting idempotents, are subsequently given below as follows:

- $\forall r \in R$, $r = e_1 + e_2$ or $r = e_1 - e_2$ for some two commuting $e_1, e_2 \in \mathrm{Id}(R)$ (see [10]).
- $\forall r \in R$, $r = e_1 + e_2$ or $r = -e_1 - e_2$ for some two commuting $e_1, e_2 \in \mathrm{Id}(R)$ (see [5]).
- $\forall r \in R$, $r = e_1 + e_2 + e_3$ for some three commuting $e_1, e_2, e_3 \in \mathrm{Id}(R)$ (see [4] and [9]).
- $\forall r \in R$, $r = e_1 + e_2 + e_3$ or $r = -e_1$ for some three commuting $e_1, e_2, e_3 \in \mathrm{Id}(R)$ (see [2]).
- $\forall r \in R$, $r = e_1 + e_2 + e_3$ or $r = e_1 - e_2$ for some three commuting $e_1, e_2, e_3 \in \mathrm{Id}(R)$ (see [4])
- $\forall r \in R$, $r = e_1 + e_2 + e_3$ or $r = -e_1 - e_2$ for some three commuting $e_1, e_2, e_3 \in \mathrm{Id}(R)$ (see [1]).
- $\forall r \in R$, $r = e_1 + e_2 + e_3$ or $r = -e_1 - e_2 - e_3$ for some three commuting $e_1, e_2, e_3 \in \mathrm{Id}(R)$ (see [3]).

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• $\forall r \in R, \quad r = e_1 + e_2 + e_3$ or $r = e_1 + e_2 - e_3$ for some three commuting $e_1, e_2, e_3 \in \mathrm{Id}(R)$ (see [6]).
• $\forall r \in R, \quad r = e_1 + e_2 + e_3$ or $r = e_1 - e_2 - e_3$ for some three commuting $e_1, e_2, e_3 \in \mathrm{Id}(R)$ (see [6]).
Actually, the rings from the last two bullets are rings lying in the classes $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively.

In all of the aforementioned variations, the ring is of necessity commutative, which suggest us to state at the end of the article two conjectures which are of some interest and importance.

Our working tactic is somewhat to develop the techniques utilized in [1–6] as well as to build some new methods inspired by the specification of the ring structure. Especially, we shall careful study the rings from the class $\mathcal{R}_3$, stated above in Definitions 1.1, by characterizing them up to an isomorphism.

2. Main Results

We start here with the following useful technicality.

Proposition 2.1. *Any ring R from the class $\mathcal{R}_3$ decomposes as $R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are either zero rings or rings belonging to the class $\mathcal{R}_3$ such that $4 = 0$ in R_1, $3 = 0$ in R_2 and $5 = 0$ in R_3.*

Proof. Let us write $3 = e_1 + e_2 - e_3$. Observing that $e_1 - e_3 = e_1(1 - e_3) - e_3(1 - e_1)$ is a difference of two orthogonal commuting idempotents, we can assume with no harm in generality that $e_1e_3 = 0$. Moreover, since $e_3(1 - e_1)$ remains an idempotent, we may also assume that $e_2e_3 = 0$.

Thus, squaring the equality for 3, one infers that $6 = 2e_1e_2 + 2e_3$ which multiplying by e_3 gives that $4e_3 = 0$. Furthermore, a multiplication of the same equality by e_1e_2 ensures that $4e_1e_2 = 0$ and, finally, the multiplication of the same by 2 riches us that $12 = 0$.

Writing next $3 = e_1 - e_2 - e_3$, as above demonstrated, we can assume without loss of generality that $e_1e_2 = e_1e_3 = 0$. Multiplying the equality of 3 by e_1 leads to $2e_1 = 0$. On the other side, squaring the equality for 3 assures that $12 = 2e_2e_3$ and the multiplication of this with e_2e_3 forces that $10e_2e_3 = 0$. Therefore, $12.5 = 60 = 4.3.5 = 0$, as wanted.

Consequently, the Chinese Remainder Theorem now applies to conclude that $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are either zero or rings again from the class $\mathcal{R}_3$ with characteristics ≤ 4, 3 and 5, respectively. $\square$

The following assertion is pivotal, strengthening [1, Proposition 2.2].

Lemma 2.2. *Suppose that R is a ring of characteristic 5. Then the following four conditions are equivalent:*
(i) *$x^3 = x$ or $x^4 = 1$, $\forall x \in R$.*
(ii) *$x^3 = -x$ or $x^4 = 1$, $\forall x \in R$.*
(iii) *$x^3 = x$ or $x^3 = -x$, $\forall x \in R$.*
(iv) *R is isomorphic to the field $\mathbb{Z}_5$.*

Proof. (i) $\Rightarrow$ (iii). For an arbitrary $y \in R$ satisfying $y^4 = 1$ but $y^3 \neq y$, considering the element $y^2 - 1$, it must be that $(y^2 - 1)^4 = 1$ or $(y^2 - 1)^3 = y^2 - 1$. In the first case we receive $y^2 = -1$ and thus $y^3 = -y$, as required, while in the second one we arrive at $y^2 = 1$ and so $y^3 = y$ which is against our initial assumption.

(ii) $\Rightarrow$ (iii). The same trick as that in the previous implication will work, assuming now that $y^3 \neq -y$.

(iii) $\iff$ (iv). Let P be the subring of R generated by 1, and thus note that $P \cong \mathbb{Z}_5$. We claim that $P = R$, so we assume in a way of contradiction that there exists $b \in R \setminus P$. With no loss of generality, we shall also assume that $b^3 = b$ since $b^3 = -b$ obviously implies that $(2b)^3 = 2b$ as $5 = 0$ and $b \notin P \iff 2b \notin P$.

Let us now $(1+b)^3 = -(1+b)$. Hence $b = b^3$ along with $5 = 0$ enable us that $b^2 = 1$. This allows us to conclude that $(1 + 2b)^3 \neq \pm(1 + 2b)$, however. In fact, if $(1 + 2b)^3 = 1 + 2b$, then one deduces that $2b = 3 \in P$ which is manifestly untrue. If now $(1 + 2b)^3 = -1 - 2b$, then one infers that $2b = 2 \in P$ which is obviously false. That is why, only $(1 + b)^3 = 1 + b$ holds. This, in turn, guarantees that $b^2 = -b$. Moreover, $b^3 = b$ is equivalent to $(-b)^3 = -b$ as well as $b^3 = -b$ to $(-b)^3 = -(-b)$ and thus, by what we have proved so far applied to $-b \notin P$, it follows that $-b = b^2 = (-b)^2 = -(-b) = b$.

Consequently, $2b = 0 = 6b = b \in P$ because $5 = 0$, which is the wanted contradiction. We thus conclude that $P = R$, as expected.

Conversely, it is trivial that the elements of $\mathbb{Z}_5 = \{0, 1, 2, 3, 4 \mid 5 = 0\}$ are solutions of one of the equations $x^3 = x$ or $x^3 = -x$.

(iv) $\Rightarrow$ (i), (ii). It is self-evident that all elements of $\mathbb{Z}_5 = \{0, 1, 2, 3, 4 \mid 5 = 0\}$ satisfy one of the equations $x^3 = x$ or $x^4 = 1$ as well as one of $x^3 = -x$ or $x^4 = 1$. $\square$

We now come to the following.

Theorem 2.3. *A ring R lies in the class $\mathcal{R}_3$ if, and only if, it is commutative and $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are rings for which*

(1) *$R_1 = \{0\}$, or $R_1/J(R_1)$ is a boolean factor-ring with nil $J(R_1) = 2\,\mathrm{Id}(R_1)$ such that $4 = 0$;*

(2) *$R_2 = \{0\}$, or R_2 is a subdirect product of a family of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$;*

(3) *$R_3 = \{0\}$, or $R_3 \cong \mathbb{Z}_5$.*

Proof. Necessity. Appealing to Proposition 2.1, there is a decomposition $R \cong R_1 \times R_2 \times R_3$, where the direct factors R_1, R_2 and R_3 still belong to the class $\mathcal{R}_3$. What we need to do is to describe explicitly these three rings.

Describing R_1: Here $4 = 0$. Since $2 \in J(R_1)$, we elementarily observe that the quotient-ring $R_1/J(R_1)$ is of characteristic 2 ring from the class $\mathcal{R}_3$. Thus it has to be a boolean ring. What it needs to show is the equality $J(R_1) = 2\,\mathrm{Id}(R_1)$. In fact, given $z \in J(R_1)$, we write $z = e_1 + e_2 - e_3$ with $e_1e_3 = e_2e_3 = 0$, or $z = e_1 - e_2 - e_3$ with $e_1e_2 = e_1e_3 = 0$, for some three commuting idempotents e_1, e_2, e_3 in R_1. In the first case, $ze_3 = -e_3$ still lies in $J(R_1)$, so that $e_3 = 0$. Hence $z = e_1 + e_2$ implying that $z(1 - e_2) = e_1(1 - e_2) \in J(R_1) \cap \mathrm{Id}(R_1) = \{0\}$ and thus that $e_1 = e_1e_2$. By a reason of symmetry, $e_2 = e_1e_2$ whence $e_1 = e_2$ giving up that $z = 2e_1 \in 2\,\mathrm{Id}(R_1)$, as needed.

In the second case, $ze_1 = e_1 \in J(R_1) \cap \mathrm{Id}(R_1) = \{0\}$ and hence $z = -e_2 - e_3 = -(e_2 + e_3)$. Similarly, as in the previous case, $z = -2e_2 = 2e_2 \in 2\,\mathrm{Id}(R_1)$ because $4 = 0$, as required.

Describing R_2: Here $3 = 0$. In fact, by the same token as in the preceding situation for R_1, we have that $J(R_2) = 2\,\mathrm{Id}(R_2)$ or $J(R_2) = -2\,\mathrm{Id}(R_2)$. If for any $j \in J(R_2)$ we write $j = 2i$ for some $i \in \mathrm{Id}(R_2)$, then $-j + 3i = i \in J(R_2) \cap \mathrm{Id}(R_2) = \{0\}$ whence $i = 0 = j$. Symmetrically, if $j = -2i$, then $j + 3i = i \in J(R_2) \cap \mathrm{Id}(R_2) = \{0\}$ and hence $i = 0 = j$, as required. Furthermore, since $3 = 0$, it easily follows that $x^3 = x$ for all $x \in R_2$ and thus [7] is applicable to get the wanted description of R_2.

Describing R_3: Here $5 = 0$. For any $x \in R_3$ we write that $x = e_1 + e_2 - e_3$ with $e_1e_3 = e_2e_3 = 0$, or $x = e_1 - e_2 - e_3$ with $e_1e_2 = e_1e_3 = 0$. In the first case, squaring the equality for x gives that $x^2 - x = 2(e_1e_2 + e_3)$ which allows us to deduce that $(x^2 - x)^2 = 2(x^2 - x)$ since $e_1e_2 + e_3$ is obviously an idempotent as e_1e_2 and e_3 are orthogonal idempotents. We, therefore, have that $x^4 - 2x^3 - x^2 + 2x = 0$. In the second case, again by squaring the equality for x, one derives that $x^2 + x = 2(e_1 + e_2e_3)$ which enables us that $(x^2 + x)^2 = 2(x^2 + x)$ because $e_1 + e_2e_3$ is obviously an idempotent as e_1 and e_2e_3 are orthogonal idempotents. We, consequently, have that $x^4 + 2x^3 - x^2 - 2x = 0$. One also observes that by the substitution $x \to -x$ the first equation will imply the second equation, and vice versa.

Furthermore, replacing $x \to 2x$ and $x \to 3x$ in the equation $x^4 - 2x^3 - x^2 + 2x = 0$, we derive that $x^4 - x^3 + x^2 - x = 0$ and that $x^4 + x^3 + x^2 + x = 0$, respectively. The same replacements in the equation $x^4 + 2x^3 - x^2 - 2x = 0$ lead respectively to $x^4 + x^3 + x^2 + x = 0$ and $x^4 - x^3 + x^2 - x = 0$, which are definitely the same equations in a rotating way, arising from the map $x \to -x$.

The next four main combinations must be considered:

Combination 1. $x^4 - x^3 + x^2 - x = 0$ with $x^4 + x^3 + x^2 + x = 0$ implies that $2x^3 = -2x$, which multiplying it by 3 implies that $x^3 = -x$ because $5 = 0$.

Combination 2. $x^4 - 2x^3 - x^2 + 2x = 0$ with $x^4 + x^3 + x^2 + x = 0$ implies that $3x^3 + 2x^2 - x = 0$.

Combination 3. $x^4 - 2x^3 - x^2 + 2x = 0$ with $x^4 - x^3 + x^2 - x = 0$ implies that $x^3 + 2x^2 - 3x = 0$.

Now, combining $3x^3 + 2x^2 - x = 0$ and $x^3 + 2x^2 - 3x = 0$, we get once again that $2x^3 = -2x$, i.e., $x^3 = -x$.

Similar arguments work for the other initial equation $x^4 + 2x^3 - x^2 - 2x = 0$ getting also that $x^3 = -x$ which as noticed above arisen from $x \to -x$.

Combination 4. $x^4 + 2x^3 - x^2 - 2x = 0$ with $x^4 - 2x^3 - x^2 + 2x = 0$ implies that $4x^3 = 4x$, that is, $x^3 = x$ since $5 = 0$.

After taking into account these four possibilities, one concludes that it must be $x^3 = x$ or $x^3 = -x$ after all. That is why, Lemma 2.2 (iii) finally tells us to obtain the wanted description of R_3 as being isomorphic to $\mathbb{Z}_5$.

Concerning the commutativity of the whole ring R, since R_2 and R_3 are obviously commutative, what remains to show is that this property holds for R_1. This, however, follows by the usage of [4, Theorem 2.2].

Sufficiency. A direct consultation with [7] informs us that every element of R_2 is a sum of two idempotents. Likewise, as in [1, 4] or [5], each element in R_1 is a sum of three idempotents. Since R_3 has only five elements, we are, therefore, in a position to exploit the same manipulation as that in the corresponding results from [1, 4] and [5] getting that the direct product $R_1 \times R_2 \times R_3$ belongs to the class $\mathcal{R}_3$, as expected. $\square$

It will definitely be somewhat interesting to examine now the equalities $r = e_1 + e_2 - e_3$ and $r = e_1 - e_2 - e_3$ in an arbitrary ring R separately, comparing them with the equation $r = e_1 + e_2 + e_3$ in R which was independently explored in [4] and [9], respectively. Specifically, the latter rings were defined in [4] to be members from the class $\mathcal{K}$. Inspired by this, let we define the rings R for which $r = e_1 + e_2 - e_3$ to lie in the class $\mathcal{K}_1$, and the rings for which $r = e_1 - e_2 - e_3$ in the class $\mathcal{K}_2$.

Proposition 2.4. *Any ring R either from the class $\mathcal{K}_1$ or $\mathcal{K}_2$ decomposes as $R_1 \times R_2$, where R_1, R_2 are rings again from the same ring class such that $2 = 0$ in R_1 and $3 = 0$ in R_2.*

Proof. Firstly, writing $3 = e_1 + e_2 - e_3$, we obtain as in the first part of Proposition 2.1 that $12 = 4.3 = 0$, as asked for.

Secondly, writing $2 = e_1 - e_2 - e_3$, we may assume as in the second part of Proposition 2.1 that $e_1e_2 = e_1e_3 = 0$. Thus $2e_1 = e_1$ yields that $e_1 = 0$. Therefore, $2 = -e_2 - e_3$ implies by squaring that $6 = 2e_2e_3$. As a final step, $2e_2e_3 = -e_2e_3 - e_2e_3$, i.e., $4e_2e_3 = 0$ insuring that $12 = 4.3 = 0$, as pursued.

Furthermore, the Chinese Remainder Theorem is applicable to get the desired decomposition. $\square$

So, we now arrive at the following.

Theorem 2.5. *A ring R is either from the class $\mathcal{K}_1$ or $\mathcal{K}_2$ if, and only if, it is commutative and $R \cong R_1 \times R_2$, where R_1, R_2 are rings for which*

(1) *$R_1 = \{0\}$, or $R_1/J(R_1)$ is a boolean quotient-ring with nil $J(R_1) = 2\,\mathrm{Id}(R_1)$ such that $4 = 0$;*

(2) *$R_2 = \{0\}$, or R_2 is a subdirect product of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$.*

Proof. Necessity. According to Proposition 2.4, there is a decomposition $R \cong R_1 \times R_2$, where the direct factors R_1 and R_2 still belong to either of classes $\mathcal{K}_1$ or $\mathcal{K}_2$. What we need to do is to describe in an explicit form these two rings.

Describing R_1: Here $4 = 0$. Since $2 \in J(R_1)$, we routinely see that the factor-ring $R_1/J(R_1)$ is of characteristic 2 ring from one of the classes $\mathcal{K}_1$ or $\mathcal{K}_2$. Thus it has to be a boolean ring. What suffices to prove is the equality $J(R_1) = 2\,\mathrm{Id}(R_1)$ which can be handled analogously to the corresponding part of Theorem 2.3.

Describing R_2: Here $3 = 0$. We claim that $J(R_2) = \{0\}$. In fact, as in the preceding case, we have that $J(R_2) = 2\,\mathrm{Id}(R_2)$ or $J(R_2) = -2\,\mathrm{Id}(R_2)$. If for any $j \in J(R_2)$ we write $j = 2i$ for some $i \in \mathrm{Id}(R_2)$, then $-j + 3i = i \in J(R_2) \cap \mathrm{Id}(R_2) = \{0\}$ whence $i = 0 = j$. Symmetrically, if $j = -2i$, then $j + 3i = i \in J(R_2) \cap \mathrm{Id}(R_2) = \{0\}$ and hence $i = 0 = j$, as required. Furthermore, since $3 = 0$, it easily follows that $x^3 = x$ for all $x \in R_2$ and thus [7] is working to get the wanted description of R_2.

The commutativity of the former ring R follows in the same way as in Theorem 2.3 above.

Sufficiency. It follows by adapting the same idea as in the "sufficiency part" of Theorem 2.3. $\square$

Now, to close all possible variations of equalities which depend on idempotents, we shall say that the ring R belongs to the class $\mathcal{P}$, provided that for any r from R the equalities $r = e_1 + e_2 - e_3$ or $r = -e_1 - e_2$ are valid for some commuting idempotents $e_1, e_2, e_3 \in \mathrm{Id}(R)$. This is tantamount to $r = e_1 - e_2 - e_3$ or $r = e_1 + e_2$ via the substitution $r \to -r$ and re-numerating.

One sees that the direct product $\mathbb{Z}_4 \times \mathbb{Z}_5 \notin \mathcal{P}$ by considering the element $(1, 3)$, where $1 = 1+0-0 = 1 + 1 - 1$ whereas $3 = -1 - 1$. Contrastingly, for the element $(2, 3)$ we have $2 = 1 + 1 = -1 - 1$ and $3 = -1 - 1$. However, the ring $\mathbb{Z}_4 \times \mathbb{Z}_5 \in \mathcal{R}_3$ which shows that these two classes are different.

What is currently offer by us is the following slight enlargement of the preceding Theorem 2.5 and of results from [2, 5] and [10].

Theorem 2.6. *The ring R lies in the class $\mathcal{P}$ if, and only if, $R \cong R_1 \times R_2 \times R_3$, where*

(1) $R_1 = \{0\}$*, or $R_1/J(R_1)$ is a boolean ring such that $J(R_1) = 2\,\mathrm{Id}(R_1)$ with $4 = 0$;*
(2) $R_2 = \{0\}$*, or R_2 is a subdirect product of a family of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$;*
(3) $R_3 = \{0\}$ *which must be fulfilled when $J(R_1) \neq \{0\}$, or $R_3 \cong \mathbb{Z}_5$.*

Proof. Necessity. We claim that $60 = 4.3.5 = 0$ in R, and thus the Chinese Remainder Theorem applies to infer the wanted decomposition of R into $R_1 \times R_2 \times R_3$ with $R_1, R_2, R_3 \in \mathcal{P}$ such that $4 = 0$ in R_1, $3 = 0$ in R_2 and $5 = 0$ in R_3.

In fact, write $3 = e_1 + e_2 - e_3$ with $e_1e_3 = e_2e_3 = 0$. Therefore, $3e_3 = -e_3$ yields that $4e_3 = 0$. Also, $3e_1e_2 = e_1e_2 + e_1e_2$ gives $e_1e_2 = 0$. On the other hand, squaring the equality for 3 forces that $6 = 2(e_1e_2 + e_3) = 2e_3$. Finally, $6.2 = 4.3 = 0$, as expected. Writing now $3 = -e_1 - e_2$, we obtain $3e_1e_2 = -e_1e_2 - e_1e_2$ amounts to $5e_1e_2 = 0$. The squaring of the equality for 3 ensures that $12 = 2e_1e_2$ whence $12.5 = 4.3.5 = 0$, as promised.

Furthermore, describing separately these three direct factors, one has that:

About R_1: Here $4 = 0$. Since $2 \in J(R_1)$, it is self-evident that the quotient $R_1/J(R_1)$ is a ring of characteristic 2 also belonging to the class $\mathcal{P}$, and thus it is necessarily a boolean ring. As for the equality concerning $J(R_1)$, given $z \in J(R_1)$, we may write that $z = e_1+e_2-e_3$ or that $z = -e_1-e_2$ for some three commuting idempotents $e_1, e_2, e_3 \in R_1$. In the first case, as above demonstrated, we may assume with no harm of generality that $e_1.e_3 = e_2.e_3 = 0$. Hence $-ze_3 = e_3 \in J(R_1) \cap \mathrm{Id}(R_1) = \{0\}$, that is, $e_3 = 0$. Thus the record $z = e_1+e_2$ riches us that $z(1-e_2) = e_1(1-e_2) \in J(R_1) \cap \mathrm{Id}(R_1) = \{0\}$, i.e., $e_1 = e_1e_2$. In a way of similarity $e_2 = e_1e_2$ and, finally, $e_1 = e_2$. Consequently, $z = 2e_1 \in 2\,\mathrm{Id}(R_1)$, as pursued. In the second case, $z = -(e_1 + e_2)$ and processing by the same token as in the former case, one concludes that $z \in -2\,\mathrm{Id}(R_1) = 2\,\mathrm{Id}(R_1)$ since $4 = 0$. This substantiates the desired equality after all.

About R_2: Here $3 = 0$. So, as $R_2 \in \mathcal{P}$, it is plainly checked that each element x in R_2 satisfies the equation $x^3 = x$. Furthermore, a simple consultation with [7] assures that R is a subdirect product of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$, as stated.

About R_3: Here $5 = 0$. Writing $x = e_1+e_2-e_3$ or $x = -e_1-e_2$ for some three commuting idempotents $e_1, e_2, e_3 \in R_3$. In the first case, additionally assuming without loss of generality that $e_1.e_3 = e_2.e_3 = 0$, one deduces that $x^2 - x = 2(e_1e_2 + e_3)$. But since the expression in the brackets is obviously an idempotent too being the sum of two orthogonal idempotents, we derive that $(x^2 - x)^2 = 2(x^2 - x)$. This, in turn, yields that $x^4 - 2x^3 - x^2 + 2x = 0$. In the second case, one obtains $x^2 + x = 2e_1e_2$ enabling us that $(x^2 + x)^2 = 2(x^2 + x)$. This, in turn, implies that $x^4 + 2x^3 - x^2 - 2x = 0$. Actually, one easily sees that these two equations arise one from other via the substitution $x \to -x$. Since the equations are the same as in the corresponding part of Theorem 2.3, we may process analogically to finish the conclusion that R_3 is the simple five element field, as formulated.
Sufficiency. Identical arguments to these from the "sufficiency part" of Theorem 2.3 work to deduce the wanted assertion. □

As a concluding discussion, we state:

Remark 2.7. Comparing the results established above with these from [6], it seems that the relationships

$$\mathcal{R}_1 \equiv \mathcal{R}_2 \equiv \mathcal{R}_3,$$

showing the equivalences between the three ring classes $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$, hold. Likewise, these three classes surprisingly coincide with the class of rings from [3] for which each element is the sum or the minus sum of three commuting idempotents.

Besides, concerning the classes $\mathcal{K}$, $\mathcal{K}_1$ and $\mathcal{K}_2$, it seems also by comparison of the already established results with these from [4] that these three classes curiously do coincide.

We close our comments by observing that in the proof of [3, Proposition 2.3], the ring $P[b]$ with $b^4 = b$ is the quotient ring of the ring $P[t]/\langle t(t-1)(t-2)(t-4)\rangle$ which, in its turn, is the direct sum of four copies of the field P. It follows immediately that if $P[b] \neq P$, then the requirements of this proposition do not hold. The same idea can be successfully applied to Case 3 and especially to Case 4 in the proof of necessity of Theorem 2.4 from [3]. Nevertheless, the methodology illustrated in [3], although somewhat elusive, is rather more transparent.

On the other vein, in 'Sufficiency' of the proof of [5, Theorem 2.4] on line 2 the phrase *is also a ring* should be written and read as *in the presence of points* (1)*,* (2) *and* (3) *is also a ring*, which is, definitely, an involuntarily omission.

In ending, we pose the following two conjectures:

Conjecture 1. If every element of a ring is a sum of (a fixed number of) commuting idempotents, then this ring is commutative itself.

Conjecture 2. If each element of a ring is expressed as a linear combination over $\mathbb{Z}$ of (a fixed number of) commuting idempotents, then that ring is necessarily commutative.

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