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ROUGH STATISTICAL CONVERGENCE ON TRIPLE SEQUENCE OF RANDOM VARIABLES IN PROBABILITY

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Abstract. This paper aims a further improvement from the works of Phu [9], Aytar [1] and Ghosal [7]. We propose a new apporach to extend the application area of rough statistical convergence usually used in triple sequence of real numbers to the theory of probability distributions. The introduction of this concept in probability of rough statistical convergence, rough strong Cesàro summable, rough lacunary statistical convergence, rough $N_\theta-$ convergence, rouugh $\lambda-$ statistical convergence and rough strong $(V,\lambda)-$ summable generalize the convergence analysis to accommodate any form of distribution of random variables. Among these six concepts in probability only three convergences are distinct rough statistical convergence, rough lacunary statistical convergence and rough $\lambda-$ statistical convergence where rough strong Cesàro summable is equivalent to rough statistical convergence, rough $N_\theta-$ convergence is equivalent to rough lacunary statistical convergence, rough strong $(V,\lambda)-$ summable is equivalent to rough $\lambda-$ statistical convergence. Basic properties and interrelations of above mentioned three distinct convergences are investigated and some observations are made in these classes and in this way we show that rough statistical convergence in probability is the more generalized concept compared to the usual rough statistical convergence.

1. Introduction

In probability theory, a new type of convergence called statistical convergence in probability was introduced in Ghosal [7]. Let $(X_{mnk})_{m,n,k\in\mathbb{N}}$ be a triple sequence of random variables where each X_{mnk} is defined on the same sample spaces W (for each (m,n,k)) with respect to a given class of events Δ and a given probability function $P:\Delta\to\mathbb{R}^3$. Then the triple sequence (X_{mnk}) is said to be statistical convergent in probability to a random variable $X:W\to\mathbb{R}^3$ if for any $\epsilon,\delta>0$

$$\lim_{uvw\to\infty}\frac{1}{uvw}\left|\left\{m\le u, n\le v,\, k\le w : P\big(|X_{mnk}-l\ge\delta\big)\right\}\right|=0.$$

In this case we write $X_{mnk}\to^{S^P} l$. The class of all triple sequences of random variables which are statistical convergence in probability is denoted by S^P.

In this paper we introduce new notions namely rough statistical convergence in probability, rough strong Cesàro summable in probability rough lacunary statistical convergence in probability, rough $N_\theta-$ convergence in probability rough strong $(V,\lambda)-$ summable in probability and rough $\lambda-$ statistical convergence in probability. Among these six concepts in probability only three convergences are distinct-rough statistical convergence in probability, rough lacunary statistical convergence in probability and rough $\lambda-$ statistical convergence in probability, rough $N_\theta-$ convergence in probability is equivalent to rough lacunary statistical convergence in probability, rough strong $(V,\lambda)-$ summable in probability is equivalent to rough $\lambda-$ statistical convergence in probability. Basic properties and interrelations of above mentioned three distinct convergences are investigated and make some observations about these classes.

The idea of statistical convergence was introduced by H. Steinhaus and also independently by H. Fast for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which parallels the theory of ordinary convergence. Later on the notion was investigated by Tripathy ([13], [15]), Tripathy and Sen [14], Tripathy and Baruah [16], Tripathy and Goswami ([17–20]) and others.

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Let K be a subset of the set $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, and let us denote the set $\{(m,n,k)\in K : m \leq u,\ n \leq v,\ k \leq w\}$ by K_{uvw}. Then the natural density of K is given by $\delta(K) = \lim_{uvw\to\infty} \frac{|K_{uvw}|}{uvw}$, where $|K_{uvw}|$ denotes the number of elements in K_{uvw}. Clearly, every finite subset has natural density zero, and we have $\delta(K^c) = 1 - \delta(K)$, where $K^c = \mathbb{N} - K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric (X,d). Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}$, $m,n,k \in \mathbb{N}$.

A triple sequence $x = (x_{mnk})$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st-\lim\ x = 0$, provided that the set

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk}| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence x.

If a triple sequence is statistically convergent, then for every $\in > 0$, infinitely many terms of the sequence may remain outside the $\in -$ neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence $x = (x_{mnk})$ satisfies some property P for all m,n,k except a set of natural density zero, then we say that the triple sequence x satisfies P for almost all (m,n,k) and we abbreviate this by a.a. (m,n,k).

Let $\left(x_{m_i n_j k_\ell}\right)$ be a sub sequence of $x = (x_{mnk})$. If the natural density of the set $K = \{(m_i, n_j, k_\ell) \in \mathbb{N}^3 : (i,j,\ell) \in \mathbb{N}^3\}$ is different from zero, then $\left(x_{m_i n_j k_\ell}\right)$ is called a non-thin subsequence of a triple sequence x.

$c \in \mathbb{R}$ is called a statistical cluster point of a triple sequence $x = (x_{mnk})$ provided that the natural density of the set

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - c| < \epsilon\right\}$$

is different from zero for every $\in > 0$. We denote the set of all statistical cluster points of the sequence x by Γ_x.

A triple sequence $x = (x_{mnk})$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk}|^{1/m+n+k} \geq M\right\}\right) = 0.$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by Phu [9], who introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [8] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. Different types of notions of triple sequence were introduced and investigated at the initial by Sahiner et al. [10, 11], Esi et al. [2–4], Dutta et al. [5], Subramanian et al. [12], Debnath et al. [6] and many others.

Throughout the paper let r be a nonnegative real number.

2. Triple Rough Statistical Convergence in Probability

Definition 2.1. Let r be a non-negative real number. A triple sequence (x_{mnk}) is said to be rough convergent to l with respect to the roughness degree r (or shortly: $r-$ convergent to x) if for every $\in > 0$, there exist some numbers u, v and w such that

$$\left|x_{mnk} - l\right| < r + \in \quad \text{for all} \;\; m \geq u,\ n \geq v,\ k \geq w$$

and denoted by $x_{mnk} \rightarrow^r l$. if we take $r = 0$, then we obtain the ordinary convergence.

Definition 2.2. Let r be a non-negative real number. A triple sequence (x_{mnk}) is said to be rough statistically convergent to l with respect to the roughness of degree r (or shortly: $r-$ statistically convergent to l) if for every $\epsilon > 0$, the set

$$\left\{ (m,n,k) \in \mathbb{N}^3 : |x_{mnk} - l| \geq r + \in \right\}$$

has asymptotic density zero or equivalently, if the condition $S - \lim_{mnk\to\infty} \sup |x_{mnk} - l| \leq r$ is satisfied and we denote by $x_{mnk} \rightarrow_r^{S^P} l$.

If we take $r = 0$, then we obtain the ordinary statistical convergence.

Definition 2.3. Let r be a non-negative real number. A triple sequence of random variables (X_{mnk}) is said to be rough statistically convergent in probability to a random variable $X : W \rightarrow \mathbb{R}^3$ with respect to the roughness of degree r (or shortly: $r-$ statistically convergent in probability to l) if for each $\in, \delta > 0$,

$$\lim_{uvw\to\infty} \frac{1}{(uvw)} \left| \left\{ m \leq u,\, n \leq v,\, k \leq w : P\big(|X_{mnk} - l| \geq r + \in \big) \geq \delta \right\} \right| = 0,$$

or, equivalently,

$$\lim_{uvw\to\infty} \frac{1}{(uvw)} \left| \left\{ m \leq u,\, n \leq v,\, k \leq w : 1 - P\big(|X_{mnk} - l| < r + \in \big) \geq \delta \right\} \right| = 0,$$

and we write $X_{mnk} \rightarrow_r^{S^P} l$. The class of all $r-$ statistically convergent triple squences of random variables in probability will be simply denoted by rS^P.

Theorem 2.4. *If $X_{mnk} \rightarrow_r^{S^P} l_1$ and $Y_{mnk} \rightarrow_r^{S^P} l_2$ then $P\{|l_1 - l_2| \geq r\} = 0$.*

Proof. Let ϵ, δ be any two positive real numbers and let

$$(u,v,w) \in \left\{ (m,n,k) \in \mathbb{N}^3 : P\Big(|X_{mnk} - l_1| \geq r + \frac{\epsilon}{2} \Big) < \frac{\delta}{2} \right\}$$
$$\bigcap \left\{ (m,n,k) \in \mathbb{N}^3 : P\Big(|X_{mnk} - l_2| \geq r + \frac{\epsilon}{2} \Big) < \frac{\delta}{2} \right\} \text{(existence of } (u,v,w)$$

is guaranteed since asymptotic density of both the sets is equal to 1. Then

$$P\Big(\big|l_1 - l_2\big| \geq r + \epsilon \Big) \leq P\left(\big|X_{mnk} - l_1\big| \geq r + \frac{\epsilon}{2} \right) + P\left(\big|X_{mnk} - l_2\big| \geq r + \frac{\epsilon}{2} \right) < \delta.$$

This implies $P\left(|l_1 - l_2| \geq r\right) = 0$. □

Remark 2.5.

(i) If $X_{mnk} \rightarrow_r^{S^P} l_1$ and $X_{mnk} \rightarrow_r^{S^P} l_2$, then $P\{l_1 = l_2\} = 1$ (here $r = 0$).

(ii) If $X_{mnk} \rightarrow_r^{S^P} l_1$ and $X_{mnk} \rightarrow_r^{S^P} l_2$, then $\{P\{l_1 - l_2\} < r\} = 1$.

Definition 2.6. A discrete random variable X is said to be one-point distribution at the point c if the spectrum consists of a single point c and $P(X = c) = 1$. Here c is a parameter of the one point distribution.

Theorem 2.7. *If a triple sequence of constants $x_{mnk} \rightarrow_r^{S} l$ then $x_{mnk} \rightarrow_r^{S^P} l$.*

Proof. Here for every (u,v,w), x_{mnk} can be regarded as a random variable with one element X_{mnk} in the corresponding spectrum. Let $\in$ be a positive real number. Since $x_{mnk} \rightarrow_r^{S} l$ then

$$\lim_{uvw\to\infty} \frac{1}{(uvw)} \left| \left\{ m \leq u, n \leq v, k \leq w : \big|X_{mnk} - l\big| \geq r + \in \right\} \right| = 0,$$
$$\Longrightarrow \lim_{uvw\to\infty} \frac{1}{(uvw)} \left| \left\{ m \leq u, n \leq v, k \leq w : \big|X_{mnk} - l\big| < r + \in \right\} \right| = 1.$$

Now the event $\{w : w \in W \textit{ and } |X_{mnk}(w) - l(w)| < r+ \in\}$ is the same as the event $|x_{mnk} - l| < r+ \in$ which is here the certain event W for all

$$(u,v,w) \in \Big\{ (m,n,k) \in \mathbb{N}^3 : \big|x_{mnk} - l\big| < r+ \in \Big\}.$$

So $P\big(\{w : w \in W \text{ and } |X_{mnk}(w) - l(w)| < r+ \in \}\big) = P(|x_{mnk} - l| < r+ \in) = P(W) = 1$ for all $(u,v,w) \in \{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - l| < r+ \in\}$. Thus for any $\delta > 0$,

$$\Big\{(u,v,w) \in \mathbb{N}^3 : 1 - P\big(|x_{uvw} - l| < r+ \in \big)\Big\} \subset \mathbb{N}^3 \backslash \Big\{m \le u,\, n \le v,\, k \le w : |x_{mnk} - l| < r+ \in \Big\}$$
$$= \Big\{m \le u,\, n \le v,\, k \le w : \big|x_{mnk} - l\big| \ge r+ \in \Big\}.$$

In general converse is not true, i.e., if a triple sequence of random variables (x_{mnk}) is a rough statistical convergence in probability to a real number l then each of X_{mnk} may not have one point distribution so each X_{mnk} can not be treated as a constant which is rough statistical convergence to l i.e., rough statistical convergence in probability is the more generalized concept than usual rough statistical convergence. □

Example. Let a triple sequence of random variables (X_{mnk}) be defined by,

$$X_{mnk} \in \begin{cases} \{-10, 10\} & \text{with probability } P(X_{mnk} = -10) = P(X_{mnk} = 10), \\ & \quad \text{if } (m,n,k) = (u,v,w)^2 \text{ for some } (u,v,w) \in \mathbb{N}^3 \\ \{0, 1\} & \text{with probability } P(X_{mnk} = 0) = P(X_{mnk} = 1), \\ & \quad \text{if } (m,n,k) \neq (u,v,w)^2 \text{ for any } (u,v,w) \in \mathbb{N}^3. \end{cases}$$

Let $0 < \in < 1$ be given. Then

$$P\Big(\big|X_{mnk} - 1\big| \ge 2+ \in \Big) = \begin{cases} 1 & \text{if } (m,n,k) = (u,v,w)^2 \text{ for some } (u,v,w) \in \mathbb{N}^3 \\ 0 & \text{if } (m,n,k) \neq (u,v,w)^2 \text{ for any } (u,v,w) \in \mathbb{N}^3. \end{cases}$$

This implies $X_{mnk} \rightarrow_2^{S^P} 1$. But it is not ordinary rough statistical convergence of a triple sequence of numbers to 1.

Theorem 2.8.

(i) $X_{mnk} \rightarrow_r^{S^P} l \Longleftrightarrow X_{mnk} - l \rightarrow_r^{S^P} 0$,
(ii) $X_{mnk} \rightarrow_r^{S^P} l \Longrightarrow cX_{mnk} \rightarrow_{|c|r}^{S^P} c$ *where* $c \in \mathbb{R}$,
(iii) $X_{mnk} \rightarrow_r^{S^P} l_1$ *and* $Y_{mnk} \rightarrow_r^{S^P} l_2 \Longrightarrow X_{mnk} + Y_{mnk} \rightarrow_r^{S^P} l_1 + l_2$,
(iv) $X_{mnk} \rightarrow_r^{S^P} l_1$ *and* $Y_{mnk} \rightarrow_r^{S^P} l_2 \Longrightarrow X_{mnk} - Y_{mnk} \rightarrow_r^{S^P} l_1 - l_2$,
(v) $X_{mnk} \rightarrow_r^{S^P} 0 \Longrightarrow X_{mnk}^2 \rightarrow_{r^2}^{S^P} 0$,
(vi) $X_{mnk} \rightarrow_r^{S^P} l \Longrightarrow X_{mnk}^2 \rightarrow_{r^2+2|x|r}^{S^P} l^2$,
(vii) $X_{mnk} \rightarrow_r^{S^P} l_1$ *and* $Y_{mnk} \rightarrow_r^{S^P} l_2 \Longrightarrow X_{mnk} \cdot Y_{mnk} \rightarrow_{\frac{r}{2} + \frac{r(|l_1 + l_2| + |l_1 - l_2|)}{2}}^{S^P} l_1 \cdot l_2$,
(viii) *If* $0 \le X_{mnk} \le Y_{mnk}$ *and* $Y_{mnk} \rightarrow_r^{S^P} 0 \Longrightarrow X_{mnk} \rightarrow_r^{S^P} 0$,
(ix) $X_{mnk} \rightarrow_r^{S^P} l$, *then for each* $\in > 0$ *there exists* $(uvw) \in \mathbb{N}^3$ *such that for any* $\delta > 0$

$$\lim_{uvw \to \infty} \frac{1}{(uvw)} \Big|\Big\{m \le u,\ n \le v,\ k \le w : P\big(|X_{mnk} - l_{uvw}| \ge 2r+ \in \big) \ge \delta\Big\}\Big| = 0,$$

which is called rough statistical Cauchy condition in probability.

Proof. Let $\in, \delta$ be any positive real numbers. Then for

(i) proof is straight forward hence omitted.
(ii) If $c = 0$ then the claim is obvious. So suppose assuming $c \neq 0$, then

$$\Big\{ (m,n,k) \in \mathbb{N}^3 : P\Big(|c\, X_{mnk} - c\, l| \ge |c|\, r + \epsilon\Big) \ge \delta\Big\}$$
$$= \Big\{ (m,n,k) \in \mathbb{N}^3 : P\Big(|X_{mnk} - l| \ge r + \frac{\epsilon}{|c|}\Big) \ge \delta\Big\}.$$

Hence, $cX_{mnk} \rightarrow^{S^P}_{|c|r} cX$.

(iii) $P\Big(\Big| (X_{mnk} + Y_{mnk}) - (l_1 + l_2) \Big| \geq r + \epsilon\Big) = P\Big(\Big| (X_{mnk} - l_1) + (Y_{mnk} - l_2) \Big| \geq r + \epsilon\Big) \leq$
$P\Big(\big|X_{mnk} - l_1\big| \geq r + \frac{\epsilon}{2}\Big) + P\Big(|Y_{mnk} - l_2| \geq r + \frac{\epsilon}{2}\Big)$.

This implies

$$\Big\{ (m,n,k) \in \mathbb{N}^3 : P\Big(\big| (X_{mnk} + Y_{mnk}) - (l_1 + l_2) \big| \geq r + \epsilon\Big) \geq \delta\Big\} \subseteq$$
$$\Big\{ (m,n,k) \in \mathbb{N}^3 : P\Big(|(X_{mnk} - l_1)| \geq r + \frac{\epsilon}{2}\Big) \geq \frac{\delta}{2}\Big\} \bigcup$$
$$\Big\{ (m,n,k) \in \mathbb{N}^3 : P\Big(|(Y_{mnk} - l_2)| \geq r + \frac{\epsilon}{2}\Big) \geq \frac{\delta}{2}\Big\}.$$

Hence $X_{mnk} + Y_{mnk} \rightarrow^{S^P}_{r} l_1 + l_2$.

(iv) Similar to the proof of (iii) and therefore omitted.

(v) $\Big\{ (m,n,k) \in \mathbb{N}^3 : P\Big(\big|X^2_{mnk}\big| \geq r^2 + \delta\Big)\Big\} = \Big\{ (m,n,k) \in \mathbb{N}^3 : P\Big(\big|X^2_{mnk}\big| \geq r^2 + 2r\eta + \eta^2\Big)\Big\}$
(where $\eta = -r + \sqrt{r^2 + \delta} > 0) = \Big\{ (m,n,k) \in \mathbb{N}^3 : P\Big(\big|X^2_{mnk}\big| \geq r + \eta\Big)\Big\}$. Hence, $X^2_{mnk} \rightarrow^{S^P}_{r^2} 0$.

(vi) $X^2_{mnk} = (X_{mnk} - l)^2 + 2l\,(X_{mnk} - l) + l^2$, so $X^2_{mnk} \rightarrow^{S^P}_{2r^2+2|l|r} l^2$.

(vii) $(X_{mnk} + Y_{mnk})^2 \rightarrow^{S^P}_{r^2+2r|l_1+l_2|} (l_1 + l_2)^2$ and $(X_{mnk} - Y_{mnk})^2 \rightarrow^{S^P}_{r^2+2r|l_1-l_2|} (l_1 - l_2)^2$
$\Longrightarrow X_{mnk} \cdot Y_{mnk} = \frac{1}{4}\Big\{\Big\{(X_{mnk} + Y_{mnk})^2 - (X_{mnk} - Y_{mnk})^2\Big\}\Big\} \rightarrow^{S^P} {}_{\frac{r^2}{2} + \frac{r(|l_1+l_2|+|l_1-l_2|)}{2}}$
$\frac{1}{4}\Big\{(l_1 + l_2)^2 - (l_1 - l_2)^2\Big\} = l_1 \cdot l_2$.

(viii) Proof is straight forward hence omitted.

(ix) Choose $(u,v,w) \in \mathbb{N}^3$ be such that $P\left(|X_{uvw} - X| \geq r + \frac{\in}{2}\right) < \frac{\delta}{2}$. Then the claim is obvious from the inequality

$$P\Big(|X_{mnk} - X_{uvw}| \geq 2r + \in \Big) \leq P\Big(|X_{mnk} - X| \geq r + \frac{\in}{2}\Big) + P\Big(|X_{uvw} - X| \geq r + \frac{\in}{2}\Big)$$
$$\leq \frac{\delta}{2} + P\Big(|X_{mnk} - X| \geq r + \frac{\in}{2}\Big).$$
□

Theorem 2.9. *Let (X_{mnk}) be a triple sequence of random variables then there exists a triple sequence of real numbers (x_{mnk}) with the property that $X_{mnk} - x_{mnk} \rightarrow^{S^P}_{r} 0$. If $m\,(X_{mnk})$ is a median of X_{mnk} then $X_{mnk} - m\,(X_{mnk}) \rightarrow^{S^P}_{r} 0$ and $x_{mnk} - m\,(X_{mnk}) \rightarrow^{S^P}_{r} 0$.*

Proof. Proof is straight forward hence omitted. □

Theorem 2.10. *Let $r > 0$. Then $X_{mnk} \rightarrow^{S^P}_{r} l \Longleftrightarrow$ there exists a triple sequence of random variables (Y_{mnk}) such that $Y_{mnk} \rightarrow^{S^P}_{r} l$ and $S - \lim_{mnk\to\infty} P\,(|X_{mnk} - Y_{mnk}| > r) = 0$.*

Proof. Let $X_{mnk} \rightarrow^{S^P}_{r} l$ and $A = \big\{(m,n,k) \in \mathbb{N}^3 : P\,(|X_{mnk} - l| \geq r + \epsilon) \geq \delta\big\}$. Then $\delta\,(A) = 0$.

Now we define

$$Y_{mnk} = \begin{cases} l & \text{if } (m,n,k) \in \mathbb{N}^3 \backslash A \\ X_{mnk} + Z & \text{otherwise} \end{cases},$$

where X is a random variable and $Z \in (-r, r)$ with probability $P\,(X_{mnk} = -r) = P\,(X_{mnk} = r)$. Then it is very obvious that

$$d\Big(\Big\{ (m,n,k) \in \mathbb{N}^3 : P\big(|Y_{mnk} - l| \geq \in \big) \geq \delta\Big\}\Big) = 0 \text{ and}$$
$$d\Big(\Big\{ (m,n,k) \in \mathbb{N}^3 : P\big(|X_{mnk} - Y_{mnk}| \geq r + \in \big) \geq \delta\Big\}\Big)$$
$$\leq d\Big(\Big\{ (m,n,k) \in \mathbb{N}^3 : P\Big(|X_{mnk} - l| \geq r + \frac{\in}{2}\Big) \geq \frac{\delta}{2}\Big\}\Big)$$
$$+\, d\Big(\Big\{ (m,n,k) \in \mathbb{N}^3 : P\Big(|Y_{mnk} - l| \geq \frac{\in}{2}\Big) \geq \frac{\delta}{2}\Big\}\Big) = 0.$$

Conversely, let $Y_{mnk} \to_r^{S^P} l$ and $S - \lim_{mnk\to\infty} P\big(|X_{mnk} - Y_{mnk}| > r\big) = 0$. Then for each $\in, \delta > 0$,

$$\lim_{uvw\to\infty} \frac{1}{(uvw)} \Big|\Big\{ m \le u, n \le v, k \le w : P\Big(|Y_{mnk} - l| \ge \frac{\in}{2}\Big) \ge \frac{\delta}{2} \Big\}\Big| = 0 \text{ and}$$
$$\lim_{uvw\to\infty} \frac{1}{(uvw)} \Big|\Big\{ m \le u, n \le v, k \le w : P\Big(|X_{mnk} - Y_{mnk}| \ge r + \frac{\in}{2}\Big) \ge \frac{\delta}{2} \Big\}\Big| = 0.$$

We know the inequality

$$P\Big(|X_{mnk} - l| \ge r + \in\Big) \le P\Big(|Y_{mnk} - l| > \frac{\in}{2}\Big) + P\Big(|X_{mnk} - Y_{mnk}| \ge r + \frac{\in}{2}\Big).$$
$$\implies \Big\{ (m,n,k) \in \mathbb{N}^3 : P\Big(|X_{mnk} - l| \ge r + \in\Big) \ge \delta \Big\}$$
$$\subseteq \Big\{ (m,n,k) \in \mathbb{N}^3 : P\Big(|Y_{mnk} - l| \ge \frac{\in}{2}\Big) \ge \frac{\delta}{2} \Big\} \bigcup$$
$$\Big\{ (m,n,k) \in \mathbb{N}^3 : P\Big(|X_{mnk} - Y_{mnk}| \ge r + \frac{\in}{2}\Big) \ge \frac{\delta}{2} \Big\}.$$

Hence $X_{mnk} \to_r^{S^P} l$. □

Theorem 2.11. *If $X_{mnk} \to_r^{S^P} l$ and $g : \mathbb{R}^3 \to \mathbb{R}^3$ is a continuous function on $\mathbb{R}^3$, then there exists a triple sequence of random variables (Y_{mnk}) such that $g(Y_{mnk}) \to_r^{S^P} g(l)$ and $g\big(P(|X_{mnk} - Y_{mnk}| > r)\big) \to_r^{S^P} 0$.*

Proof. The proof is similar to Theorem 2.4 in [7] and hence omitted. □

3. Strong CesÀro Summable of a Triple Sequence of Real Numbers

Definition 3.1. A triple sequence (x_{mnk}) is said to be strong Cesàro summable to l if

$$\lim_{uvw\to\infty} \frac{1}{(uvw)} \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{k=1}^{w} |x_{mnk} - l| = 0.$$

In this case we write $x_{mnk} \to^{[C,1,1]} l$. The set of all strong Cesàro summable triple squences is denoted by either $|C,1|$ or $|C,1,1|$.

Definition 3.2. Let r be a non-negative real number. A triple sequence of random variables (X_{mnk}) is said to be rough strong Cesàro summable in probability to a random variable $X : W \to \mathbb{R}^3$ with respect to the roughness of degree r (or shortly: $r-$ strong Cesàro summable in probability to l) of for each $\epsilon > 0$,

$$\lim_{uvw\to\infty} \frac{1}{(uvw)} \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{k=1}^{w} P\left(|X_{mnk} - l| \ge r + \epsilon\right) = 0.$$

In this case we write $X_{mnk} \to_r^{[C,1,1]^P} l$.

The class of all $r-$ strong Cesàro summable triple sequences of random variables in probability will be simply denoted by $r\,[C,1,1]^P$.

Theorem 3.3. *The followings are equivalent:* (i) $X_{mnk} \to_r^{S^P} l$ (ii) $X_{mnk} \to_r^{[C,1,1]^P} l$.

Proof. (i) $\implies$ (ii). First suppose that $X_{mnk} \to_r^{S^P} l$. Then we can write

$$\frac{1}{(uvw)} \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{k=1}^{w} P\big(|X_{mnk} - l| \ge r + \in\big)$$
$$= \frac{1}{(uvw)} \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{k=1, P(|X_{mnk}-l| \ge r+\epsilon) \ge \frac{\delta}{2}}^{w} P\big(|X_{mnk} - l| \ge r + \in\big)$$
$$+ \frac{1}{(uvw)} \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{k=1, P(|X_{mnk}-l| \ge r+\epsilon) < \frac{\delta}{2}}^{w} P\big(|X_{mnk} - l| \ge r + \in\big)$$

$$\leq \frac{1}{(uvw)}\Big|\Big\{m \leq u,\ n \leq v,\ k \leq w : P\Big(\,|X_{mnk} - l| \geq r + \epsilon\Big) > \frac{\delta}{2}\Big\}\Big| + \frac{\delta}{2}.$$

(ii) $\Longrightarrow$ (i). Next suppose that condition (ii) holds. Then

$$\sum_{m=1}^{u}\sum_{n=1}^{v}\sum_{k=1}^{w} P\big(\,|X_{mnk} - l| \geq r + \epsilon\big)$$

$$\geq \sum_{m=1}^{u}\sum_{n=1}^{v}\sum_{k=1, P(|X_{mnk}-l| \geq r+\epsilon) \geq \delta}^{w} P\Big(\,|X_{mnk} - l| \geq r + \in\Big)$$

$$\geq \delta\Big|\Big\{m \leq u,\, n \leq v,\, k \leq w : P\big(\,|X_{mnk} - l| \geq r + \in\big) > \delta\Big\}\Big|.$$

Therefore

$$\frac{1}{(uvw)}\sum_{m=1}^{u}\sum_{n=1}^{v}\sum_{k=1}^{w} P\big(\,|X_{mnk} - l| \geq r + \in\big)$$

$$\geq \frac{1}{(uvw)}\Big|\Big\{m \leq u, n \leq v, k \leq w : P\big(\,|X_{mnk} - l| \geq r + \in\big) > \delta\Big\}\Big|.$$

Hence $X_{mnk} \to_r^{S^P} l$. □

4. Triple Rough Lacunary Statistical Convergence in Probability

Definition 4.1. The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$m_0 = 0,\ ]; h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty \text{ and}$$
$$n_0 = 0,\ \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty,$$
$$k_0 = 0,\ \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j$, $h_{i,\ell,j} = h_i \overline{h_\ell h_j}$ and $\theta_{i,\ell,j}$ is determine by $I_{i,\ell,j} = \big\{\,(m,n,k) : m_{i-1} < m < m_i$ and $n_{\ell-1} < n \leq n_\ell$ and $k_{j-1} < k \leq k_j\big\}$, $q_i = \frac{m_i}{m_{i-1}}$, $\overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}$, $\overline{q_j} = \frac{k_j}{k_{j-1}}$.

Definition 4.2. Let $\theta = \{m_r n_s k_t\}_{(rst)\in\mathbb{N}\bigcup 0}$ be the triple lacunary sequence. A number triple sequence (X_{mnk}) is said to be triple lacunary statistically convergent to a real number l (or shortly: $S_\theta-$ convergent to l) if for any $\in > 0$,

$$\lim_{rst\to\infty}\frac{1}{h_{rst}}\Big|\Big\{\,(m,n,k) \in I_{rst} : |X_{mnk} - l| \geq \epsilon\Big\}\Big| = 0$$

and it is denoted by $X_{mnk} \to^{S_\theta} l$, where $I_{r,s,t} = \big\{\,(m,n,k) : m_{r-1} < m < m_r$ and $n_{s-1} < n \leq n_s$ and $k_{t-1} < k \leq k_t\big\}$, $q_r = \frac{m_r}{m_{r-1}}$, $\overline{q_s} = \frac{n_s}{n_{s-1}}$, $\overline{q_t} = \frac{k_t}{k_{t-1}}$.

Definition 4.3. Let $\theta = \{m_r n_s k_t\}$ be the triple lacunary sequence. A number triple seqeunce (x_{mnk}) is said to be $N_\theta-$ convergent to a real number l if for any $\in > 0$,

$$\lim_{rst\to\infty}\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t}|x_{mnk} - l| = 0.$$

In this case we write $x_{mnk} \to^{N_\theta} l$.

Definition 4.4. Let r be a non-negative real number. A triple sequence of random variables (X_{mnk}) is said to be rough lacunary statistically convergent in probability to $X : W \to \mathbb{R}^3$ with respect to the roughness of degree r (or shortly: $r-$ lacunary statistically convergent in probability to l) if for any $\epsilon, \delta > 0$

$$\lim_{rst\to\infty}\frac{1}{h_{rst}}\Big|\Big\{\,(m,n,k) \in I_{rst} : P\big(\big|X_{mnk} - l\big| \geq r + \epsilon\big) \geq \delta\Big\}\Big| = 0,$$

and we write $X_{mnk} \to_r^{S_\theta^P} l$. The class of all $r-$ triple lacunary statistically convergent sequences of random variables in probability will be denoted simply by rS_θ^P.

Definition 4.5. Let r be a non-negative real number. A triple sequence of random variables (X_{mnk}) is said to be rough $N_\theta-$ convergent in probability to $X : W \to \mathbb{R}^3$ with respect to the roughness of degree r (or shortly: $r - N_\theta-$ convergent in probability to l) if for any $\epsilon > 0$,

$$\lim_{rst\to\infty} \frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t} P\big(|x_{mnk} - l| \geq r + \epsilon\big) = 0$$

and we write $X_{mnk} \to_r^{N_\theta^P} l$. The class of all $r - N_\theta-$ convergent triple sequence of random variables in probability will be denoted simply by rN_θ^P.

Theorem 4.6. *Let $\theta = \{m_r, n_s, k_t\}$ be a triple lacunary sequence. Then the followings are equivalent:*
(i) *(X_{mnk}) is a $r-$ triple lacunary statistically convergent in probability to l.*
(ii) *(X_{mnk}) is $r - N_\theta-$ convergent in probability to l.*

Proof. (i) $\implies$ (ii) First suppose that $X_{mnk} \to_r^{S_\theta^P} l$. Then we can write

$$\begin{aligned}
&\frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t} P\big(|x_{mnk} - l| \geq r + \epsilon\big) \\
&= \frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t, P(|x_{mnk}-l|\geq r+\epsilon)\geq \frac{\delta}{2}} P\big(|x_{mnk} - l| \geq r + \epsilon\big) \\
&+ \frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t, P(|x_{mnk}-l|\geq r+\epsilon)< \frac{\delta}{2}} P\big(|x_{mnk} - l| \geq r + \epsilon\big) \\
&\leq \frac{1}{h_{rst}} \Big|\Big\{ (m,n,k) \in I_{rst} : P\big(|X_{mnk} - l| \geq r + \epsilon\big) \geq \frac{\delta}{2}\Big\}\Big|.
\end{aligned}$$

(ii) $\implies$ (i) Next suppose that condition (ii) holds. Then

$$\begin{aligned}
&\sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t} P\big(|x_{mnk} - l| \geq r + \epsilon\big) \\
&\geq \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t, P(|x_{mnk}-l|\geq r+\epsilon)\geq \delta} P\big(|x_{mnk} - l| \geq r + \epsilon\big) \\
&\geq \delta \Big|\Big\{ (m,n,k) \in I_{rst} : P\big(|X_{mnk} - l| \geq r + \epsilon\big) \geq \delta\Big\}\Big|.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{\delta\, h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t} P\big(|x_{mnk} - l| \geq r + \epsilon\big) \\
&\geq \frac{1}{h_{rst}} \Big|\Big\{ (m,n,k) \in I_{rst} : P\big(|x_{mnk} - l| \geq r + \epsilon\big)\Big\}\Big|.
\end{aligned}$$

Hence $X_{mnk} \to_r^{S_\theta^P} l$. ☐

Theorem 4.7. *If $X_{mnk} \to_r^{S_\theta^P} l_1$ and $X_{mnk} \to_r^{S_\theta^P} l_2$ then $P\left(|l_1 - l_2| \geq r\right) = 0$.*

Proof. Similar to the proof of the Theorem 2.4 and therefore omitted. ☐

5. Triple Rough-$\lambda-$statistical Convergence in Probability

Let $\lambda = (\lambda_{uvw})$ be a non-decreasing triple sequence of positive numbers tending to ∞ such that $\lambda_{uvw+1} \leq \lambda_{uvw} + 1$, $\lambda_{111} = 1$. The collection of all such triple sequence λ is denoted by $\mathbb{D}$.

The generalized De la valeé$-$ Pousin mean is defined for the triple sequence (x_{mnk}) of real numbers by $t_{uvw}(x) = \frac{1}{\lambda_{uvw}} \sum_{(m,n,k)\in Q_{uvw}} x_{mnk}$, where $Q_{uvw} = [(uvw) - \lambda_{uvw} + 1, (uvw)]$. A triple sequence (x_{mnk}) of real numbers is said to be $[V,\lambda]-$ summable to l, if $\lim t_{uvw}(x) = l$.

Definition 5.1. A triple sequence (x_{mnk}) is said to be strong $[V,\lambda]-$ summable (or shortly: $[V,\lambda]-$ convergent) to l, if $\lim_{uvw\to\infty} \frac{1}{\lambda_{uvw}} \sum_{(m,n,k)\in Q_{uvw}} |x_{mnk} - l| = 0$. In this case we write $x_{mnk} \to^{[V,\lambda]} l$.

Definition 5.2. A triple sequence (x_{mnk}) is said to be $\lambda-$ statistically convergent (or shortly: $S_\lambda-$ convergent) to l if for any $\epsilon > 0$,

$$\lim_{uvw\to\infty} \frac{1}{\lambda_{uvw}} \Big|\Big\{ (m,n,k) \in Q_{uvw} : |x_{mnk} - l| \geq \epsilon \Big\}\Big| = 0.$$

In this case we write $S_\lambda - \lim x_{mnk} = l$ or by $x_{mnk} \to^{S_\lambda} l$.

Definition 5.3. Let r be a non-negative real number. A triple sequence of random variables (X_{mnk}) is said to be rough $[V,\lambda]-$ summable in probability to $X : W \to \mathbb{R}^3$ with respect to the roughness degree r (or shortly: $r - [V,\lambda] -$ summable in probability to l) if for any $\epsilon > 0$,

$$\lim_{uvw\to\infty} \frac{1}{\lambda_{uvw}} \sum_{(m,n,k)\in Q_{uvw}} : P\big(|X_{mnk} - l| \geq r + \epsilon \big) = 0.$$

In this case we write $X_{mnk} \to_r^{[V,\lambda]^P} l$. The class all rough $[V,\lambda]-$ summable sequences of random variables in probability will be denoted by $r\,[V,\lambda]^P$.

Definition 5.4. Let r be a non-negative real number. A triple sequence of random variables (X_{mnk}) is said to be rough $\lambda-$ statistically convergent in probability to $X : W \to \mathbb{R}^3$ with respect to the roughness degree r (or shortly: $r - \lambda-$ statistically convergent in probability to l) if for any $\epsilon, \delta > 0$,

$$\lim_{uvw\to\infty} \frac{1}{\lambda_{uvw}} \Big|\Big\{ (m,n,k) \in Q_{uvw} : P\left(|x_{mnk} - l| \geq r + \epsilon\right) \geq \delta \Big\}\Big| = 0.$$

In this case we write $X_{mnk} \to_r^{S_\lambda^P} l$. The class of all $r - \lambda-$ statistically convergent triple sequence of random variables in probability will be denoted simply by rS_λ^P.

Theorem 5.5. *For any triple sequence of random variables (X_{mnk}) the following are equivalent:*
(i) *(X_{mnk}) is $r - [V,\lambda]-$ summable in probability to l.*
(ii) *(X_{mnk}) is $r - \lambda-$ statistically convergent in probability to l.*

Proof. It can be established using the technique of Theorem 4.6, so omitted. □

Theorem 5.6. *If $X_{mnk} \to_r^{S_\lambda^P} l_1$ and $X_{mnk} \to_r^{S_\lambda^P} l_2$ then $P\left(|l_1 - l_2| \geq r\right) = 0$.*

Proof. It can be established using the technique of Theorem 2.4, so omitted. □

Theorem 5.7. *If $\lambda \in \mathbb{D}$ is such that $\lim\left(\frac{\lambda_{uvw}}{uvw}\right) = 1$, then $rS_\lambda^P \subset rS^P$.*

Proof. Let $0 < \eta < 1$ be given. Since $\lim\left(\frac{\lambda_{uvw}}{uvw}\right) = 1$, we can choose $(r,s,t) \in \mathbb{N}^3$ such that $\left|\frac{\lambda_{uvw}}{uvw} - 1\right| < \frac{\eta}{2}$ for all $u \geq r$, $v \geq s$, $w \geq t$. Now observe that for $\epsilon, \delta > 0$

$$\begin{aligned}
&\frac{1}{(uvw)}\Big|\Big\{m \leq u,\, n \leq v,\, k \leq w : P\big(|X_{mnk} - l| \geq r + \epsilon\big) \geq \delta\Big\}\Big| \\
&= \frac{1}{(uvw)}\Big|\Big\{m \leq u, n \leq v, k \leq w - \lambda_{uvw} : P\big(|X_{mnk} - l| \geq r + \epsilon\big) \geq \delta\Big\}\Big| \\
&+ \frac{1}{(uvw)}\Big|\Big\{Big|\,(m,n,k) \in Q_{uvw} : P\big(|X_{mnk} - l| \geq r + \epsilon\big) \geq \delta\Big\}\Big| \\
&\leq \frac{(uvw) - \lambda_{uvw}}{(uvw)} + \frac{1}{(uvw)}\Big|\Big\{ (m,n,k) \in Q_{uvw} : P\big(|X_{mnk} - l| \geq r + \epsilon\big) \geq \delta\Big\}\Big| \\
&\leq 1 - \Big(1 - \frac{\eta}{2}\Big) + \frac{1}{(uvw)}\Big|\Big\{ (m,n,k) \in Q_{uvw} : P\big(|X_{mnk} - l| \geq r + \epsilon\big) \geq \delta\Big\}\Big| \\
&= \frac{\eta}{2} + \frac{\lambda_{uvw}}{(uvw)} \cdot \frac{1}{\lambda_{uvw}}\Big|\Big\{ (m,n,k) \in Q_{uvw} : P\big(|X_{mnk} - l| \geq r + \epsilon\big) \geq \delta\Big\}\Big| \\
&< \frac{\eta}{2} + \frac{2}{\lambda_{uvw}}\Big|\Big\{ (m,n,k) \in Q_{uvw} : P\big(|X_{mnk} - l| \geq r + \epsilon\big) \geq \delta\Big\}\Big|
\end{aligned}$$

hold for all $u \geq r$, $v \geq s$, $w \geq t$. □

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