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ABOUT ONE CONTACT PROBLEM FOR A VISCOELASTIC HALFPLATE

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Abstract. Exact solutions of two-dimensional singular integro-differential equations related to the problems of interaction of an elastic thin infinite homogeneous inclusion with a plate for the Kelvin-Voigt linear model are considered. The Kolosov-Muskhelishvili's type formulas are obtained, and the problem is reduced to the Volterra type integral equations. Using the method of integral transformation, the boundary value problem of the theory of analytic functions is obtained. The solution of the problem is represented explicitly and asymptotic analysis is carried out.

Introduction

The theories of viscoelasticity including the Maxwell model, the Kelvin-Voigt model and the standard linear solid model were used to predict response of a material under the action of different loading conditions. Viscoelastic materials play an important role in many branches of civil and geotechnical engineering, technology and biomechanics.

Significant development of hereditary Bolzano–Volterra mechanics is determined by many technical applications in the theory of polymers, metals, plastics, concrete and in the mining engineering. The fundamentals of the theory of viscoelasticity, the methods for solving linear and nonlinear problems of the creep theory, the problems of mechanics of inhomogeneous ageing viscoelastic materials, some boundary value problems of the theory of growing solids, the contact and mixed problems of the theory of viscoelasticity for composite inhomogeneously ageing and nonlinearly ageing bodies are considered in [1, 5, 7, 11, 12, 14, 15, 20].

A complete study of various possible forms of viscoelastic relations and some aspects of the general theory of viscoelasicity are studied in [8, 9, 13, 19]. The research dealing with the material creep can be found in [2–4, 17].

In [6, 21], we have considered integro-differential equations with a variable coefficient related to the interaction of an elastic thin inclusion and a plate, when the inclusion and plate materials possess a creep property. Using the investigation of different boundary value problems of the theory of analytic functions, we have got solutions of those integro-differential equations and established asymptotics of unknown contact stresses.

1. Kolosov-Muskhelishvili's Type Formulas for One Model of the Plane Theory of Viscoelasticity

For viscoelastic bodies, following the Kelvin–Voigt model [20], the Hook's law has the form

$$\begin{aligned} X_x &= \lambda\theta + 2\mu e_{xx} + \lambda^*\dot{\theta} + 2\mu^*\dot{e}_{xx}, \\ Y_y &= \lambda\theta + 2\mu e_{yy} + \lambda^*\dot{\theta} + 2\mu^*\dot{e}_{yy}, \\ X_y &= \mu\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) + \mu^*\Big(\frac{\partial \dot{v}}{\partial x} + \frac{\partial \dot{u}}{\partial y}\Big), \end{aligned} \tag{1.1}$$

where X_x, Y_y, X_y, $u, v, \theta = e_{xx} + e_{yy}, e_{xx}, e_{yy}, e_{xy}$ are the functions of variables x, y, t. The points in the expressions $\dot{\theta}$, $\dot{e}_{xx}$, $\dot{e}_{yy}$, $\frac{\partial \dot{v}}{\partial x}$, $\frac{\partial \dot{u}}{\partial y}$ denote derivatives with respect to the time t; λ, μ are the elastic and λ^*, μ^* are viscoelastic constants.

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The components of stresses X_x, Y_y, X_y defined by the relations (1.1), when body forces are absent, just as in the classical case, must satisfy the equilibrium equations and the compatibility condition in the plane theory of elasticity [18].

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}=0, \qquad \frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}=0, \qquad \Delta\left(X_x+Y_y\right)=0.$$

Moreover, X_x, Y_y, X_y should be single-valued and continuous functions up to the boundary, together with their second derivatives with respect to the variables x, y. If these conditions are fulfilled, there exists a function $U\left(x,y,t\right)$ satisfying the biharmonic equation with respect to the variables x, y,

$$\Delta\Delta U=0, \tag{1.2}$$

through which the stresses are expressed as follows:

$$X_x=\frac{\partial^2 U}{\partial y^2}, \qquad Y_y=\frac{\partial^2 U}{\partial x^2}, \qquad X_y=-\frac{\partial^2 U}{\partial x\partial y}. \tag{1.3}$$

In view of formula (1.3), we write the first two equations of the relations (1.1) in the form

$$\begin{aligned} \lambda\theta+2\mu e_{xx}+\lambda^*\dot{\theta}+2\mu^*\dot{e}_{xx}&=\frac{\partial^2 U}{\partial y^2},\\ \lambda\theta+2\mu e_{yy}+\lambda^*\dot{\theta}+2\mu^*\dot{e}_{yy}&=\frac{\partial^2 U}{\partial x^2}. \end{aligned} \tag{1.4}$$

Summing up the above equations, we obtain the following equality

$$2\left(\lambda+\mu\right)\theta+2\left(\lambda^*+\mu^*\right)\dot{\theta}=\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\equiv\Delta U,$$

which we write as

$$\dot{\theta}+k\theta=\frac{\Delta U}{2\left(\lambda^*+\mu^*\right)}, \tag{1.5}$$

where $k=\dfrac{\lambda+\mu}{\lambda^*+\mu^*}$.

Assuming that at the moment of time t_0 (i.e., at the moment when the body is under the action of loading) $e_{xx}\left(x,y,t_0\right)=0$ and $e_{yy}\left(x,y,t_0\right)=0$, a solution of the linear first order differential equation (1.5) takes the form

$$\theta\left(x,y,t\right)=e^{-k(t-t_0)}\int\limits_{t_0}^{t}\frac{\Delta U\left(x,y,\tau\right)}{2\left(\lambda^*+\mu^*\right)}e^{k(\tau-t_0)}d\tau. \tag{1.6}$$

Equations (1.4) are given the form

$$\begin{aligned} \dot{e}_{xx}+me_{xx}&=\frac{1}{2\mu^*}\left[\frac{\partial^2 U}{\partial y^2}-\lambda\theta-\lambda^*\dot{\theta}\right]\equiv\Psi_1,\\ \dot{e}_{yy}+me_{yy}&=\frac{1}{2\mu^*}\left[\frac{\partial^2 U}{\partial x^2}-\lambda\theta-\lambda^*\dot{\theta}\right]\equiv\Psi_2, \end{aligned} \tag{1.7}$$

where $m=\dfrac{\mu}{\mu^*}$.

It follows from (1.7) that

$$\begin{aligned} \frac{\partial u}{\partial x}&=e^{-m(t-t_0)}\int\limits_{t_0}^{t}\Psi_1\left(x,y,\tau\right)e^{m(\tau-t_0)}d\tau,\\ \frac{\partial v}{\partial y}&=e^{-m(t-t_0)}\int\limits_{t_0}^{t}\Psi_2\left(x,y,\tau\right)e^{m(\tau-t_0)}d\tau. \end{aligned} \tag{1.8}$$

The functions Ψ_1, Ψ_2 introduced by the relations (1.7), in view of equation (1.5) and formula (1.6), after some transformations are reprsented as follows:

$$\Psi_1(x,y,t)=\frac{1}{2\mu^*}\left[-\frac{\partial^2 U}{\partial x^2}+\Delta U+\frac{n_2e^{-k(t-t_0)}}{4}\int\limits_{t_0}^{t}e^{k(\tau-t_0)}\Delta U d\tau-\frac{\lambda^*\Delta U}{2(\lambda^*+\mu^*)}\right],$$
$$\Psi_2(x,y,t)=\frac{1}{2\mu^*}\left[-\frac{\partial^2 U}{\partial y^2}+\Delta U+\frac{n_2e^{-k(t-t_0)}}{4}\int\limits_{t_0}^{t}e^{k(\tau-t_0)}\Delta U d\tau-\frac{\lambda^*\Delta U}{2(\lambda^*+\mu^*)}\right] \tag{1.9}$$

where $n_2=\dfrac{2(\mu\lambda^*-\lambda\mu^*)}{(\lambda^*+\mu^*)^2}$.

Following [18], we introduce the notation $\Delta U=P$, where P is a harmonic function of variables x, y, according to equation (1.2). Let Q be a conjugate to it function. Introduce the function $\varphi(z,t)$ in such a way that

$$\varphi(z,t)=p+iq=\frac{1}{4}\int(P+iQ)\,dz, \tag{1.10}$$

from which it follows that

$$P=4\frac{\partial p}{\partial x}=4\frac{\partial q}{\partial y}. \tag{1.11}$$

Substituting into equations (1.8) the values of the functions Ψ_1, Ψ_2 from (1.9) and taking into account equalities (1.11), we can represent the expressions $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial v}{\partial y}$ as

$$2\mu^*\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left\{-\int\limits_{t_0}^{t}e^{m(\tau-t_0)}\Big(\frac{\partial U}{\partial x}-n_1p\Big)d\tau+n_2\int\limits_{t_0}^{t}e^{n(\tau-t_0)}\Big(\int\limits_{t_0}^{\tau}e^{k(s-t_0)}pds\Big)d\tau\right\}e^{-m(t-t_0)}$$
$$2\mu^*\frac{\partial v}{\partial y}=\frac{\partial}{\partial y}\left\{-\int\limits_{t_0}^{t}e^{m(\tau-t_0)}\Big(\frac{\partial U}{\partial y}-n_1q\Big)d\tau+n_2\int\limits_{t_0}^{t}e^{n(\tau-t_0)}\Big(\int\limits_{t_0}^{\tau}e^{k(s-t_0)}qds\Big)d\tau\right\}e^{-m(t-t_0)},$$

where $n_1=\dfrac{2(\lambda^*+2\mu^*)}{(\lambda^*+\mu^*)}$, $n=m-k=\dfrac{\mu\lambda^*-\lambda\mu^*}{\mu^*(\lambda^*+\mu^*)}$.

Integrating the last expressions, we obtain

$$2\mu^*u=\left\{-\int\limits_{t_0}^{t}e^{m(\tau-t_0)}\Big(\frac{\partial U}{\partial x}-n_1p\Big)d\tau\right.$$
$$\left.+n_2\int\limits_{t_0}^{t}e^{n(\tau-t_0)}\Big(\int\limits_{t_0}^{\tau}e^{k(s-t_0)}pds\Big)d\tau\right\}e^{-m(t-t_0)}+f_1(y,t), \tag{1.12}$$

$$2\mu^*v=\left\{-\int\limits_{t_0}^{t}e^{m(\tau-t_0)}\Big(\frac{\partial U}{\partial y}-n_1q\Big)d\tau\right.$$
$$\left.+n_2\int\limits_{t_0}^{t}e^{n(\tau-t_0)}\Big(\int\limits_{t_0}^{\tau}e^{k(s-t_0)}qds\Big)d\tau\right\}e^{-m(t-t_0)}+f_2(x,t). \tag{1.13}$$

Omitting the functions $f_1(y,t)$, $f_2(x,t)$ due to the fact that they provide only rigid displacement at any fixed moment of time t and taking into account (1.10), it follows from equalities (1.12) and (1.13) that

$$2\mu^*(u+iv)=\left\{-\int\limits_{t_0}^{t}e^{m(\tau-t_0)}\Big(\frac{\partial U}{\partial x}+i\frac{\partial U}{\partial y}-n_1\varphi(z,\tau)\Big)d\tau\right.$$

$$+n_2\int\limits_{t_0}^{t} e^{n(\tau-t_0)}\left(\int\limits_{t_0}^{\tau} e^{k(s-t_0)}\varphi(z,s)ds\right)d\tau\Big\}e^{-m(t-t_0)}.$$

Taking into account that [18]

$$\frac{\partial U}{\partial x}+i\frac{\partial U}{\partial y}=\varphi(z,t)+z\overline{\varphi'(z,t)}+\overline{\psi(z,t)}, \tag{1.14}$$

from (1.14), after some transformations, we obtain

$$2\mu^*(u+iv)=\int\limits_{t_0}^{t} e^{m(\tau-t)}\left(\varphi(z,\tau)-z\overline{\varphi'(z,\tau)}-\overline{\psi(z,\tau)}\right)d\tau+\omega\int\limits_{t_0}^{t} e^{k(\tau-t)}\varphi(z,\tau)d\tau, \tag{1.15}$$

where $\omega=\dfrac{2\mu_*}{\lambda^*+\mu^*}$ (here "prime" means the derivative with respect to a variable z).

Thus formulas (1.14) and (1.15) are generalize Kolosov-Muskhelishvili's formulas for a viscoelastic material.

2. Statement of the Problem and Reduction to the Integral Equation

Consider now the contact problem of interaction of a semi-infinite stringer of constant rigidity (constant cross-section) and an infinite viscoelastic plate occupying the lower half-plane, when the stringer is under the action of tangential stresses $T_0(x)H(t-t_0)$ ($H(t-t_0)$ is the Heaviside function) and free from normal stresses. We have to find tangential contact stresses $T(x,t)$ along the contact line.

From equation (1.15), we get

$$2\mu^*\frac{\partial u(x,y,t)}{\partial x}=\operatorname{Re}\left\{\int\limits_{t_0}^{t} e^{m(\tau-t)}\left[\Phi(z,\tau)-\overline{\Phi(z,\tau)}-z\overline{\Phi'(z,\tau)}-\overline{\Psi(z,\tau)}\right]d\tau\right\}$$

$$+\omega\operatorname{Re}\int\limits_{t_0}^{t} e^{k(\tau-t)}\Phi(z,\tau)d\tau. \tag{2.1}$$

Since normal stresses in our case are absent, the complex potentials Φ, Ψ will have the form [18]:

$$\Phi(z,t)=\frac{1}{2\pi}\int\limits_{0}^{\infty}\frac{T(\sigma,t)}{\sigma-z}d\sigma, \tag{2.2}$$

$$\Psi(z,t)=-\frac{1}{2\pi}\int\limits_{0}^{\infty}\frac{T(\sigma,t)}{\sigma-z}d\sigma-\frac{1}{2\pi}\int\limits_{0}^{\infty}\frac{T(\sigma,t)}{(\sigma-z)^2}\sigma d\sigma. \tag{2.3}$$

If we substitute into equation (2.1) the values of the functions Φ,Ψ from formulas (2.2) and (2.3) and pass to the real values, then after some transformations we get

$$2\mu^*\frac{\partial u(x,0,t)}{\partial x}=\int\limits_{t_0}^{t} e^{m(\tau-t)}\Phi(x,\tau)d\tau+\omega\int\limits_{t_0}^{t} e^{k(\tau-t)}\Phi(x,\tau)d\tau. \tag{2.4}$$

Introducing a time operator

$$L\Phi(x,t)=\int\limits_{t_0}^{t} e^{m\tau}\Phi(x,\tau)d\tau+\omega\int\limits_{t_0}^{t} e^{nt+k\tau}\Phi(x,\tau)d\tau, \tag{2.5}$$

(2.4) will have the form

$$2\mu^*\frac{\partial u(x,0,t)}{\partial x}=e^{-mt}L\Phi(x,t). \tag{2.6}$$

On the other hand, for deformations of the stringer points we have

$$\frac{\partial u_0(x,t)}{\partial x} = \frac{1}{E}\int_0^x [T(y,t) - T_0(y)H(t-t_0)]dy, \tag{2.7}$$

and the condition for the principal vector to be equal to zero yields

$$\int_0^\infty [T(y,t) - T_0(y)H(t-t_0)]dy = 0.$$

From the condition of the stringer and plate contact

$$\frac{\partial u(x,0,t)}{\partial x} = \frac{\partial u_0(x,t)}{\partial x},$$

in view of the relations (2.6), (2.7) and (2.2), we obtain

$$e^{-mt}L\biggl(\int_0^\infty \frac{T(\sigma,t)d\sigma}{\sigma - x}\biggr) = \frac{4\pi\mu^*}{E}\int_0^x [T(y,t) - T_0(y)H(t-t_0)]\,dy. \tag{2.8}$$

Introducing the notation

$$\int_0^x [T(y,t) - T_0(y)H(t-t_0)]\,dy \equiv K(x,t), \qquad K(0,t) = K(\infty,t) = 0,$$

we write the relation (2.8) in the form

$$L\int_0^\infty \frac{K'(\sigma,t)d\sigma}{\sigma - x} = \alpha e^{mt}K(x,t) - F(x)B(t), \tag{2.9}$$

where

$$B(t) = LH(t-t_0) = \frac{1}{m}(e^{mt} - e^{mt_0}) + \frac{\omega}{k}e^{nt}(e^{kt} - e^{kt_0}), \quad \alpha = \frac{4\pi\mu^*}{E}, \quad F(x) = \int_0^\infty \frac{T_0(\sigma)d\sigma}{\sigma - x}.$$

After transformation of variables

$$\sigma = e^{\zeta}, \qquad x = e^{\xi},$$

the integral differential equation (2.9) takes the form

$$-L\int_{-\infty}^\infty \frac{K'{}_0(\zeta,t)d\zeta}{1 - e^{-(\xi-\zeta)}} = \alpha e^{\xi}e^{mt}K_0(\xi,t) - B(t)e^{\xi}F_0(\xi), \quad |\xi| < \infty \tag{2.10}$$

$$K_0(-\infty,t) = K_0(\infty,t) = 0,$$

where

$$K_0(\zeta,t) = K(e^{\zeta},t), F_0(\xi) = F(e^{\xi}).$$

Performing the generalized Fourier transformation [10] of both parts of equation (2.10), we obtain

$$\pi\, s\, cth\pi s\, L\hat{K}_0(s,t) = -\alpha e^{mt}\hat{K}_0(s-i,t) + B(t)\hat{F}(s), \tag{2.11}$$

where

$$\hat{F}(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F_0(\xi)e^{\xi}e^{i\xi s}d\xi, \quad \hat{K}_0(s,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty K_0(\xi,t)e^{\xi}e^{i\xi s}d\xi.$$

The problem can be formulated as follows: Find the function $\hat{K}_0(z,t)$, analytic in the strip $-1 < \operatorname{Im} z < 1$, (with the exception of a finite number of points lying in the strip $0 < \operatorname{Im} z < 1$ in which it may have poles), vanishing at infinity and satisfying condition (2.11). obviously, if we will be able to find the function $\hat{K}_0^-(z,t)$, holomorphic in the strip $-1 < \operatorname{Im} z < 0$, vanishing at infinity, continuously extendable on the strip boundary, by the boundary condition (2.11), then the solution of the above-formulated problem is

$$\hat{K}_0(z,t)=\begin{cases} L\hat{K}_0^-(z,t), & -1<\operatorname{Im} z<0 \\ \dfrac{-\alpha e^{mt}\hat{K}_0^-(z-i,t)+B(t)\hat{F}(t)}{\pi z cth\pi z}, & 0<\operatorname{Im} z<1 \end{cases} \tag{2.11.1}$$

To factorize the coefficient of problem (2.11), we represent the function $M(s)=s\,cth\pi s$ as follows:

$$M(s)=iscth\pi s\cdot th\frac{\pi}{2}s\cdot\frac{sh\frac{\pi}{2}(s-i)}{sh\frac{\pi}{2}s}.$$

Owing to the fact that the index of the function $M_0(s)=\ cth\pi s\, th\frac{\pi}{2}s$ equals zero and $\ln\left[cth\pi s\, th\frac{\pi}{2}s\right]$ is integrable on the real axis, we can represent the function $M_0(s)$ in the form

$$M_0(s)=\frac{\mathrm{X}_0(s-i)}{\mathrm{X}_0(s)},$$

where $\mathrm{X}_0(z)=\exp\left(-\frac{1}{2i}\int\limits_{-\infty}^{\infty}\ln\left[cth\pi s\, th\frac{\pi}{2}s\right]cth\pi(t-z)dz\right)$.

Then introducing the notation

$$\tilde{K}(s,t)=\frac{is\hat{K}_0(s,t)}{sh\frac{\pi}{2}s\cdot\mathrm{X}_0(s)},$$

the relation (2.11) can rewritten as

$$\frac{1+is}{\alpha_0}L\,\tilde{K}(s,t)+\tilde{K}(s-i,t)e^{mt}=G(s)B(t), \tag{2.12}$$

where

$$G(s)=\frac{1}{\pi}\frac{(1+is)\hat{F}_0(s)}{\alpha_0 sh\frac{\pi}{2}(s-i)\cdot\mathrm{X}_0(s-i)}.$$

Using the known representation [16]

$$\frac{1+is}{\alpha_0}=\frac{\mathrm{X}_1(s-i)}{\mathrm{X}_1(s)},\quad \mathrm{X}_1(z)=\exp(-iz\ln\alpha_0)\Gamma(1+iz)$$

the condition (2.12) takes the form

$$L\,A(s,t)+A(s-i,t)e^{mt}=G_1(s)B(t), \tag{2.13}$$

where

$$A(s,t)=\frac{\tilde{K}(s,t)}{\mathrm{X}_1(s)}\qquad G_1(s)=\frac{G(s)}{\mathrm{X}_1(s-i)}.$$

Performing the generalized Fourier transformation of both sides of equation (2.13), we obtain the Volterra second kind integral equation

$$L\hat{A}(u,t)+e^{-u}\hat{A}(u,t)e^{mt}=\hat{G}_1(u)B(t). \tag{2.14}$$

Taking into account the form of the operator L, from the last equation, we have

$$\hat{A}(u,t_0)=0. \tag{2.15}$$

Having differentiated the relation (2.14) by using the notation (2.5), and get

$$(\omega+1)e^{mt}\hat{A}(u,t)+\omega n\int\limits_{t_0}^{t}e^{nt+k\tau}\hat{A}(u,\tau)d\tau+e^{-u}\dot{\hat{A}}(u,t)e^{mt}$$
$$+me^{-u}\hat{A}(u,t)e^{mt}=\hat{G}_1(u)\dot{B}(t). \tag{2.16}$$

At the point $t=t_0$ the previous expression yields

$$\dot{\hat{A}}(u,t_0)=(1+\omega)\hat{G}_1(u)e^{u}. \tag{2.17}$$

Multiplying both parts of equation (2.16) by e^{-mt} and differentiating with respect to the variable t, after some transformations, we obtain

$$\ddot{\hat{A}}(u,t) + a(u)\dot{\hat{A}}(u,t) + b(u)\hat{A}(u,t) = \gamma \hat{G}_1(u)e^u, \tag{2.18}$$

where

$$a(u) = (\omega+1)e^u + m + k, \qquad b(u) = \gamma e^u + mk, \qquad \gamma = \frac{\lambda + 3\mu}{\lambda^* + \mu^*}.$$

The discriminant of the corresponding characteristic equation will have the form

$$D = [(\omega+1)e^u + n]^2 - 4n\omega e^u.$$

It is not difficult to show that the above discriminant is always positive, and a generalized solution of the inhomogeneous differential equation (2.18) has the form

$$\hat{A}(u,t) = \bar{\hat{A}}(u,t) + \tilde{\hat{A}}(u,t), \tag{2.19}$$

where

$$\bar{\hat{A}}(u,t) = c_1(u)e^{-p_1(u)t} + c_2(u)e^{-p_2(u)t}, \qquad p_1(u), p_2(u) > 0$$

is a general solution of the homogeneous equation corresponding to equation (2.18), and $\tilde{\hat{A}}(u,t) = \dfrac{\gamma \hat{G}_1(u)e^u}{\gamma e^u + mk}$ is a particular solution of that equation.

From (2.19), using initial conditions (2.15) and (2.17) and defining coefficients $c_1(u)$ and $c_2(u)$, we obtain the solution of the differential equation (2.18) in the form

$$\hat{A}(u,t) = \hat{G}_1(u)G_2(u,t) + \frac{\hat{G}_1(u)}{1 + \delta e^{-u}},$$

where

$$G_2(u,t) = \left[\frac{(\gamma p_2(u) - (1+\omega)b(u))e^{-p_1(u)(t-t_0)} + ((1+\omega)b(u) - \gamma p_1(u))e^{-p_2(u)(t-t_0)}}{(p_1(u) - p_2(u))b(u)}\right]e^u, \tag{2.20}$$

$$\delta = \frac{mk}{\gamma} = \frac{\mu(\lambda+\mu)}{\gamma\mu^*(\lambda^*+\mu^*)}.$$

Using Parceval's generalized formula, the inverse Fourier transformation [10] yields

$$A(s,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G_1(y)\hat{G}_2(y-s,t)dy + \frac{i}{2}\int_{-\infty}^{\infty} \frac{G_1(y)e^{i(y-s)\ln\delta}dy}{sh\pi(y-s)},$$

where $A(s,t)$ is the boundary value on the real axis of the function $A(z,t)$, ($z = s + iy$) holomorphic on the strip $-1 < \operatorname{Im} z < 1$, with the exception of the point $z = \dfrac{i}{2}$, at which it has the first order pole continuously extendable in the strip boundary and vanishing at infinity (see formula (2.11.1)).

Respectively,

$$\hat{K}_0(z,t) = A(z,t)\mathrm{X}(z),$$

where $\mathrm{X}(z) = \dfrac{\mathrm{X}_0(z)\mathrm{X}_1(z)}{iz} sh\dfrac{\pi}{2}z.$

Performing again the inverse Fourier generalized transformation and getting back to our variables, we obtain

$$K'(x,t) = T(x,t) - T_0(x)H(t-t_0) = \frac{x^{-1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(s,t)\mathrm{X}(s)e^{-is\ln x}ds.$$

Using Cauchy's formula and residue theorem for the holomorphic in the strip $-1 < \operatorname{Im} z < 1$ function $A(z,t)$, we obtain the following asymptotic estimates for the unknown tangential contact stress

$$K'(x,t) = O(x^{-1/2}), \qquad x \to 0-$$

$$K'(x,t) = O(x^{-2}), \qquad x \to \infty.$$

As for the behaviour of the tangential contact stress and other mechanical values concerning time $t \geq t_0$, it is clearly seen from the expression (2.20).

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