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CHARACTERIZATION OF SETS OF SINGULAR ROTATIONS FOR A CLASS OF DIFFERENTIATION BASES

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Abstract. We study the dependence of differential properties of an indefinite integral on rotations of the coordinate system. Namely, the following problem is studied: For a summable function f of what kind can be the set of rotations γ for which $\int f$ is not differentiable with respect to the γ-rotation of a given basis B? The result obtained in the paper implies a solution of the problem for any homothecy invariant differentiation basis B of two-dimensional intervals which has symmetric structure.

1. Definitions and Notation

A collection B of open bounded and non-empty subsets of $\mathbb{R}^n$ is called a *differentiation basis* (briefly: *basis*) if for every $x \in \mathbb{R}^n$ there exists a sequence (R_k) of sets from B such that $x \in R_k$ $(k \in \mathbb{N})$ and $\lim\limits_{k\to\infty} \operatorname{diam} R_k = 0$.

For a basis B by $B(x)$ $(x \in \mathbb{R}^n)$ it will be denoted the collection of all sets from B containing the point x.

Let B be a basis. For $f \in L(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the upper and lower limits of the integral means $\frac{1}{|R|}\int_R f$, where R is an arbitrary set from $B(x)$ and $\operatorname{diam} R \to 0$, are called the *upper and the lower derivatives with respect to B of the integral of f at the point x*, and are denoted by $\overline{D}_B(\int f, x)$ and $\underline{D}_B(\int f, x)$, respectively. If the upper and the lower derivatives coincide, then their common value is called the *derivative of $\int f$ at the point x* and denoted by $D_B(\int f, x)$. We say that B *differentiates* $\int f$ (or $\int f$ *is differentiable with respect to* B) if $\overline{D}_B(\int f, x) = \underline{D}_B(\int f, x) = f(x)$ for almost all $x \in \mathbb{R}^n$. If this is true for each f in the class of functions $F \subset L(\mathbb{R}^n)$ we say that B *differentiates* F. By F_B denote the class of all functions $f \in L(\mathbb{R}^n)$ the integrals of which are differentiable with respect to B. The *maximal operator M_B corresponding to B* is defined as follows: $M_B(f)(x) = \sup_{R\in B(x)} \frac{1}{|R|}\int_R |f|$, where $f \in L(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

A basis B is called *translation invariant* (*homothecy invariant*) if for any set R from B and any translation (homothecy) $M : \mathbb{R}^n \to \mathbb{R}^n$ the set $M(R)$ also belongs to B. It is easy to check that each homothecy invariant basis is translation invariant also. Let us call a basis B convex if each set $R \in B$ is convex.

Denote by $\mathbf{I} = \mathbf{I}(\mathbb{R}^n)$ the basis consisting of all n-dimensional intervals. Differentiation with respect to $\mathbf{I}$ is called *strong differentiation*.

Let us call a basis B *non-standard* if there exists a function $f \in L(\mathbb{R}^n)$ the integral of which is not differentiable with respect to B (i.e. if B does not differentiate $L(\mathbb{R}^n)$).

The basis $\mathbf{I}$ is non-standard (see, e.g., [3, Ch. IV, §1]). Note that (see, [3, Appendix III]) a homothecy invariant basis B of multi-dimensional intervals is non-standard if and only if $\sup\{I \in B : l^I/l_I\} = \infty$, where l^I and l_I are the lengthes of the biggest and of the smallest edges of an interval I, respectively. Moreover, a clear geometrical criterion for the non-standartness it is known also for translation invariant bases of multi-dimensional intervals (see [14, 16]).

By $\Gamma(\mathbb{R}^n)$ denote the collection of all rotations in $\mathbb{R}^n$.

Let B be a basis in $\mathbb{R}^n$ and $\gamma \in \Gamma(\mathbb{R}^n)$. The *$\gamma$-rotated basis B* is defined as follows: $B(\gamma) = \{\gamma(R) : R \in B\}$.

Denote by ρ_k $(k = 0, 1, 2, 3)$ the rotation of the plane by the angle $\pi k/2$.

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Let us call a set $E \subset \Gamma(\mathbb{R}^2)$ *symmetric* if for any $\gamma \in E$ the rotations $\rho_1 \circ \gamma, \rho_2 \circ \gamma$ and $\rho_3 \circ \gamma$ also belong to the set E.

Let us call a translation invariant basis B of two-dimensional intervals *symmetric* if the bases $B(\rho_1), B(\rho_2)$ and $B(\rho_3)$ are equal to B. Obviously, the basis $\mathbf{I}(\mathbb{R}^2)$ is symmetric.

The set of two-dimensional rotations $\Gamma(\mathbb{R}^2)$ can be identified with the circumference $\mathbb{T} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$, if to a rotation γ we put into correspondence the point $\gamma((1,0))$. The distance $d(\gamma, \sigma)$ between rotations $\gamma, \sigma \in \Gamma(\mathbb{R}^2)$ is assumed to be equal to the length of the smallest arch of the circumference $\mathbb{T}$ connecting the points $\gamma((1,0))$ and $\sigma((1,0))$.

A class of functions F is called *invariant with respect to a class of transformations of a variable* Λ if $(f \in F, \lambda \in \Lambda) \Rightarrow f \circ \lambda \in F$.

2. Introduction

The dependence of properties of functions of several variables on rotations of the system of coordinates (that is, on a transformation of the variables that is a rotation) has been studied by various authors.

Zygmund posed the following problem (see, [3, Ch. IV, §2]): Is it possible to improve an arbitrary function $f \in L(\mathbb{R}^2)$ by means of a rotation of the coordinate system to achieve strong differentiability of the integral of f? In [7] Marstrand gave a negative answer to this problem by constructing a non-negative function $f \in L(\mathbb{R}^2)$ such that $\overline{D}_{\mathbf{I}}\left(\int f \circ \gamma, x\right) = \infty$ a.e. for every $\gamma \in \Gamma(\mathbb{R}^2)$. In the works [6, 10, 13] and [11] the result of Marstrand was extended to bases of quite general type.

As established by Lepsveridze [5], Oniani [8] and Stokolos [15], the property of strong differentiability (that is, the class $F_{\mathbf{I}}$) is not invariant with respect to linear changes of variables and, in particular, to rotations. A similar result was proved by Dragoshanskii [2] for the class of continuous functions of two variables whose Fourier series (Fourier integral) is Pringsheim convergent almost everywhere.

In [11] non-invariance of a class F_B with respect to rotations was proved for any non-standard translation invariant basis B of multi-dimensional intervals.

Suppose B is a translation invariant basis. Then it is easy to verify that the differentiation of the integral of a "rotated" function $f \circ \gamma$ with respect to B at a point x is equivalent to the differentiation of the integral of f with respect to the "rotated" basis $B(\gamma^{-1})$ at the point $\gamma^{-1}(x)$. Consequently, we can reduce the study of the behavior of functions $f \circ \gamma$ $(\gamma \in \Gamma(\mathbb{R}^n))$ with respect to the basis B to the study of the behavior of f with respect to the rotated bases $B(\gamma)$ $(\gamma \in \Gamma(\mathbb{R}^n))$. Below we will use this approach.

If for a translation invariant basis B the class F_B is not invariant with respect to the rotations then there exists a function $f \in L(\mathbb{R}^n)$ having non-homogeneous behaviour with respect to rotated bases $B(\gamma)$ $(\gamma \in \Gamma(\mathbb{R}^n))$, more exactly, $\int f$ is not differentiable with respect to $B(\gamma)$ for some rotations and $\int f$ is differentiable with respect $B(\gamma)$ for some other rotations. Thus, for f some rotations γ are "singular" and some other rotations γ are "regular". In this connection naturally arises the problem: *Of what kind can be the sets of singular and of regular rotations for a fixed function*? Note that by duality argument we can restrict ourselves by studying sets of singular rotations.

In connection to the posed problem let us formulate rigor definition of a set of singular rotations: Suppose B is a translation invariant basis in $\mathbb{R}^n$ and $E \subset \Gamma(\mathbb{R}^n)$. Let us call E a W_B*-set* if there exists a function $f \in L(\mathbb{R}^n)$ with the following two properties: 1) $f \notin F_{B(\gamma)}$ for every $\gamma \in E$; 2) $f \in F_{B(\gamma)}$ for every $\gamma \notin E$.

Let us formulate also the definition of a set of "strongly" singular rotations: Suppose B is a translation invariant basis in $\mathbb{R}^n$ and $E \subset \Gamma(\mathbb{R}^n)$. Let us call E an R_B*-set* if there exists a function $f \in L(\mathbb{R}^n)$ with the following two properties: 1) $\overline{D}_{B(\gamma)}\left(\int f, x\right) = \infty$ a.e. for every $\gamma \in E$; 2) $f \in F_{B(\gamma)}$ for every $\gamma \notin E$.

Now the problem can be formulated as follows: *For a given translation invariant basis B what kind of sets are W_B-sets(R_B-sets)*?

Note that for a standard basis B, i.e. for a basis B differentiating $L(\mathbb{R}^n)$, the problem is trivial. Here note also that if a translation invariant basis B of two-dimensional intervals is symmetric then every W_B-set and every R_B-set is symmetric.

In [1] for an arbitrary translation invariant basis B in $\mathbb{R}^2$ it was established the following three structural properties of sets of singular rotations: 1) Each W_B-set is of type $G_{\delta\sigma}$; 2) Each R_B-set is of type G_δ; 3) At most countable union of R_B-sets is a W_B-set.

Sets of singular rotations for the case of strong differentiability process on the plane (i.e., for the case $B = \mathbf{I}(\mathbb{R}^2)$) was characterized by G. Karagulyan [4] proving that: 1) a set $E \subset \Gamma(\mathbb{R}^2)$ is a $W_{\mathbf{I}(\mathbb{R}^2)}$-set if and only if E is symmetric and of type $G_{\delta\sigma}$; 2) a set $E \subset \Gamma(\mathbb{R}^2)$ is an $R_{\mathbf{I}(\mathbb{R}^2)}$-set if and only if E is symmetric and of type G_δ.

Our purpose is to show that the idea in Karagulyan's construction works for bases of two-dimensional intervals of quite general type.

3. Result

For a translation invariant convex basis B let us define the following function $\sigma_B(\lambda) = \overline{\lim\limits_{\varepsilon\to 0}} |\{M_B(\chi_{V_\varepsilon}) > \lambda\}|/|V_\varepsilon|$ $(0 < \lambda < 1)$, where V_ε is the ball with the centre at the origin and with the radius ε. Here and below everywhere χ_E denotes the characteristic function of a set E. We call σ_B a *spherical halo function of* B. It is easy to check that if B is homothecy invariant, then $\sigma_B(\lambda) = |\{M_B(\chi_V) > \lambda\}|$, where V is the unit ball.

We say that a translation invariant convex basis B has the *non-regular spherical halo function* if $\overline{\lim\limits_{\lambda\to 0}} \lambda\sigma_B(\lambda) = \infty$.

Theorem 1. *Let B be a non-standard translation invariant basis of two-dimensional intervals which is symmetric and has the non-regular spherical halo function. Then*:

- *a set $E \subset \Gamma(\mathbb{R}^2)$ is a W_B-set if and only if E is symmetric and of type $G_{\delta\sigma}$;*
- *a set $E \subset \Gamma(\mathbb{R}^2)$ is an R_B-set if and only if E is symmetric and of type G_δ.*

In [11] (see Lemma 2.4) it was shown that every non-standard homothecy invariant convex basis B has the non-regular spherical halo function. Taking into account this fact, we obtain from Theorem 1 the following corollary.

Corollary 1. *Let B be a non-standard homothecy invariant basis of two-dimensional intervals which is symmetric. Then for W_B-sets and R_B-sets characterizations analogous to the ones given in Theorem* 1 *are true.*

4. Auxiliary Propositions

By $\mathfrak{B}_{\mathrm{TI}}$ and $\mathfrak{B}_{\mathrm{HI}}$ we will denote the classes of all translation invariant and homothecy invariant bases in $\mathbb{R}^2$, respectively. By $\mathfrak{B}_{\mathrm{I}}$ it will be denoted the class of all bases consisting of two-dimensional intervals. The lower left vertex of an interval $I \subset \mathbb{R}^2$ denote by $a(I)$. For a set $A \subset \mathbb{R}^n$ with the centre of symmetry at a point x and for a number $\alpha > 0$ we denote by αA the dilation of A with the coefficient α, i.e. the set $\alpha A = \{x + \alpha(y - x) : y \in A\}$.

Let $B \in \mathfrak{B}_{\mathrm{I}}$. For a square interval Q and $\lambda \in (0,1)$ by $\Omega_B(Q,\lambda)$ denote the collection of all intervals $I \in B$ with the properties: $a(I) = a(Q), I \supset Q$ and $|Q|/|I| > \lambda$. The set $E_B(Q,\lambda)$ will be defined as the union of all intervals from the collection $\Omega_B(Q,\lambda)$. Obviously, $\frac{1}{|I|}\int_I \chi_Q > \lambda$ for each $I \in \Omega_B(Q,\lambda)$ and $E_B(Q,\lambda) \subset \{M_B(\chi_Q) > \lambda\}$.

Lemma 1. *Let $B \in \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{I}}$, Q be a square interval and $0 < \lambda < 1$. Then $|E_B(Q,\lambda)| \geq c(|\{M_B(\chi_Q) > \lambda\}| - 18|Q|/\lambda)$, where c is a positive absolute constant.*

Proof. Without loss of generality let us assume that Q is a square interval of the type $(-\varepsilon,\varepsilon)^2$. Let Θ be the collection of all intervals $I \in B$ such that $\frac{1}{|I|}\int_I \chi_Q > \lambda$. Obviously, $\{M_B(\chi_Q) > \lambda\} = \bigcup_{I\in\Theta} I$.

Denote by Θ_0 the collection of all intervals $I \in \Theta$ having at least one side with the length smaller than 2ε. It is easy to check that every $I \in \Theta_0$ is contained in the union of the intervals $(-3\varepsilon, 3\varepsilon) \times (-\varepsilon - 2\varepsilon/\lambda, \varepsilon + 2\varepsilon/\lambda)$ and $(-\varepsilon - 2\varepsilon/\lambda, \varepsilon + 2\varepsilon/\lambda) \times (-3\varepsilon, 3\varepsilon)$. Consequently, $|\bigcup_{I\in\Theta_0} I| < 18|Q|/\lambda$.

Let $\mathbb{R}^2_k$ $(k \in \overline{1,4})$ be the k-th coordinate quarter. Denote by Θ_k $(k \in \overline{1,4})$ the collection of all intervals $I \in \Theta \setminus \Theta_0$ for which $|I \cap \mathbb{R}^2_k| = \max\{|I \cap \mathbb{R}^2_m| : m \in \overline{1,4}\}$. Obviously, $\Theta = \bigcup_{k=0}^4 \Theta_k$. The unions $\bigcup_{I\in\Theta_k} I$ and $\bigcup_{I\in\Theta_m} I$ are symmetric with respect to Ox_2 if $k = 1, m = 2$ or $k = 3, m = 4$ and are symmetric with respect to Ox_1 if $k=2, m=3$ or $k=4, m=1$. Hence, the sets $\bigcup_{I\in\Theta_k} I$ $(k\in\overline{1,4})$ have one and the same measure. Consequently,

$$\Big|\bigcup_{I\in\Theta_1} I\Big| \geq \frac{1}{4}\Big(\Big|\bigcup_{I\in\Theta} I\Big| - \Big|\bigcup_{I\in\Theta_0} I\Big|\Big) \geq \frac{1}{4}\bigg(|\{M_B(\chi_Q)>\lambda\}| - \frac{18|Q|}{\lambda}\bigg). \tag{1}$$

For arbitrary $I\in\Theta_1$ let us consider the translation T for which $T(I)\in\Omega_B(Q,\lambda)$. It is clear that $I\subset 2T(I)$. Consequently, $\bigcup_{I\in\Theta_1} I\subset\bigcup_{I\in\Omega_B(Q,\lambda)} 2I$. Therefore, by (1): $|\bigcup_{I\in\Omega_{B(Q,\lambda)}} 2I| \geq \frac{1}{4}(|\{M_B(\chi_Q)>\lambda\}| - 18|Q|/\lambda)$. On the other hand, by virtue of the inclusion $\bigcup_{I\in\Omega_B(Q,\lambda)} 2I\subset\{M_{\mathbf{I}(\mathbb{R}^2)}(\chi_A)\geq 1/4\}$, where $A=\bigcup_{I\in\Omega_B(Q,\lambda)} I$, and the strong maximal inequality (see, e.g., [3, Ch. II, §3]), we have: $|\bigcup_{I\in\Omega_B(Q,\lambda)} 2I| \leq C|\bigcup_{I\in\Omega_B(Q,\lambda)} I|$, where C is a positive absolute constant. From the last two estimations it follows the validity of the lemma. □

Lemma 2. *Let $B\in\mathfrak{B}_{\mathrm{TI}}\cap\mathfrak{B}_{\mathrm{I}}$ and $0<\lambda<1$. If $\sigma_B(\lambda)>144/\lambda$, then for every $\varepsilon>0$ there is a square interval Q such that* $\operatorname{diam} Q<\varepsilon$ *and $|E_B(Q,\lambda)|\geq c\sigma_B(\lambda)|Q|/8$, where c is the constant from Lemma* 1.

Proof. Taking into account the definition of the spherical halo function σ_B, we can find a ball $V_\delta=\{x\in\mathbb{R}^2:\operatorname{dist}(x,O)<\delta\}$ such that $\delta<\varepsilon/4$ and $|\{M_B(\chi_{V_\delta})>\lambda\}|/|V_\delta|>\sigma_B(\lambda)/2$. Let us consider the square interval Q superscribed around V_δ, i.e. $Q=(-\delta,\delta)^2$. Then $\operatorname{diam} Q<\varepsilon$ and $|\{M_B(\chi_Q)>\lambda\}|\geq|\{M_B(\chi_{V_\delta})>\lambda\}|>\sigma_B(\lambda)|V_\delta|/2>\sigma_B(\lambda)|Q|/4$. Now, taking into account the estimation $\sigma_B(\lambda)>144/\lambda$, by virtue of Lemma 1, we write: $|E_B(Q,\lambda)|\geq c(\sigma_B(\lambda)|Q|/4-18|Q|/\lambda)\geq c\sigma_B(\lambda)|Q|/8$. This proves the lemma. □

Suppose, $S=(0,\varepsilon)\times(0,\varepsilon)$, $0<\alpha\leq\pi/4$ and $n\in\mathbb{N}$. For each $k\in\overline{1,n}$ let us define the points $P_k^+(S,\alpha)$, $P_k^-(S,\alpha)$ and the balls $V_k^+(S,\alpha,n)$, $V_k^-(S,\alpha,n)$ as follows:

$$P_k^+(S,\alpha)=\Big(\frac{\varepsilon}{2^k},\frac{\varepsilon}{2^k}\tan(\alpha)\Big),\quad P_k^-(S,\alpha)=\Big(\frac{\varepsilon}{2^k},-\frac{\varepsilon}{2^k}\tan(\alpha)\Big),$$
$$V_k^+(S,\alpha,n)=\Big\{x\in\mathbb{R}^2:\operatorname{dist}(x,P_k^+(S,\alpha))<\frac{\varepsilon}{4^n}\tan(\alpha)\Big\},$$
$$V_k^-(S,\alpha,n)=\Big\{x\in\mathbb{R}^2:\operatorname{dist}(x,P_k^-(S,\alpha))<\frac{\varepsilon}{4^n}\tan(\alpha)\Big\}.$$

Suppose, Q and S are square intervals with $Q\supset S$ and $a(Q)=a(S)=(0,0)$, $h>1, 0<\alpha\leq\pi/4$ and $n\in\mathbb{N}$. Let $\xi=\xi_{Q,h,S,\alpha,n}$ be the function which is proportional to the function $\sum_{k=1}^n\chi_{V_k^+(S,\alpha,n)}-\sum_{k=1}^n\chi_{V_k^-(S,\alpha,n)}$, and satisfies the following conditions: $\{\xi>0\}=\bigcup_{k=1}^n V_k^+(S,\alpha,n)$, $\{\xi<0\}=\bigcup_{k=1}^n V_k^-(S,\alpha,n)$ and $\|\xi\|_L=2\|h\chi_Q\|_L$. The function $\xi_{Q,h,S,\alpha,n}$ we will call (S,α,n)-*oscillator corresponding to the function* $h\chi_Q$. It is easy to see that:

1) the balls $V_k^+(S,\alpha,n)$ are disjoint and contained in the square S;
2) the balls $V_k^-(S,\alpha,n)$ are disjoint and contained in the square $S^-=(0,\varepsilon)\times(-\varepsilon,0)$;
3) $\int_{V_k^+(S,\alpha,n)}\xi=-\int_{V_k^-(S,\alpha,n)}\xi=h|Q|/n$ for each $k\in\overline{1,n}$.

For $\gamma\in\Gamma(\mathbb{R}^2)$ and $\varepsilon>0$ denote $V[\gamma,\varepsilon]=\{\rho\in\Gamma(\mathbb{R}^2):\operatorname{dist}(\rho,\gamma)\leq\varepsilon\}$.

For a basis B by $\overline{M}_B$ denote the following type maximal operator: $\overline{M}_B(f)(x)=\sup_{R\in B(x)}\frac{1}{|R|}\int_R f$ $(f\in L(\mathbb{R}^n)$, $x\in\mathbb{R}^n)$.

Lemma 3. *Let $B\in\mathfrak{B}_{\mathrm{TI}}\cap\mathfrak{B}_{\mathrm{I}}$. Suppose Q and S are square intervals with $Q\supset S$ and $a(Q)=a(S)=(0,0)$, $h>1, 0<\alpha\leq\pi/4$ and $n\in\mathbb{N}$. Then for the oscillator $\xi=\xi_{Q,h,S,\alpha,n}$ it is valid the following estimation*: $\frac{1}{|\gamma(I)|}\int_{\gamma(I)}\xi>1$ *for every $I\in\Omega_B(Q,1/h)$ and $\gamma\in V[\rho_0,\alpha/2]$; consequently, $\{\overline{M}_{B(\gamma)}(\xi)>1\}\supset\gamma(E_B(Q,1/h))$ for every $\gamma\in V[\rho_0,\alpha/2]$.*

Proof. Let $I\in\Omega_B(Q,1/h)$ and $\gamma\in V[\rho_0,\alpha/2]$. Using simple geometry it is easy to see that $\gamma(I)\supset\{\xi>0\}$ and $\gamma(I)\cap\{\xi<0\}=\emptyset$. Consequently, taking into account the properties of the oscillator ξ, we write: $\frac{1}{|\gamma(I)|}\int_{\gamma(I)}\xi=\frac{1}{|I|}\int_{\{\xi>0\}}\xi=\|h\chi_Q\|_L/|I|=h|Q|/|I|>1$. The lemma is proved. □

Remark 1. On the basis of Lemmas 1 and 3 the oscillator $\xi = \xi_{Q,h,S,\alpha,n}$ may be interpreted as the transformation of the function $h\chi_Q$ that conserves values of integral means with respect to the bases $B(\gamma)$ for rotations γ belonging to the neighbourhood $V[\rho_0, \alpha/2]$. In particular, if it is known that the set $\{M_B(h\chi_Q) > 1\}$ has a big measure, then the sets $\{\overline{M}_{B(\gamma)}(\xi) > 1\}$ have big measures of the same order for every $\gamma \in V[\rho_0, \alpha/2]$.

The following Lemma was shown in [4] (see Lemma 2) and plays an essential role in achieving differentiation effect for desired rotations.

Lemma A. *Let S be a square interval, $0 < \alpha < \pi/12$ and $n \in \mathbb{N}$. Then for arbitrary rectangle R the sides of which compose with the line Ox_1 angles greater than 3α it is valid the estimation $|\nu_+ - \nu_-| \le 2$, where ν_+ is a number of all points $P_k^+(S,\alpha)$ $(k \in \overline{1,n})$ belonging to R and ν_- is a number of all points $P_k^-(S,\alpha)$ $(k \in \overline{1,n})$ belonging to R.*

For a square $S = (0,\varepsilon)^2$ by $\Delta(S)$ denote the union of the strips $(-7\varepsilon, 7\varepsilon) \times \mathbb{R}$ and $\mathbb{R} \times (-7\varepsilon, 7\varepsilon)$.

For a basis B let $\widehat{M}_B$ be the following type maximal operator: $\widehat{M}_B(f)(x) = \sup_{R \in B(x)} \frac{1}{|R|} \left| \int_R f \right|$ $(f \in L(\mathbb{R}^n), x \in \mathbb{R}^n)$.

For a non-empty set $E \subset \Gamma(\mathbb{R}^2)$ and a number $\varepsilon > 0$ denote $V[E,\varepsilon] = \{\gamma \in \Gamma(\mathbb{R}^2) : \operatorname{dist}(\gamma, E) \le \varepsilon\}$.

Below the set of the rotations ρ_0, ρ_1, ρ_2 and ρ_3 will be denoted by Π.

Lemma 4. *Let Q be a square interval with $a(Q) = (0,0)$, $h > 1$ and $0 < \alpha < \pi/12$. Then for every square interval $S \subset Q$ with $a(S) = (0,0)$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that for the oscillator $\xi = \xi_{Q,h,S,\alpha,n}$ it is valid the following inclusion: $\{\widehat{M}_{\mathbf{I}(\gamma)}(\xi) \ge \varepsilon\} \subset \gamma(\Delta(S))$ for every $\gamma \notin V[\Pi, 3\alpha]$.*

Proof. Suppose $x \notin \gamma(\Delta(S)), \gamma \notin V[\Pi, 3\alpha]$, $R \in \mathbf{I}(\gamma)(x)$ and $R \cap \operatorname{supp} \xi \ne \emptyset$. For $n \in \mathbb{N}$ denote by $N_+, N_-, N_+^*, N_-^*, N_+^{**}$ and N_-^{**} the sets of indexes $k \in \overline{1,n}$ satisfying conditions $V_k^+(S,\alpha,n) \cap R \ne \emptyset$, $V_k^-(S,\alpha,n) \cap R \ne \emptyset$, $P_k^+(S,\alpha) \in R$, $P_k^-(S,\alpha) \in R$, $V_k^+(S,\alpha,n) \subset R$ and $V_k^-(S,\alpha,n) \subset R$, respectively.

It is easy to see that if n is big enough, then every line l composing an angle with the axis Ox_1 greater than 3α may intersect at most one among balls $V_k^+(S,\alpha,n)(V_k^-(S,\alpha,n))$. Below we will assume that n has the just mentioned property. Consequently, the boundary of the rectangle R may intersect at most 4 among balls $V_k^+(S,\alpha,n)(V_k^-(S,\alpha,n))$. Thus, there are true the following estimations: $\operatorname{card}(N_+ \setminus N_+^*) + \operatorname{card}(N_+^* \setminus N_+^{**}) \le 4$ and $\operatorname{card}(N_- \setminus N_-^*) + \operatorname{card}(N_-^* \setminus N_-^{**}) \le 4$. Herewith, by virtue of Lemma A: $|\operatorname{card} N_+^* - \operatorname{card} N_-^*| \le 2$.

Let us estimate $|\int_R \xi|$. We have

$$\Big|\int_R \xi\Big| = \Big|\sum_{k\in N_+} \int_{V_k^+(S,\alpha,n)\cap R} \xi + \sum_{k\in N_-} \int_{V_k^-(S,\alpha,n)\cap R} \xi\Big|$$
$$\le \Big|\sum_{k\in N_+^*} \int_{V_k^+(S,\alpha,n)\cap R} \xi + \sum_{k\in N_-^*} \int_{V_k^-(S,\alpha,n)\cap R} \xi\Big|$$
$$+\Big|\sum_{k\in N_+} \int_{V_k^+(S,\alpha,n)\cap R} \xi - \sum_{k\in N_+^*} \int_{V_k^+(S,\alpha,n)\cap R} \xi\Big|$$
$$+\Big|\sum_{k\in N_-} \int_{V_k^+(S,\alpha,n)\cap R} \xi - \sum_{k\in N_-^*} \int_{V_k^+(S,\alpha,n)\cap R} \xi\Big| = a_1 + a_2 + a_3.$$

The term a_1 can be estimated as follows

$$a_1 \le \Big|\sum_{k\in N_+^*} \int_{V_k^+(S,\alpha,n)} \xi + \sum_{k\in N_-^*} \int_{V_k^-(S,\alpha,n)} \xi\Big|$$
$$\le \sum_{k\in N_+^*\setminus N_+^{**}} \int_{V_k^+(S,\alpha,n)} \xi + \sum_{k\in N_-^*\setminus N_-^{**}} \int_{V_k^-(S,\alpha,n)} |\xi| = a_{1,1} + a_{1,2} + a_{1,3}.$$

By virtue of equalities $\int_{V_k^+(S,\alpha,n)} \xi = -\int_{V_k^-(S,\alpha,n)} \xi = h|Q|/n$ ($k \in \overline{1,n}$), we write:

$$a_{1,1} = |\operatorname{card} N_+^* - \operatorname{card} N_-^*| \frac{h|Q|}{n},$$

$$a_{1,2} \le \sum_{k \in N_+^* \setminus N_+^{**}} \int_{V_k^+(S,\alpha,n)} \xi = \operatorname{card}(N_+^* \setminus N_+^{**}) \frac{h|Q|}{n},$$

$$a_{1,3} \le \sum_{k \in N_-^* \setminus N_-^{**}} \int_{V_k^-(S,\alpha,n)} |\xi| = \operatorname{card}(N_-^* \setminus N_-^{**}) \frac{h|Q|}{n},$$

$$a_2 \le \sum_{k \in N_+ \setminus N_+^*} \int_{V_k^+(S,\alpha,n)} \xi = \operatorname{card}(N_+ \setminus N_+^*) \frac{h|Q|}{n},$$

$$a_3 \le \sum_{k \in N_- \setminus N_-^*} \int_{V_k^-(S,\alpha,n)} |\xi| = \operatorname{card}(N_- \setminus N_-^*) \frac{h|Q|}{n}.$$

Consequently,

$$\Big| \int_R \xi \Big| \le a_{1,1} + a_{1,2} + a_{1,3} + a_2 + a_3 \le \frac{10h|Q|}{n}.$$

Since $x \notin \gamma(\Delta(S)), R \in \mathbf{I}(\gamma)(x)$ and $R \cap \operatorname{supp} \xi \neq \emptyset$, it is easy to check that the side lengths of R are not less than the length of the sides of S. Therefore, $|R| \ge |S|$. Hence,

$$\frac{1}{|R|} \Big| \int_R \xi \Big| \le \frac{10h|Q|}{n|S|}.$$

The last estimation implies that if n is big enough, then for every $\gamma \notin V[\Pi, 3\alpha]$ it is valid the needed inclusion: $\{\widehat{M}_{\mathbf{I}(\gamma)}(\xi) \ge \varepsilon\} \subset \gamma(\Delta(S))$. The lemma is proved. □

Remark 2. On the basis of Lemma 4 the oscillator $\xi = \xi_{Q,h,S,\alpha,n}$ may be considered as the transformation of the function $h\chi_Q$ that decreases values of integral means with respect to the bases $\mathbf{I}(\gamma)$ for rotations γ not belonging to the neighbourhood $V[\Pi, 3\alpha]$.

Let us define an oscillator for more general parameters. Suppose, Q and S are square intervals with $Q \supset S$ and $a(Q) = a(S) = (0,0)$, $h > 1, 0 < \alpha \le \pi/4, n \in \mathbb{N}, \gamma \in \Gamma(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$. Denote by T the translation: $T(y) = y - x$. The *oscillator* $\xi_{Q,h,S,\alpha,n,\gamma,x}$ define as the function $(\xi_{Q,h,S,\alpha,n} \circ \gamma^{-1}) \circ T$.

For $\gamma \in \Gamma(\mathbb{R}^2)$ the set of the rotations $\gamma, \rho_1 \circ \gamma, \rho_2 \circ \gamma$ and $\rho_3 \circ \gamma$ will be denoted by Π_γ.

From Lemmas 3 and 4 we can easily obtain the following two assertions.

Lemma 5. *Let $B \in \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{I}}$. Suppose, Q and S are square intervals with $Q \supset S$ and $a(Q) = a(S) = (0,0)$, $h > 1, 0 < \alpha \le \pi/4$, $n \in \mathbb{N}$, $\gamma \in \Gamma(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$. Then for the oscillator $\xi = \xi_{Q,h,S,\alpha,n,\gamma,x}$ it is valid the following condition: $\frac{1}{|\gamma^*(I)+x|} \int_{\gamma^*(I)+x} \xi > 1$ for every $I \in \Omega_B(Q, 1/h)$ and $\gamma^* \in V[\gamma, \alpha/2]$; consequently, $\{\overline{M}_{B(\gamma^*)}(\xi) > 1\} \supset \gamma^*(E_B(Q, 1/h)) + x$ for every $\gamma^* \in V[\gamma, \alpha/2]$.*

Lemma 6. *Let Q be a square interval with $a(Q) = (0,0)$, $h > 1$ and $0 < \alpha < \pi/12$. Then for every square interval $S \subset Q$ with $a(S) = (0,0)$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that for every $\gamma \in \Gamma(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$ the oscillator $\xi = \xi_{Q,h,S,\alpha,n,\gamma,x}$ satisfies the following inclusion:*

$$\{\widehat{M}_{\mathbf{I}(\gamma^*)}(\xi) \ge \varepsilon\} \subset \gamma^*(\Delta(S)) + x \quad \text{for every} \quad \gamma^* \notin V[\Pi_\gamma, 3\alpha].$$

Recall that a one-dimensional interval I is called *dyadic* if it has the form $(k/2^m, (k+1)/2^m)$, where $k, m \in \mathbb{Z}$. A square interval Q is called *dyadic* if it is a product of two dyadic intervals.

The length of the sides of a square Q denote by $d(Q)$. If $d(Q) = 1/2^m$, then let us call the number m an *order of a dyadic square Q*.

Suppose Q and S are square intervals with $Q \supset S$ and $a(Q) = a(S) = (0,0)$, $h > 1, 6hd(Q) \leq 1, 0 < \alpha < \pi/12$, $n \in \mathbb{N}$ and $\gamma \in \Gamma(\mathbb{R}^2)$. For this parameters we will define the function $f_{Q,h,S,\alpha,n,\gamma}$ below.

Let $W(Q,h)$ be the smallest square interval concentric with Q containing the square $6hQ$ and having $d(W)$ of the type $1/2^j$ $(j \in \mathbb{Z})$. Note that by virtue of the condition $6hd(Q) \leq 1$, we have: $d(W) \leq 1$. Let us decompose the unit square $(0,1)^2$ into pair-wise non-overlapping square intervals congruent to $W(Q,h)$ and the obtained squares denote by $W_1, \dots, W_k$. By $x_1, \dots, x_k$ denote the centres of $W_1, \dots, W_k$, respectively. The order of the dyadic squares $W_1, \dots, W_k$ denote by $m(Q,h)$.

The function $f_{Q,h,S,\alpha,n,\gamma}$ define as follows: $f_{Q,h,S,\alpha,n,\gamma} = \sum_{j=1}^{k} \xi_{Q,h,S,\alpha,n,\gamma,x_j}$. It is clear that $\operatorname{supp} f_{Q,h,S,\alpha,n,\gamma} \subset (0,1)^2$.

Let Θ be a some collection of rectangles and Δ be a subinterval of $(0,\infty)$. Then by Θ_Δ denote the collection of all rectangles $R \in \Theta$ the side lengths of which belong to the interval Δ.

Let B be a some basis consisting of rectangles and Δ be a subinterval of $(0,\infty)$. Then by M_B^Δ and $\overline{M}_B^\Delta$ denote the following type operators: $M_B^\Delta(f)(x) = \sup_{R \in B(x)_\Delta} \frac{1}{|R|}\int_R |f|$ and $\overline{M}_B^\Delta(f)(x) = \sup_{R \in B(x)_\Delta} \frac{1}{|R|}\int_R f$, where $f \in L(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Let $B \in \mathfrak{B}_{\mathrm{I}} \cap \mathfrak{B}_{\mathrm{TI}}$ and Q be a square interval. By $\sigma_{B,Q}$ denote the function defined as follows: $\sigma_{B,Q}(\lambda) = |E_B(Q,\lambda)|/|Q|$ $(0 < \lambda < 1)$.

By $\mathbf{P}$ it will be denoted the basis of all two-dimensional rectangles.

Lemma 7. *Let $B \in \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{I}}$. Suppose, Q and S are square intervals with $Q \supset S$ and $a(Q) = a(S) = (0,0)$, $h > 1$, $6hd(Q) \leq 1$, $0 < \alpha < \pi/12$, $n \in \mathbb{N}$, $\gamma \in \Gamma(\mathbb{R}^2)$, $W = W(Q,h)$ and $m = m(Q,h)$. Then the function $f = f_{Q,h,S,\alpha,n,\gamma}$ has the following properties*:

1) $\|f\|_L < 1/h$;

2) *for every $\gamma^* \in V[\gamma, \alpha/2]$ there is a set $A(\gamma^*)$ such that*:

(a) $A(\gamma^*) \subset \{\overline{M}_{B(\gamma^*)}^{[d(Q),d(W)]}(f) > 1\}$;

(b) $|A(\gamma^*)| \geq \sigma_{B,Q}(1/h)/(300h^2)$;

(c) $A(\gamma^*)$ *is uniformly distributed in the dyadic squares of order m contained in $(0,1)^2$, i.e. if $W_1, \dots, W_k$ are all dyadic squares of order m contained in $(0,1)^2$, then the sets $A(\gamma^*) \cap W_k$ are congruent*;

(d) $A(\gamma^*)$ *is a union of dyadic squares of the fixed order, moreover, the order is one and the same for every $\gamma^* \in V[\gamma, \alpha/2]$*;

3) $|\{M_{\mathbf{P}}^{(0,d(Q))}(f) > 0\}| < 1/h^2$;

4) $M_{\mathbf{P}}^{(d(W),\infty)}(f)(x) < 2/h$ *for every $x \in \mathbb{R}^2$.*

Proof. Let W_j, x_j and $\xi_{Q,h,S,\alpha,n,\gamma,x_j}$ $(j \in \overline{1,k})$ be parameters from the definition of the function $f_{Q,h,S,\alpha,n,\gamma}$. Denote $\xi_j = \xi_{Q,h,S,\alpha,n,\gamma,x_j}$ $(j \in \overline{1,k})$.

Using the inclusion $6hQ \subset W$ it is easy to see that $\|f\|_L = \sum_{j=1}^{k} \|\xi_j\|_L = \sum_{j=1}^{k} 2h|Q| = 2h|Q|k = 2h\frac{|Q|}{|W|}k|W| \leq 2h \cdot \frac{1}{36h^2} \cdot 1 < 1/h$.

Let $I \in \Omega_B(Q,1/h), j \in \overline{1,k}$ and $\gamma^* \in V[\gamma,\alpha/2]$. It is easy to check that the side lengthes of I belong to the interval $[d(Q), hd(Q)]$. Consequently, taking into account the inclusion $6hQ \subset W$, we have: $\gamma^*(I) + x_j \subset W_j$. Thus, the rectangle $\gamma^*(I) + x_j$ does not intersect supports of functions ξ_ν with $\nu \neq j$. Therefore, by virtue of Lemma 5, $\frac{1}{|\gamma^*(I)+x_j|}\int_{\gamma^*(I)+x_j} f = \frac{1}{|\gamma^*(I)+x_j|}\int_{\gamma^*(I)+x_j} \xi_j > 1$. Now taking into account estimation $6hd(Q) \leq d(W)$, we conclude that for every $\gamma^* \in V[\gamma,\alpha/2]$,

$$\bigcup_{j=1}^{k} \bigcup_{I \in \Omega_B(Q,1/h)} (\gamma^*(I) + x_j) \subset \{\overline{M}_{B(\gamma^*)}^{[d(Q),d(W)]}(f) > 1\}. \tag{2}$$

For a set $E \subset \mathbb{R}^2$ by $E(\nu)$ $(\nu \in \mathbb{Z})$ let us denote the union of all dyadic squares of order ν contained in E.

Since the set $E_B(Q,1/h)$ is open and the sets $\gamma^*(E_B(Q,1/h))$ ($\gamma^* \in V[\gamma,\alpha/2]$) are congruent, then it is possible to find a number $\nu > m$ (see, e.g., [10, Lemma 7] for details) for which

$$|\gamma^*(E_B(Q,1/h))(\nu)| \geq |\gamma^*(E_B(Q,1/h))|/2 = |E_B(Q,1/h)|/2 \tag{3}$$

for every $\gamma^* \in V[\gamma,\alpha/2]$.

Let us define the set $A(\gamma^*)$ ($\gamma^* \in V[\gamma,\alpha/2]$) as the union of the translations: $\gamma^*(E_B(Q,1/h))(\nu)+x_j$ ($j \in \overline{1,k}$). By virtue of the inclusions $\gamma^*(I)+x_j \subset W_j$ we obtain:

$$\gamma^*(E_B(Q,1/h))(\nu) + x_j \subset \gamma^*(E_B(Q,1/h)) + x_j \subset W_j, \tag{4}$$

for every $\gamma^* \in V[\gamma,\alpha/2]$ and $j \in \overline{1,k}$.

From (3), (4) and the obvious inclusion $W \subset 12hQ$, for arbitrary $\gamma^* \in V[\gamma,\alpha/2]$ we write

$$|A(\gamma^*)| = \sum_{j=1}^{k} |\gamma^*(E_B(Q,1/h))(\nu) + x_j| \geq k\frac{|E_B(Q,1/h)|}{2}$$
$$= k|I|\frac{|Q|}{|I|}\frac{|E_B(Q,1/h)|}{2|Q|} \geq 1\cdot\frac{1}{144h^2}\cdot\frac{\sigma_{B,Q}(1/h)}{2} \geq \frac{\sigma_{B,Q}(1/h)}{300h^2}.$$

This proves the property (b) of the sets $A(\gamma^*)$. The properties (a), (c) and (d) directly follow from the definition of the sets $\gamma^*(E_B(Q,1/h))(\nu)$ and the relations (2) and (4).

Let $x \notin \bigcup_{j=1}^{k} 5(\gamma(Q)+x_j)$. Then it is easy to see that $\operatorname{dist}(x,\operatorname{supp} f) \geq 2d(Q)$. Therefore, for every $R \in \mathbf{P}(x)_{(0,d(Q))}$ we have: $\int_R f = 0$, and consequently, $M_{\mathbf{P}}^{(0,d(Q))}(f)(x) = 0$. Thus, $\{M_{\mathbf{P}}^{(0,d(Q))}(f) > 0\} \subset \bigcup_{j=1}^{k} 5(\gamma(Q)+x_j)$. By virtue of the last inclusion,

$$|\{M_{\mathbf{P}}^{(0,d(Q))}(f) > 0\}| \leq 25k|Q| = 25k|W|\frac{|Q|}{|W|} < 25\cdot 1\cdot\frac{1}{36h^2} < \frac{1}{h^2}.$$

Let $x \in \mathbb{R}^2$ and $R \in \mathbf{P}(x)_{(d(W),\infty)}$. By N denote the set of all numbers $j \in \overline{1,k}$ for which $W_j \cap R \neq \emptyset$. It is easy to check that $\bigcup_{j\in N} W_j \subset 5R$. This inclusion implies that $(\operatorname{card} N)|I| = \sum_{j\in N}|I_j| \leq 25|R|$. Thus, $\operatorname{card} N \leq 25|R|/|I|$. Now we can write,

$$\int\limits_R |f| \leq \sum_{j\in N}\int\limits_{W_j}|f| = \sum_{j\in N}\int\limits_{W_j}|\xi_j| = \sum_{j\in N} 2h|Q|$$
$$= (\operatorname{card} N)2h|Q| \leq 50h\frac{|R||Q|}{|W|} = 50h|R|\frac{1}{36h^2} < \frac{3}{2h}|R|.$$

The obtained estimation implies that $M_{\mathbf{P}}^{(d(W),\infty)}(f)(x) < 2/h$ for every $x \in \mathbb{R}^2$. The lemma is proved. □

Lemma 8. *Let Q be a square interval with $a(Q) = (0,0)$, $h > 1$ and $0 < \alpha < \pi/12$. Then for every $\varepsilon > 0$ and $k \in \mathbb{N}$ there are a square interval $S \subset Q$ with $a(S) = (0,0)$ and a number $n \in \mathbb{N}$ such that for every $\gamma \in \Gamma(\mathbb{R}^2)$ and $x_1,\dots,x_k \in \mathbb{R}^2$ the functions $\xi_j = \xi_{Q,h,S,\alpha,n,\gamma,x_j}$ ($j \in \overline{1,k}$) satisfy the following estimation*:

$$\Big|\Big\{\widehat{M}_{\mathbf{I}(\gamma^*)}\Big(\sum_{j=1}^{k}\xi_j\Big) \geq \varepsilon\Big\} \cap (0,1)^2\Big| < \varepsilon \quad \textit{for every} \quad \gamma^* \notin V[\Pi_\gamma, 3\alpha].$$

Proof. Let us choose a square interval $S \subset Q$ with $a(S) = (0,0)$ so that $28\sqrt{2}\operatorname{diam} S < \varepsilon/k$, and using Lemma 6 let us choose a number $n \in \mathbb{N}$ so that for every $\gamma \in \Gamma(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$ the oscillator $\xi = \xi_{Q,h,S,\alpha,n,\gamma,x}$ satisfies the following condition: $\{\widehat{M}_{\mathbf{I}(\gamma^*)}(\xi) \geq \varepsilon/k\} \subset \gamma^*(\Delta(S)) + x$ for every $\gamma^* \notin V[\Pi_\gamma, 3\alpha]$.

Suppose, $\gamma \in \Gamma(\mathbb{R}^2)$, $x_1,\dots,x_k \in \mathbb{R}^2$ and $\xi_j = \xi_{Q,h,S,\alpha,n,\gamma,x_j}$ ($j \in \overline{1,k}$). Let us consider an arbitrary $\gamma^* \notin V[\Pi_\gamma, 3\alpha]$. Then taking into account the estimation $\widehat{M}_{\mathbf{I}(\gamma^*)}\Big(\sum_{j=1}^{k}\xi_j\Big) \leq \sum_{j=1}^{k}\widehat{M}_{\mathbf{I}(\gamma^*)}(\xi_j)$, we have

$$\Big\{\widehat{M}_{\mathbf{I}(\gamma^*)}\Big(\sum_{j=1}^{k}\xi_j\Big) \geq \varepsilon\Big\} \subset \bigcup_{j=1}^{k}\{\widehat{M}_{\mathbf{I}(\gamma^*)}(\xi_j) \geq \varepsilon/k\} \subset \bigcup_{j=1}^{k}(\gamma^*(\Delta(S)) + x_j). \tag{5}$$

Note that: 1) For any strip Δ it is true the estimation: $|\Delta \cap (0,1)^2| \leq \sqrt{2}$ (width of Δ); 2) $\gamma^*(\Delta(S)) + x_j$ $(j \in \overline{1,k})$ is a union of two strips with the widthes less than $14\,\mathrm{diam}\,S$. Consequently, on the basis of choosing of S, for each j we write: $|(\gamma^*(\Delta(S))+x_j)\cap(0,1)^2| \leq 2(\sqrt{2}\,14\,\mathrm{diam}\,S) < \varepsilon/k$. Hence, using (5) we obtain that $|\{\widehat{M}_{\mathbf{I}(\gamma^*)}(\sum_{j=1}^{k}\xi_j) \geq \varepsilon\} \cap (0,1)^2| < \varepsilon$. The lemma is proved. □

Lemma 9. *Let $I \subset \mathbb{R}$ be an open interval. For every $s > 1$ and $\varepsilon \in (0,1)$ there are pairwise non-overlapping closed intervals $I_k \subset I$ $(k \in \mathbb{N})$ such that $I = \bigcup_{k=1}^{\infty} I_k$, $|I_k| < \varepsilon|I|$ $(k \in \mathbb{N})$, $sI_k \subset I$ $(k \in \mathbb{N})$ and $\sum_{k=1}^{\infty}\chi_{sI_k}(x) \leq c(s)$ $(x \in I)$, where $c(s)$ is a constant depending only on the parameter s.*

Proof. Let x_0 be a midpoint of I and for a number $t \in (0,1)$ let us consider the points $x_m = \sup I - t^m|I|/2$, $x_{-m} = \inf I + t^m|I|/2$ $(m \in \mathbb{N})$. It is easy to check that if t is quite close to 1 then the intervals $[x_m, x_{m+1}]$ $(m \in \mathbb{Z})$ generate the needed decomposition of I. □

Lemma 10. *For an arbitrary non-empty symmetric set $E \subset \Gamma(\mathbb{R}^2)$ of type G_δ there are sequences of rotations (γ_k) and numbers (α_k) from the interval $(0,\pi/12)$ such that $\overline{\lim\limits_{k\to\infty}}\, V[\gamma_k, \alpha_k/2] = \overline{\lim\limits_{k\to\infty}}\, V[\Pi_{\gamma_k}, 3\alpha_k] = E$.*

Proof. For an interval $I \subset [0,2\pi)$ denote $I_{\mathbb{T}} = \{(\cos(t),\sin(t)) : t \in I\}$ and $\Gamma_I = \{\gamma \in \Gamma(\mathbb{R}^2) : \gamma((1,0)) \in I_{\mathbb{T}}\}$.

First let us prove the following statement: For an arbitrary non-empty set $W \subset \Gamma_{[0,\pi/2)}$ of G_δ type there are sequences of rotations (σ_m) and numbers (β_m) from the interval $(0,\pi/12)$ such that $\overline{\lim\limits_{m\to\infty}}\, V[\sigma_m, \beta_m/2] = \overline{\lim\limits_{m\to\infty}}\, V[\sigma_m, 3\beta_m] = W$.

Without loss of generality we can assume that $\rho_0 \notin W$, i.e. $W \subset \Gamma_{(0,\pi/2)}$. Using identification of $\Gamma_{(0,\pi/2)}$ with the interval $(0,\pi/2)$ by the mapping $\Gamma_{(0,\pi/2)} \ni \gamma \mapsto \mathrm{dist}(\gamma,\rho_0) \in (0,\pi/2)$ we can formulate our statement in the following equivalent way: For an arbitrary non-empty set $V \subset (0,\pi/2)$ of G_δ type there exists a sequence of closed intervals $I_m \subset (0,\pi/2)$ such that $|I_m| < \pi/12$ and $\overline{\lim\limits_{m\to\infty}}\, I_m = \overline{\lim\limits_{m\to\infty}}\,(6I_m) = V$.

Consider a sequence of open sets $G_n \subset (0,\pi/2)$ with $G_1 \supset G_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} G_n = V$. Let $\{I_p^{(n)}\}$ be the collection of open intervals decomposing G_n. For each n and p let us consider a sequence of closed intervals $(I_{p,q}^{(n)})_{q\in\mathbb{N}}$ corresponding to the parameters $s = 6, \varepsilon = 1/12$ and $I = I_p^{(n)}$ according to Lemma 9. If we enumerate the intervals $I_{p,q}^{(n)}$ by one index $m \in \mathbb{N}$, then it is easy to see that the obtained sequence of intervals (I_m) will satisfy the needed conditions. This proves the statement.

Now let us consider an arbitrary non-empty symmetric set $E \subset \Gamma(\mathbb{R}^2)$ of G_δ type. Let (σ_m) and (β_m) be sequences corresponding to the set $E \cap \Gamma_{[0,\pi/2)}$ according to the above proved statement. By $\lceil x \rceil$ $(x \in \mathbb{R})$ denote the number $\min\{n \in \mathbb{Z} : x \leq n\}$. Then it is easy to check that the sequences: $\gamma_k = \rho_{(k-1)(\mathrm{mod}\ 4)} \circ \sigma_{\lceil k/4 \rceil}$, $\alpha_k = \beta_{\lceil k/4 \rceil}$ $(k \in \mathbb{N})$, will satisfy the needed conditions. □

5. Proof of Theorem 1

Let B be a basis satisfying the conditions of the theorem. In the introduction it was mentioned that the following three statements are true: 1) Each W_B-set is of type $G_{\delta\sigma}$ and each R_B-set is of type G_δ; 2) Every W_B-set and every R_B-set is symmetric; 3) Not more than countable union of R_B-sets is a W_B-set.

Taking into account three statements above it suffices to prove that an arbitrary symmetric set $E \subset \Gamma(\mathbb{R}^2)$ of type G_δ is an R_B-set. If E is empty, then the statement is trivial. Thus let us consider the case of a non-empty set E.

By virtue of Lemma 10 there are sequences $\gamma_k \in \Gamma(\mathbb{R}^2)$ and $\alpha_k \in (0,\pi/12)$ such that $\overline{\lim\limits_{k\to\infty}}\, V[\gamma_k, \alpha_k/2] = \overline{\lim\limits_{k\to\infty}}\, V[\Pi_{\gamma_k}, 3\alpha_k] = E$.

Taking into account non-regularity of the spherical halo function σ_B and the estimation $\sigma_B(1/h) \leq Ch\ln h$ $(h \geq 2)$ (which is valid by virtue of strong maximal inequality (see, e.g., [3, Ch. II, §3]) it is not difficult to choose sequences (h_j) and (η_j) with the properties: $h_j \geq 2$, $0 < \eta_j < h_j$, $\lim\limits_{j\to\infty} h_j = \infty$, $\lim\limits_{j\to\infty} \eta_j = \infty$, $\sigma_B(1/h_j) > 144h_j$, $\sigma_B(1/h_j)/h_j^2 < 1$, $\sum_{j=1}^{\infty} \sigma_B(1/h_j)/h_j^2 = \infty$ and $\sum_{j=1}^{\infty} \eta_j/h_j < \infty$.

On the basis of divergence of the series $\sum_j \sigma_B(1/h_j)/h_j^2$ we can choose numbers $1 = j_0 < j_1 < j_2 < \cdots$ so that $\prod_{j=j_{k-1}}^{j_k-1}\left(1 - \frac{c}{2400}\frac{\sigma_B(1/h_j)}{h_j^2}\right) < \frac{1}{2^k}$ for every $k \in \mathbb{N}$. Here c is the constant from Lemma 1.

Denote $J_k = \{j \in \mathbb{N} : j_{k-1} \leq j \leq j_k - 1\}$ $(k \in \mathbb{N})$.

Using Lemmas 7 and 8 we can find sequences of square intervals (Q_j) and (S_j) with $a(Q_j) = a(S_j) = (0,0)$ and a sequence of natural numbers (n_j) for which the functions $f_j = f_{Q_j,h_j,S_j,\alpha_j,n_j,\gamma_j}$, $g_j = \eta_j f_j$ $(j \in \mathbb{N})$ satisfy the following conditions:

1) $\|g_j\| = \eta_j\|f_j\| < \eta_j/h_j$;

2) $d(W_1) > d(Q_1) > d(W_2) > d(Q_2) > \cdots$. Here $W_j = W(Q_j,h_j)$ is a square interval from the definition of the function $f_{Q,h,S,\alpha,n,\gamma}$;

3) there are sets $A_j(\gamma)$ $(k \in \mathbb{N}, \gamma \in V[\gamma_k,\alpha_k/2], j \in J_k)$ such that:

(a) $A_j(\gamma) \subset \{\overline{M}_{B(\gamma)}^{[d(Q_j),d(W_j)]}(f_j) > 1\} = \{\overline{M}_{B(\gamma)}^{[d(Q_j),d(W_j)]}(g_j) > \eta_j\}$;

(b) $|A_j(\gamma)| \geq c\sigma_B(1/h_j)/(2400h_j^2)$;

(c) $A_j(\gamma)$ is uniformly distributed in the dyadic squares of order $m_j = m(Q_j,h_j)$ contained in $(0,1)^2$, i.e. if $W_1,\dots,W_\nu$ are all dyadic squares of order m_j contained in $(0,1)^2$, then the sets $A_j(\gamma) \cap W_i$ $(i \in \overline{1,\nu})$ are congruent. Here $m(Q_j,h_j)$ is the number from the definition of the function $f_{Q,h,S,\alpha,n,\gamma}$;

(d) $A_j(\gamma)$ is an union of dyadic squares of the order $m_j^* > m_j$, where m_j^* does not depend on $\gamma \in V[\gamma_k,\alpha_k/2]$;

4) the numbers m_j and m_j^* from the conditions 3)–(c) and 3)–(d) satisfy inequalities: $m_1 < m_1^* < m_2 < m_2^* < \cdots$;

5) $|\{M_{\mathbf{P}}^{(0,d(Q_j))}(g_j) > 0\}| = |\{M_{\mathbf{P}}^{(0,d(Q_j))}(f_j) > 0\}| < 1/h_j^2$ for every $j \in \mathbb{N}$;

6) $M_{\mathbf{P}}^{(d(W_j),\infty)}(g_j)(x) = \eta_j M_{\mathbf{P}}^{(d(W_j),\infty)}(f_j)(x) < 2\eta_j/h_j$ for every $j \in \mathbb{N}$ and $x \in \mathbb{R}^2$;

7) $|\{\widehat{M}_{\mathbf{I}(\gamma)}(f_j) \geq 1/(\eta_j 2^j)\} \cap (0,1)^2| < 1/(\eta_j 2^j)$ for every $k \in \mathbb{N}, \gamma \notin V[\Pi_{\gamma_k},3\alpha_k]$ and $j \in J_k$. Consequently, $|\{\widehat{M}_{\mathbf{I}(\gamma)}(g_j) \geq 1/2^j\} \cap (0,1)^2| < 1/2^j$ for every $k \in \mathbb{N}, \gamma \notin V[\Pi_{\gamma_k},3\alpha_k]$ and $j \in J_k$.

Set $g = \sum_{j=1}^{\infty} g_j$. First note that $\|g\|_L \leq \sum_{j=1}^{\infty}\|g_j\|_L < \sum_{j=1}^{\infty}\eta_j/h_j < \infty$. Thus, g is a summable function. Suppose $\gamma \notin E$. Let us prove that $\mathbf{I}(\gamma)$ differentiates $\int g$. Since $\operatorname{supp} g \subset (0,1)^2$, then $\mathbf{I}(\gamma)$ differentiates $\int g$ at every point $x \notin [0,1]^2$. Further, denote

$$T_j = \{\widehat{M}_{\mathbf{I}(\gamma)}(g_j) \geq 1/2^j\} \cap (0,1)^2, \quad T = \varlimsup_{j\to\infty} T_j.$$

We have that $\gamma \notin \varlimsup\limits_{k\to\infty} V[\Pi_{\gamma_k},3\alpha_k]$. Consequently, there is $k_0 \in \mathbb{N}$ for which $\gamma \notin V[\Pi_{\gamma_k},3\alpha_k]$ for every $k \geq k_0$. The last condition on the basis of the estimation 7) implies: $|T_j| < 1/2^j$ for every $j \geq j_{k_0}$. Now taking into account that $|T_j| \leq 1$ $(j \in \mathbb{N})$ we have: $\sum_{j=1}^{\infty}|T_j| < \infty$. Consequently, $|T| = 0$. Thus, for arbitrary given point $x \in (0,1)^2 \setminus T$ there is $j^* \in \mathbb{N}$ for which $\widehat{M}_{\mathbf{I}(\gamma)}(g_j)(x) < 1/2^j$ for every $j > j^*$. Now taking into account boundedness of the functions g_j we write: $\widehat{M}_{\mathbf{I}(\gamma)}(g)(x) \leq \sum_{j=1}^{\infty}\widehat{M}_{\mathbf{I}(\gamma)}(g_j)(x) \leq \sum_{j=1}^{j^*}\widehat{M}_{\mathbf{I}(\gamma)}(g_j)(x) + \sum_{j=j^*+1}^{\infty} 1/2^j < \infty$. Thus, $(0,1)^2 \setminus T \subset \{\widehat{M}_{\mathbf{I}(\gamma)}(g) < \infty\}$. Note that by virtue of the result of Besicovitch (see, e.g., [3, Ch. IV, §3]) the sets $\{g < \overline{D}_B(\int g,\cdot) < \infty\}$ and $\{-\infty < \underline{D}_B(\int g,\cdot) < g\}$ have zero measure. Therefore, taking into account the last inclusion, we conclude that $\mathbf{I}(\gamma)$ differentiates $\int g$.

Suppose $\gamma \in E$. Then $\gamma \in \varlimsup\limits_{k\to\infty} V[\gamma_k,\alpha_k/2]$. Thus, the set $N = \{k \in \mathbb{N} : \gamma \in V[\gamma_k,\alpha_k/2]\}$ is infinite. Let $k \in N$. Taking into account the properties 3)–(c), 3)–(d) and 4) it is easy to see that the sets $A_j(\gamma)$ $(j \in J_k)$ are probabilistically independent. Therefore,

$$\Big|\bigcup_{j\in J_k} A_j(\gamma)\Big| = 1 - \Big|\bigcap_{j\in J_k}((0,1)^2 \setminus A_j(\gamma))\Big| = 1 - \prod_{j\in J_k}(1 - |A_j(\gamma)|).$$

Now using 3)–(b) and taking into account the choice of the numbers j_k, we obtain: $|\bigcup_{j\in J_k} A_j(\gamma)| > 1 - 1/2^k$. From this estimation we conclude: if A denotes the upper limit of the sequence of the sets $\bigcup_{j\in J_k} A_j(\gamma)$ $(k \in N)$, then A is of full measure in $(0,1)^2$, i.e. $|(0,1)^2 \setminus A| = 0$.

Let F be the upper limit of the sequence of the sets $\{M_{\mathbf{P}}^{(0,d(Q_j))}(g_j) > 0\}$ $(j \in \mathbb{N})$. By virtue of the property 5), $\sum_{j=1}^{\infty} |\{M_{\mathbf{P}}^{(0,d(Q_j))}(g_j) > 0\}| < \infty$. Therefore the set F is of zero measure.

For any $x \in A \setminus F$ let us prove the equality $\overline{D}_{B(\gamma)}(\int g, x) = +\infty$. It will imply that the equality is valid for almost every point from $(0,1)^2$.

We can find an infinite set $N^* \subset N$, a sequence $j(k) \in J_k$ $(k \in N^*)$ and a number $j(0) \in \mathbb{N}$ with the properties: i) $x \in A_{j(k)}(\gamma)$ for every $k \in N^*$; ii) $x \notin \{M_{\mathbf{P}}^{(0,d(Q_j))}(g_j) > 0\}$ for every $j > j(0)$. We can assume that $j(k) > j(0)$ $(k \in N^*)$.

For every $k \in N^*$ we can find a rectangle $R_k \in B(\gamma)(x)_{[d(Q_{j(k)}),d(W_{j(k)})]}$ for which $\frac{1}{|R_k|}\int_{R_k} g_{j(k)} > \eta_{j(k)}$. Let us estimate the integral means on R_k of the functions g_j with $j \neq j(k)$. Taking into account the property 2), we have: $\frac{1}{|R_k|}\int_{R_k} g_j = 0$ if $j(0) < j < j(k)$ and $\frac{1}{|R_k|}\int_{R_k} g_j < \eta_j/h_j$ if $j > j(k)$. Consequently,

$$\frac{1}{|R_k|}\int\limits_{R_k} g = \frac{1}{|R_k|}\int\limits_{R_k} g_{j(k)} - \sum_{j=1}^{j(0)} \frac{1}{|R_k|}\int\limits_{R_k} g_j - \sum_{j=j(0)+1}^{j(k)-1} \frac{1}{|R_k|}\int\limits_{R_k} g_j$$
$$- \sum_{j=j(k)+1}^{\infty} \frac{1}{|R_k|}\int\limits_{R_k} g_j > \eta_{j(k)} - \sum_{j=1}^{j(0)} \|g_j\|_{L^\infty} - \sum_{j=j(k)+1}^{\infty} \frac{\eta_j}{h_j}.$$

Thus, the rectangles R_k $(k \in N^*)$ satisfy conditions: $R_k \in B(\gamma)(x)$ $(k \in N^*)$, $\operatorname{diam} R_k \to 0$ $(N^* \ni k \to \infty)$ and $\frac{1}{|R_k|}\int_{R_k} g \to +\infty$ $(N^* \ni k \to \infty)$. Therefore, $\overline{D}_{B(\gamma)}\big(\int g, x\big) = +\infty$.

Summarizing above established properties of the function g we have: i) $g \in L(\mathbb{R}^2)$ and $\operatorname{supp} g \subset (0,1)^2$; ii) $\overline{D}_{B(\gamma)}\big(\int g, x\big) = +\infty$ a.e. on $(0,1)^2$ for every $\gamma \in E$; iii) $\mathbf{I}(\gamma)$ differentiates $\int g$ for every $\gamma \notin E$.

Set $f(x_1,x_2) = \sum_{i,j\in\mathbb{Z}} g(x_1+i, x_2+j)/2^{i+j}$ $((x_1,x_2) \in \mathbb{R}^2)$. Then we can easily check that f satisfies the conditions providing E to be an R_B-set. The theorem is proved.

Remark 3. The function f constructed in the proof of Theorem 1 for any rotation $\gamma \notin E$ satisfies stronger condition than it is required. Namely, $\int f$ is differentiable with respect to the basis $\mathbf{I}(\gamma)$ which is broader than the basis $B(\gamma)$.

Remark 4. The function f constructed in the proof of Theorem 1 takes values of both signs. For non-negative summable functions the problem of characterization of singular rotation's sets is open even for the case of the basis $\mathbf{I}(\mathbb{R}^2)$. Some partial results in this direction are obtained in [8] and [12].

Remark 5. For the multidimensional case the problem of characterization of $W_{\mathbf{I}(\mathbb{R}^n)}$-sets and $R_{\mathbf{I}(\mathbb{R}^n)}$-sets is open. Note that a class of $R_{\mathbf{I}(\mathbb{R}^n)}$-sets is found in [9].

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