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ON (CO)HOMOLOGICAL PROPERTIES OF REMAINDERS OF STONE-ČECH COMPACTIFICATIONS

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Abstract. In the paper are defined the Čech border homology and cohomology groups of closed pairs of normal spaces and showed that they give intrinsic characterizations of Čech (co)homology groups based on finite open coverings, cohomological coefficients of cyclicity, small and large cohomological dimensions of remainders of Stone-Čech compactifications of metrizable spaces.

Introduction

The investigation and discussion presented in this paper are centered around the following problem:

Find necessary and sufficient conditions under which a space of given class has a compactification whose remainder has the given topological property (cf. [35], Problem I, p. 332, and Problem II, p. 334).

This problem for different topological invariants and properties was studied by several authors (see [1–3, 5–8, 11–14, 19–25, 27, 30–36]).

The present paper is motivated by this general problem. Specifically, we study this problem for the properties: Čech (co)homology groups based on finite open covers, cohomological coefficients of cyclicity and cohomological dimensions of remainders of Stone-Čech compactifications of metrizable spaces are given groups and given numbers, respectively.

In the paper we define the Čech type covariant and contravariant functors which coefficients in an abelian group G,

$$\check{\mathrm{H}}^{\infty}_{n}(-,-;G) : \mathscr{N}^{\mathbf{2}}_{p} \to \mathscr{A}b$$

and

$$\hat{\mathrm{H}}^{n}_{\infty}(-,-;G) : \mathscr{N}^{\mathbf{2}}_{p} \to \mathscr{A}b,$$

from the category $\mathscr{N}^2_p$ of closed pairs of normal spaces and proper maps to the category $\mathscr{A}b$ of abelian groups and homomorphisms. The construction of these functors is based on all border open covers of pairs $(X,A) \in ob(\mathscr{N}^{\mathbf{2}}_p)$ (see Definition 1.1 and Definition 1.2).

One of our main results of the paper is the following theorem (see Theorem 2.1). Let $\mathscr{M}^2_p$ be the category of closed pairs of metrizable spaces and proper maps. For each closed pair $(X,A) \in ob(\mathscr{M}^{\mathbf{2}}_p)$, one has

$$\check{H}^{f}_{n}(\beta X \setminus X, \beta A \setminus A; G) = \check{H}^{\infty}_{n}(X,A;G)$$

and

$$\hat{H}^{n}_{f}(\beta X \setminus X, \beta A \setminus A; G) = \hat{H}^{n}_{\infty}(X,A;G),$$

where $\check{H}^{f}_{n}(\beta X \setminus X, \beta A \setminus A; G)$ and $\hat{H}^{n}_{f}(\beta X \setminus X, \beta A \setminus A; G)$ are Čech homology and cohomology groups based on all finite open covers of $(\beta X \setminus X, \beta A \setminus A)$, respectively (see [17, Ch. IX, p. 237]).

We also investigate the border cohomological coefficient of cyclicity η^{∞}_{G}, border small and large cohomological dimensions $d^{f}_{\infty}(X;G)$ and $D^{f}_{\infty}(X;G)$ and prove the following relations (see Theorem 2.3,

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Theorem 3.2 and Theorem 3.6):

$$\eta^{\infty}_G(X,A)=\eta_G(\beta X\setminus X,\beta A\setminus A),$$
$$d^f_{\infty}(X;G)=d_f(\beta X\setminus X;G),$$
$$D^f_{\infty}(X;G)=D_f(\beta X\setminus X;G),$$

where $\eta_G(\beta X\setminus X,\beta A\setminus A)$, $d_f(\beta X\setminus X;G)$ and $D_f(\beta X\setminus X;G)$ are well known cohomological coefficient of cyclicity [10,29], small cohomological dimension and large cohomological dimension [28] of remainders $(\beta X\setminus X,\beta A\setminus A)$ and $\beta X\setminus X$, respectively.

Without any further reference we will use definitions and results from the monographs General Topology [18], Algebraic Topology [17] and Dimension Theory [28].

1. On Čech Border (Co)homology Groups

In this section we give an outline of a generalization of Čech homology theory by replacing the set of all finite open coverings in the definition of Čech (co)homology group $(\hat{H}^n_f(X,A;G))$ $\check{H}^f_n(X,A;G)$ (see [17, Ch. IX, p. 237]) by a set of all finite open families with compact enclosures. For this aim we give the following definitions.

An indexed family of subsets of set X is a function α from an indexed set V_α to the set 2^X of subsets of X. The image $\alpha(v)$ of index $v\in V_\alpha$ is denoted by α_v. Thus the indexed family α is the family $\alpha=\{\alpha_v\}_{v\in V_\alpha}$. If $|V_\alpha|<\aleph_0$, then we say that α family is a finite family.

Let V'_α be a subset of set V_α. A family $\{\alpha_v\}_{v\in V'_\alpha}$ is called a subfamily of family $\{\alpha_v\}_{v\in V_\alpha}$.

By $\alpha=\{\alpha_v\}_{v\in(V_\alpha,V'_\alpha)}$ we denote the family consisting of family $\{\alpha_v\}_{v\in V_\alpha}$ and its subfamily $\{\alpha_v\}_{v\in V'_\alpha}$.

Definition 1.1. (see [33]). A finite family $\alpha=\{\alpha_v\}_{v\in V_\alpha}$ of open subsets of normal space X is called a border cover of X if its enclosure $K_\alpha=X\setminus\bigcup\limits_{v\in V_\alpha}\alpha_v$ is a compact subset of X.

Definition 1.2. (cf. [33]). A finite open family $\alpha=\{\alpha_v\}_{v\in(V_\alpha,V^A_\alpha)}$ is called a border cover of closed pair $(X,A)\in\mathscr{N}^2$ if there exists a compact subset K_α of X such that $X\setminus K_\alpha=\bigcup\limits_{v\in V_\alpha}\alpha_v$ and $A\setminus K_\alpha\subseteq\bigcup\limits_{v\in V^A_\alpha}\alpha_v$.

The set of all border covers of (X,A) is denoted by $\mathrm{cov}_\infty(X,A)$. Let $K^A_\alpha=K_\alpha\cap A$. Then the family $\{\alpha_v\cap A\}_{v\in V^A_\alpha}$ is a border cover of subspace A.

Definition 1.3. Let $\alpha,\beta\in\mathrm{cov}_\infty(X,A)$ be two border covers of (X,A) with indexing pairs (V_α,V^A_α) and (V_β,V^A_β), respectively. We say that the border cover β is a refinement of border cover α if there exists a refinement projection function $p:(V_\beta,V^A_\beta)\to(V_\alpha,V^A_\alpha)$ such that for each index $v\in V_\beta$ $(v\in V^A_\beta)$ $\beta_v\subset\alpha_{p(v)}$.

It is clear that $\mathrm{cov}_\infty(X,A)$ becomes a directed set with the relation $\alpha\le\beta$ whenever β is a refinement of α.

Note that for each $\alpha\in\mathrm{cov}_\infty(X,A)$, $\alpha\le\alpha$, and if for each $\alpha,\beta,\gamma\in\mathrm{cov}_\infty(X,A)$, $\alpha\le\beta$ and $\beta\le\gamma$, then $\alpha\le\gamma$.

Let $\alpha,\beta\in\mathrm{cov}_\infty(X,A)$ be two border covers with indexing pairs (V_α,V^A_α) and (V_β,V^A_β), respectively. Consider a family $\gamma=\{\gamma_v\}_{v\in(V_\gamma,V^A_\gamma)}$, where $V_\gamma=V_\alpha\times V_\beta$ and $V^A_\gamma=V^A_\alpha\times V^A_\beta$. Let $v=(v_1,v_2)$, where $v_1\in V_\alpha$, $v_2\in V_\beta$. Assume that $\gamma_v=\alpha_{v_1}\cap\beta_{v_2}$. The family $\gamma=\{\gamma_v\}_{v\in(V_\gamma,V^A_\gamma)}$ is a border cover of (X,A) and $\gamma\ge\alpha,\beta$.

For each border cover $\alpha\in\mathrm{cov}^\infty(\mathrm{X},\mathrm{A})$ with indexing pair (V_α,V^A_α), by (X_α,A_α) denote the nerve α, where A_α is the subcomplex of simplexes s of complex X_α with vertices of V^A_α such that $\mathrm{Car}_\alpha(\mathrm{s})\cap\mathrm{A}\neq\emptyset$, where $\mathrm{Car}_\alpha(s)$ is the carrier of simplex s (see [17, pp. 234]). The pair (X_α,A_α) is a simplicial pair. Moreover, any two refinement projection functions $p,q:\beta\to\alpha$ induce contiguous simplicial maps of simplicial pairs $p^\beta_\alpha,q^\beta_\alpha:(X_\beta,A_\beta)\to(X_\alpha,A_\alpha)$ (see [17, pp. 234–235]).

Using the construction of formal homology theory of simplicial complexes [17, Ch. VI] we can define the unique homomorphisms

$$p^{\beta}_{\alpha *} : H_n(X_\beta, A_\beta : G) \to H_n(X_\alpha, A_\alpha; G)$$

and

$$p^{\beta *}_{\alpha} : H^n(X_\alpha, A_\alpha : G) \to H_n(X_\beta, A_\beta; G),$$

where G is any abelian coefficient group.

Note that $p^{\alpha}_{\alpha *} = 1_{H_n(X_\alpha, A_\alpha : G)}$ and $p^{\alpha *}_{\alpha} = 1_{H^n(X_\alpha, A_\alpha : G)}$. If $\gamma \geq \beta \geq \alpha$ than

$$p^{\gamma}_{\alpha *} = p^{\beta}_{\alpha *} \cdot p^{\gamma}_{\beta *}$$

and

$$p^{\gamma *}_{\alpha} = p^{\gamma *}_{\beta} \cdot p^{\beta *}_{\alpha}.$$

Thus, the families

$$\{H_n(X_\alpha, A_\alpha; G), p^{\beta}_{\alpha *}, \mathrm{cov}_\infty(X, A)\}$$

and

$$\{H^n(X_\alpha, A_\alpha; G), p^{\beta *}_{\alpha}, \mathrm{cov}_\infty(X, A)\}$$

form inverse and direct systems of groups.

The inverse and direct limit groups of above defined inverse and direct systems are denoted by symbols

$$\check{H}^{\infty}_{n}(X, A; G) = \varprojlim \{H_n(X_\alpha, A_\alpha; G), p^{\beta}_{\alpha *}, \mathrm{cov}_\infty(X, A)\}$$

and

$$\hat{H}^{n}_{\infty}(X, A; G) = \varinjlim \{H^n(X_\alpha, A_\alpha; G), p^{\beta *}_{\alpha}, \mathrm{cov}_\infty(X, A)\}$$

and called n-dimensional Čech border homology group and n-dimensional Čech border cohomology group of pair (X, A) with coefficients in abelian group G, respectively.

For a pair $(X, A) \in ob(\mathscr{N}^2_p)$ and a proper map $f : (X, A) \to (Y, B)$ of pairs, the induced homomorphisms

$$f^{\infty}_{*} : \check{H}^{\infty}_{n}(X, A; G) \to \check{H}^{\infty}_{n}(Y, B; G)$$

and

$$f^{*}_{\infty} : \hat{H}^{n}_{\infty}(X, A; G) \to \hat{H}^{n}_{\infty}(Y, B; G),$$

and the boundary and coboundary homomorphisms

$$\partial^{\infty}_{n} : \check{H}^{\infty}_{n}(X, A; G) \to \check{H}^{\infty}_{n-1}(A; G)$$

and

$$\delta^{n}_{\infty} : \hat{H}^{n-1}_{\infty}(A; G) \to \hat{H}^{n}_{\infty}(X, A; G)$$

are defined. For details of these definitions, see Eilenberg and Steenrod [17].

We have the following theorems.

Theorem 1.4. *There exist the covariant and contravariant functors*

$$\check{\mathrm{H}}^{\infty}_{*}(-, -; G) : \mathscr{N}^2_p \to \mathscr{A}b$$

and

$$\hat{\mathrm{H}}^{*}_{\infty}(-, -; G) : \mathscr{N}^2_p \to \mathscr{A}b$$

given by formulas

$$\check{\mathrm{H}}^{\infty}_{*}(-, -; G)(X, A) = \check{H}^{\infty}_{*}(X, A; G), \quad (X, A) \in ob(\mathscr{N}^2_p)$$
$$\check{\mathrm{H}}^{\infty}_{*}(-, -; G)(f) = f^{\infty}_{*}, \quad f \in \mathrm{Mor}_{\mathscr{N}^2_{\mathrm{p}}}((X, A), (Y, B))$$

and

$$\hat{\mathrm{H}}^{*}_{\infty}(-, -; G)(X, A) = \hat{H}^{*}_{\infty}(X, A; G), \quad (X, A) \in ob(\mathscr{N}^2_p)$$
$$\hat{\mathrm{H}}^{*}_{\infty}(-, -; G)(f) = f^{*}_{\infty}, \quad f \in \mathrm{Mor}_{\mathscr{N}^2_{\mathrm{p}}}((X, A), (Y, B)).$$

Theorem 1.5. *Let $f:(X,A)\to(Y,B)$ be a proper map. Then hold the following equalities*

$$(f_{|A})_*^\infty \cdot \partial_n^\infty = \partial_n^\infty \cdot f_*^\infty$$

and

$$\delta_\infty^{n-1} \cdot (f_{|A})_\infty^* = f_\infty^* \cdot \delta_\infty^{n-1}.$$

Theorem 1.6. *Let $(X,A)\in ob(\mathcal{N}_p^2)$ and let $i:A\to X$ and $j:X\to(X,A)$ be the inclusion maps. Then the Čech border cohomology sequence*

$$\cdots \longrightarrow \check{H}_\infty^{n-1}(A;G) \xrightarrow{\delta_\infty^{n-1}} \check{H}_\infty^n(X,A;G) \xrightarrow{j_\infty^*} \check{H}_\infty^n(X;G) \xrightarrow{i_\infty^*} \check{H}_\infty^n(A;G) \longrightarrow \cdots$$

is exact while the Čech border homology sequence

$$\cdots \longleftarrow \hat{H}_{n-1}^\infty(A;G) \xleftarrow{\partial_n^\infty} \hat{H}_n^\infty(X,A;G) \xleftarrow{j_*^\infty} \hat{H}_n^\infty(X;G) \xleftarrow{i_*^\infty} \hat{H}_n^\infty(A;G) \longleftarrow \cdots$$

is partially exact.

Theorem 1.7. *Let $(X,A)\in ob(\mathcal{N}_p^2)$ and G be an abelian group. If U is open in X and $\bar{U}\subset \mathrm{int}A$, then the inclusion map $i:(X\setminus U, A\setminus U)\to(X,A)$ induces isomorphisms*

$$i_*^\infty : \check{H}_n^\infty(X\setminus U, A\setminus U) \to \check{H}_n^\infty(X,A;G)$$

and

$$j_\infty^* : \hat{H}_\infty^n(X,A;G) \to \hat{H}_\infty^n(X\setminus U, A\setminus U)$$

Theorem 1.8. *If X is a compact space, then for each $n\neq 0$,*

$$\check{H}_n^\infty(X;G) = 0 = \check{H}_\infty^n(X;G)$$

and

$$\hat{H}_0^\infty(X;G) = G = \hat{H}_\infty^0(X;G).$$

Theorem 1.9. *Let (X,A,B) be a triple of normal space X and its closed subsets A and B with $B\subset A$. Then the Čech border homology sequence*

$$\cdots \longleftarrow \check{H}_{n-1}^\infty(A,B;G) \xleftarrow{\bar{\partial}_n^\infty} \check{H}_n^\infty(X,A;G) \xleftarrow{\bar{j}_*^\infty} \check{H}_n^\infty(X,B;G) \xleftarrow{\bar{i}_*^\infty} \check{H}_n^\infty(A,B;G) \longleftarrow \cdots$$

and the Čech border cohomology sequence

$$\cdots \longrightarrow \hat{H}_\infty^{n-1}(A,B;G) \xrightarrow{\bar{\delta}_\infty^n} \hat{H}_\infty^n(X,A;G) \xrightarrow{\bar{j}_\infty^*} \hat{H}_\infty^n(X,B;G) \xrightarrow{\bar{i}_\infty^*} \hat{H}_\infty^n(A,B;G) \longrightarrow \cdots$$

are partially exact and exact, respectively. Here $\bar{\partial}_n^\infty = j_{n-1}^{'\infty}\cdot\partial_n^\infty$, $\bar{\delta}_\infty^n = \delta_\infty^n \cdot j_\infty^{'n-1}$ and $j_{n-1}^{'\infty}$, $j_\infty^{'n-1}$, $\bar{j}_\infty^*$, $\bar{j}_*^\infty$, and $\bar{i}_\infty^*$, $\bar{i}_*^\infty$ are the homomorphisms induced by the inclusion maps $j^{'}:A\to(A,B)$, $\bar{i}:(A,B)\to(X,B)$, $\bar{j}:(X,B)\to(X,A)$.

The proofs of formulated theorems are similar to the proofs of corresponding theorems of Eilenberg and Steenrod (see [17], Ch. IX, Theorem 3.4, Theorem 4.3, Theorem 4.4, Theorem 5.1, Theorem 6.1, Theorem 7.6) and hence they will be omitted.

2. On Čech (Co)homology Groups and Coefficients of Cyclicity of Remainders of Stone-Čech Compactifications

Now we are mainly interested in the following problem: how to characterize the Čech homology and cohomology groups, and coefficients of cyclicity of remainders of Stone-Čech compactifications of metrizable spaces.

Our main result about the connection between Čech (co)homology groups of remainders and Čech border (co)homology groups of spaces is the following theorem:

Theorem 2.1. *Let $(X, A) \in ob(\mathscr{M}_p^2)$ and let $(\beta X, \beta A)$ be the pair of Stone-Čech compactifications of X and A. Then*

$$\check{H}_n^f(\beta X \setminus X, \beta A \setminus A; G) = \check{H}_n^{\infty}(X, A; G)$$

and

$$\hat{H}_f^n(\beta X \setminus X, \beta A \setminus A; G) = \hat{H}_{\infty}^n(X, A; G).$$

Proof. Let $\alpha = \{\alpha_v\}_{v\in(V_\alpha, V_\alpha^{\beta A\setminus A})}$ and $\alpha' = \{\alpha'_w\}_{w\in(W_{\alpha'}, W_{\alpha'}^{\beta A\setminus A})}$ be the closed covers of pairs $(\beta X\setminus X, \beta A\setminus A)$ and $\alpha \geq \alpha'$. By Lemma 4 of [33] there exist open swellings $\beta_1 = \{\beta_v^1\}_{v\in(V_\alpha, V_\alpha^{\beta A\setminus A})}$ and $\beta' = \{\beta'_w\}_{w\in(W_{\alpha'}, W_{\alpha'}^{\beta A\setminus A})}$ of α and α' in βX, respectively. Assume that $\alpha_v \subseteq \alpha'_{w_k}$, $k = 1, 2, \ldots, m_v$. Let

$$\beta_v = \beta_v^1 \cap \Big(\bigcap_{k=1}^{m_v} \beta'_{w_k}\Big), \qquad v \in V_\alpha.$$

Note that $\alpha_v \subset \beta_v \subset \beta_v^1$ for each $v \in V_\alpha$. It is clear that $\beta = \{\beta_v\}_{v\in(V_\alpha, V_\alpha^A)}$ is a swelling of $\alpha = \{\alpha_v\}_{v\in(V_\alpha, V_\alpha^{\beta A\setminus A})}$ and $\beta \geq \beta'$.

The swelling in βX of closed cover α of $(\beta X \setminus X, \beta A \setminus A)$ is denoted by $s(\alpha)$. Let S be the set of all swellings of such kind.

Now define an order $\geq'$ in S. By definition,

$$s(\alpha') \geq' s(\alpha) \Leftrightarrow s(\alpha') \geq s(\alpha) \ \wedge \ \alpha' \geq \alpha.$$

It is clear that S is directed by $\geq'$. Let $((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)})$ be the nerve of $s(\alpha) \in S$ and $p_{s(\alpha)s(\alpha')}$ be the projection simplicial map induced by the refinement $\alpha' \geq \alpha$. Consider an inverse system

$$\{H_n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G), p_{s(\alpha)*}^{s(\alpha')}, S\}$$

and a direct system

$$\{H^n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G), p_{s(\alpha)}^{s(\alpha')*}, S\}.$$

Let $\varphi : S \to \mathrm{cov}_{\mathrm{f}}^{\mathrm{cl}}(\beta\mathrm{X} \setminus \mathrm{X}, \beta\mathrm{A} \setminus \mathrm{A})$ be the function in the set of closed finite covers of pair $(\beta X \setminus X, \beta A \setminus A)$ given by formula

$$\varphi(s(\alpha)) = \alpha, \qquad s(\alpha) \in S.$$

Note that φ is an increasing function and

$$\varphi(S) = \mathrm{cov}_{\mathrm{f}}^{\mathrm{cl}}(\beta\mathrm{X} \setminus \mathrm{X}, \beta\mathrm{A} \setminus \mathrm{A}).$$

For each index $s(\alpha) \in S$, we have

$$H_n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G) = H_n((\beta X \setminus X)_\alpha, (\beta A \setminus A)_\alpha; G)$$

and

$$H^n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G) = H^n((\beta X \setminus X)_\alpha, (\beta A \setminus A)_\alpha; G).$$

It is known that for normal spaces the Čech (co)homology groups based on finite open covers and on finite closed covers are isomorphic. By Theorems 3.14 and 4.13 of [17, Ch. VIII] we have

$$\check{H}_n^f(\beta X \setminus X, \beta A \setminus A; G) \approx \varprojlim \{H_n((\beta X\setminus X)_{s(\alpha)}, (\beta A\setminus A)_{s(\alpha)}; G), p_{s(\alpha)*}^{s(\alpha')}, S\} \tag{2.1}$$

and

$$\hat{H}_f^n(\beta X \setminus X, \beta A \setminus A; G) \approx \varinjlim \{H^n((\beta X\setminus X)_{s(\alpha)}, (\beta A\setminus A)_{s(\alpha)}; G), p_{s(\alpha)}^{s(\alpha')*}, S\}. \tag{2.2}$$

For each swelling $s(\alpha) = \{s(\alpha)_v\}_{v\in(V_\alpha, V_\alpha^{\beta A\setminus A})} \in S$, the family

$$s(\alpha) \wedge X = \{s(\alpha)_v \cap X\}_{v\in(V_\alpha, V_\alpha^{\beta A\setminus A})}$$

is a border cover of (X, A).

Let $\psi : S \to \mathrm{cov}_\infty(X, A)$ be the function defined by formula

$$\psi(s(\alpha)) = s(\alpha) \wedge X, \qquad s(\alpha) \in S.$$

The function ψ increases and $\psi(S)$ is a cofinal subset of $\mathrm{cov}_\infty(X,A)$. Note that the correspondence

$$((\beta X\setminus X)_{s(\alpha)},(\beta A\setminus A)_{s(\alpha)})\to(X_{s(\alpha)\wedge X},A_{s(\alpha)\wedge X}):s(\alpha)_v\to s(\alpha)_v\cap X,\quad v\in V_\alpha$$

induces an isomorphism of pairs of simplicial complexes. Thus, for each $s(\alpha)\in S$, we have the isomorphisms

$$H_n((\beta X\setminus X)_{s(\alpha)},(\beta A\setminus A)_{s(\alpha)};G)=H_n(X_{s(\alpha)\wedge X},A_{s(\alpha)\wedge X};G)$$

and

$$H^n((\beta X\setminus X)_{s(\alpha)},(\beta A\setminus A)_{s(\alpha)};G)=H^n(X_{s(\alpha)\wedge X},A_{s(\alpha)\wedge X};G).$$

By Theorems 3.15 and 4.13 of [17, Ch.VIII], we have

$$\check{H}_n^\infty(X,A;G)=\varprojlim\{H_n((\beta X\backslash X)_{s(\alpha)},(\beta A\backslash A)_{s(\alpha)};G),p_{s(\alpha)*}^{s(\alpha')},S\} \tag{2.3}$$

and

$$\hat{H}^n_\infty(X,A;G)=\varinjlim\{H_n((\beta X\backslash X,\beta A\backslash A)_{s(\alpha)};G),p_{s(\alpha)}^{s(\alpha')*},S\}. \tag{2.4}$$

From (2.1), (2.2), (2.3) and (2.4) it follows that

$$\check{H}_n^\infty(X,A;G)=\check{H}_n^f(\beta X\setminus X,\beta A\setminus A;G)$$

and

$$\hat{H}^n_\infty(X,A;G)=\hat{H}^n_f(\beta X\setminus X,\beta A\setminus A;G).$$ □

The cohomological coefficient of cyclicity $\eta_G(X,A)$ of pair (X,A) was defined and investigated by S. Novak [29] and M. F. Bokstein [10].

Now give the following definition and result.

Definition 2.2. Let G be an abelian group and n nonnegative integer. A border cohomological coefficient of cyclicity of pair $(X,A)\in ob(\mathscr{M}_p^2)$ with respect to G denoted by $\eta_G^\infty(X,A)$ is n, if $\hat{H}^m_\infty(X,A;G)=0$ for all $m>n$ and $\hat{H}^n_\infty(X,A;G)\neq 0$.

Finally, $\eta_G^\infty(X,A)=+\infty$ if for every m there is $n\geq m$ with $\hat{H}^n_\infty(X,A;G)\neq 0$.

Theorem 2.3. *For each pair* $(X,A)\in ob(\mathscr{M}_p^2)$,

$$\eta_G^\infty(X,A)=\eta_G(\beta X\setminus X,\beta A\setminus A).$$

Proof. This is an immediate consequence of Theorem 2.1. Indeed, let $\eta_G(\beta X\setminus X,\beta A\setminus A)=n$. Then for each $m>n$, $\hat{H}^m_f(\beta X\setminus X,\beta A\setminus A;G)=0$ and $\hat{H}^n_f(\beta X\setminus X,\beta A\setminus A;G)\neq 0$. From the isomorphism

$$\hat{H}^k_f(\beta X\setminus X,\beta A\setminus A;G)=\hat{H}^k_f(X,A;G)$$

it follows that $\hat{H}^m_\infty(X,A;G)=0$ for each $m>n$, and $\hat{H}^n_\infty(X,A;G)\neq 0$. Thus, $\eta_G^\infty(X,A)=n=\eta_G(\beta X\setminus X,\beta A\setminus A)$. □

3. On Cohomological Dimensions of Remainders of Stone-Čech Compactifications

The theory of cohomological dimension has become an important branch of dimension theory since A. Dranishnikov solved P. S. Alexandrov's problem [16] and he and other authors developed the theory of extension dimension.

Our next aim is to study some questions of theory of cohomological dimension. In particular, we in this section give a description of cohomological dimension of remainder of Stone-Čech compactification of metrizable space.

Following Y. Kodama (see the appendix of [28]) and T. Miyata [26] we give the following definition.

Definition 3.1. The border small cohomological dimension $d^f_\infty(X;G)$ of normal space X with respect to group G is defined to be the smallest integer n such that, whenever $m\geq n$ and A is closed in X, the homomorphism $i^*_{A,\infty}:\hat{H}^m_\infty(X;G)\to\hat{\mathrm{H}}^m_\infty(A;G)$ induced by the inclusion $i:A\to X$ is an epimorphism.

The border small cohomological dimension of X with coefficient group G is a function $d^f_\infty:\mathscr{N}\to\mathrm{N}\cup\{0,+\infty\}:X\to n$, where $d^f_\infty(X;G)=n$ and N is the set of all positive integers.

Theorem 3.2. *Let X be a metrizable space. Then the following equality*

$$d^f_\infty(X;G) = d_f(\beta X \setminus X;G)$$

holds, where $d_f(\beta X \setminus X;G)$ is the small cohomological dimension of $\beta X \setminus X$ (see [28], *p.* 199).

Proof. Let A be a closed subset of X. Assume that $d_f(\beta X \setminus X;G) = n$. Then for each $m \geq n$ the homomorphism $i^*_{\beta X\setminus X,\infty} : \hat{H}^m_f(\beta X \setminus X;G) \to \hat{H}^m_f(\beta A \setminus A;G)$ is an epimorphim. Consider the following commutative diagram

$$\begin{array}{ccc} \hat{H}^m_\infty(X;G) & \approx & \hat{H}^m_f(\beta X \setminus X;G) \\ {\scriptstyle i^*_{A,\infty}}\downarrow & & \downarrow{\scriptstyle i^*_{\beta A\setminus A}} \\ \hat{H}^m_\infty(A;G) & \approx & \hat{H}^m_f(\beta A \setminus A;G) \end{array} \tag{3.1}$$

It is clear that the homomorphim

$$i^*_{A,\infty} : \hat{H}^m_f(X;G) \to \hat{H}^m_f(A;G)$$

also is an epimorphim for each $m \geq n$. Thus,

$$d^f_\infty(X;G) \leq n = d_f(\beta X \setminus X;G). \tag{3.2}$$

Let $d^f_\infty(X;G) = n$. To see the reverse inequality, let B be a closed subset of $\beta X \setminus X$ and let $m \geq n$.

Consider an open in $\beta X \setminus X$ neighbourhood U of B. There exists an open neighbourhood V of B in $\beta X \setminus X$ such that $\bar{V}^{\beta X\setminus X} \subset U$. By Lemma 5 of [33] we can find an open set W in βX such that $W \cap (\beta X \setminus X) = V$ and $\bar{W}^{\beta X} \cap (\beta X \setminus X) \subseteq U$. Let $A = \bar{W}^{\beta X} \cap X$. It is clear that $\beta A = \bar{A}^{\beta X}$.

We have

$$\bar{W}^{\beta X} = \overline{W \cap X}^{\beta X} \subset \overline{\bar{W}^{\beta X} \cap X}^{\beta X} \subset \overline{\bar{W}^{\beta X}}^{\beta X} = \bar{W}^{\beta X}.$$

Consequently, $\beta A = \overline{\bar{W}^{\beta X} \cap X}^{\beta X} = \bar{W}^{\beta X}$. This shows that

$$B \subset \beta A \cap (\beta X \setminus X) \subset U.$$

Hence, we have

$$B \subset \beta A \setminus A \subset U.$$

Thus, for each closed set B of $\beta X \setminus X$ and its open neighbourhood U in $\beta X \setminus X$ there exists a closed subset A in X such that $B \subset \beta A \setminus A \subset U$.

Let $a \in H^n_f(B;G)$. There is a closed finite cover α of B such that an element $a_\alpha \in H^m(N(\alpha);G)$ represents the element a.

Using Lemma 4 of [33] we can find the swellings $\tilde{\alpha}$ and $\tilde{\tilde{\alpha}}$ of α in B and $\beta X \setminus X$, respectively, such that $\tilde{\tilde{\alpha}}_{|B} = \tilde{\alpha}$. Let U be the union of elements of $\tilde{\tilde{\alpha}}$. There is a closed set A of X with $B \subset \beta A \setminus A \subset U$. The nerves $N(\alpha)$, $N(\tilde{\alpha})$ and $N(\tilde{\tilde{\alpha}}_{|\beta A\setminus A})$ are isomorphic. We can assume that

$$H^n(N(\alpha);G) = H^n(N(\tilde{\alpha});G) = H^n(N(\tilde{\tilde{\alpha}}_{|\beta A\setminus A});G).$$

Hence, the element a_α also belongs to the group $H^n(N(\tilde{\tilde{\alpha}}_{|\beta A\setminus A});G)$. Consequently, it represents some element b of $\hat{H}^n(\beta A \setminus A;G)$.

The inclusion $i_A : A \to X$ induces an epimorphism $i^*_{A,\infty} : \hat{H}^m_\infty(X;G) \to \hat{H}^m(A;G)$. From diagram (3.1) it follows that the homomorphism $i^*_{\beta A\setminus A} : \hat{H}^m(\beta X \setminus X;G) \to \hat{H}^m(\beta A \setminus A;G)$ is an epimorphism. Consequently, there is an element $c \in \hat{H}^m(\beta X \setminus X;G)$ such that $i^*_{\beta A\setminus A}(c) = b$. The homomorphism $j^*_B : \hat{H}^m(\beta A \setminus A;G) \to \check{H}^m(B;G)$ induced by the inclusion $j_B : B \to \beta A \setminus A$ satisfies the condition $j^*_B(b) = a$. From equality $i_{\beta A\setminus A} \cdot j_B = i_B$ it follows that $i^*_B(c) = a$.

Thus the inclusion $i_B : B \to \beta X\setminus X$ also induces an epimorphism $i^*_B : \check{H}^m(\beta X\setminus X;G) \to \check{H}^m(B;G)$. Hence, we obtaine

$$d_f(\beta X \setminus X;G) \leq n = d^f_\infty(X;G). \tag{3.3}$$

From the inequalities (3.2) and (3.3) it follows that

$$d^f_\infty(X;G) = d_f(\beta X \setminus X;G).$$

□

Theorem 3.3. *Let A be a closed subspace of a normal space X. Then*

$$d^f_\infty(A;G) \le d^f_\infty(X;G).$$

Proof. Let B be an arbitrary closed subset of A and $j_B : B \to A$, $i_A : A \to X$ and $k_B : B \to X$ be the inclusion maps. Note that $k_B = i_A \cdot j_B$. The induced homomorphisms $k^*_{B,\infty} : \hat{H}^n_\infty(X;G) \to \hat{H}^n_\infty(B;G)$, $i^*_{A,\infty} : \hat{H}^n_\infty(X;G) \to \hat{H}^n_\infty(A;G)$ and $j^*_{B,\infty} : \hat{H}^n_\infty(A;G) \to \hat{H}^n_\infty(B;G)$ satisfy the equality $k^*_{B,\infty} = j^*_{B,\infty} \cdot i^*_{A,\infty}$.

Let $n = d^\infty_f(X;G)$. For each $m \ge n$, the homomorphisms $k^*_{B,\infty} : \hat{H}^m_\infty(X;G) \to \hat{H}^m_\infty(B;G)$ and $i^*_{A,\infty} : \hat{H}^m_\infty(X;G) \to \hat{H}^m_\infty(A;G)$ are epimorphisms. Hence, the homomorphism $j^*_{B,\infty} : \hat{H}^m_\infty(A;G) \to \hat{H}^m_\infty(B;G)$ is also an ephimorphism for each $m \ge n$. Thus, $d^\infty_f(A;G) \le n = d^\infty_f(X;G)$. □

Corollary 3.4. *For each closed subspace A of a metrizable space X,*

$$d^f_\infty(A;G) \le d_f(\beta X \setminus X;G).$$

Definition 3.5. The border large cohomological dimension $D^f_\infty(X;G)$ of normal space X with respect to group G is defined to be the largest integer n such that $\hat{H}^n_\infty(X,A;G) \ne 0$ for some closed set A of X.

The border large cohomological dimension of X with coefficient group G is a function $D^f_\infty : \mathscr{N} \to \mathrm{N} \cup \{0, +\infty\} : X \to n$, where $D^f_\infty(X;G) = n$ and N is the set of all positive integers.

Theorem 3.6. *For each metrizable space X, one has*

$$D^f_\infty(X;G) = D_f(\beta X \setminus X;G),$$

where $D_f(\beta X \setminus X;G)$ is the large cohomological dimension of $\beta X \setminus X$ (see [28], *p.* 199).

Proof. Let $D_f(\beta X \setminus X;G) = n$. Consider an arbitrary closed subspace A of X. The remainder $\beta A \setminus A$ is a closed subset of $\beta X \setminus X$. By the assumption, we have $\hat{H}^m(\beta X \setminus X, \beta A \setminus A;G) = 0$ for each $m > n$. Theorem 2.1 implies that $\hat{H}^m_\infty(X,A;G) = 0$ for each $m > n$ and $A \subset X$. Thus,

$$D^f_\infty(X;G) \le n = D_f(\beta X \setminus X;G). \tag{3.4}$$

Let $D^f_\infty(X;G) = n$. Assume that $D_f(\beta X \setminus X;G) = n_1 > n$. Then there is a closed set B in $\beta X \setminus X$ such that $\hat{H}^{n_1}(\beta X \setminus X, B;G) \ne 0$. Using Lemma 4 of [33] and the proof of Theorem 3.2 we can show that there is a closed set A of X such that $B \subset \beta A \setminus A$, and $\hat{H}^{n_1}(\beta X \setminus X, \beta A \setminus A;G) \ne 0$. By Theorem 2.1 $\hat{H}^{n_1}_\infty(X,A;G) \ne 0$. But it is not possible because $D^\infty_f(X;G) = n$. Therefore, $n_1 \le n$. Thus,

$$D_f(\beta X \setminus X;G) \le n = D^f_\infty(X;G). \tag{3.5}$$

The inequalities (3.4) and (3.5) imply

$$D^f_\infty(X;G) = D_f(\beta X \setminus X;G).$$ □

Theorem 3.7. *If A is a closed subset of normal space X, then*

$$D^f_\infty(A;G) \le D^f_\infty(X;G).$$

Proof. By Theorem 1.9, for each closed set B of A, there is the exact Čech border cohomological sequence

$$\cdots \longrightarrow \hat{H}^{m-1}_\infty(A;G) \xrightarrow{\bar{\delta}^m_\infty} \hat{H}^m_\infty(X,A;G) \xrightarrow{\bar{j}^*_\infty} \hat{H}^m_\infty(X,B;G) \xrightarrow{\bar{i}^*_\infty} \hat{H}^m_\infty(A,B;G) \longrightarrow \cdots$$

It is clear that, if $m > D^f_\infty(X;G)$, then $\hat{H}^m_\infty(X,A;G) = \hat{H}^m_\infty(X,B;G) = 0$. Consequently, $\hat{H}^m_\infty(A,B;G) = 0$. Thus, we have

$$D^f_\infty(A;G) \le D^f_\infty(X;G).$$ □

Corollary 3.8. *For each closed subspace A of metrizable space X, one has*

$$D^f_\infty(A;G) \le D_f(\beta X \setminus X;G).$$

Theorem 3.9. *If X is a normal space then*

$$d^f_\infty(X;G) \leq D^f_\infty(X;G).$$

Proof. Let A be a closed subset of normal space X. Consider the exact Čech border cohomological sequence of pair (X, A)

$$\cdots \longrightarrow \hat{H}^{m-1}_\infty(A,B;G) \xrightarrow{\delta^m_\infty} \hat{H}^m_\infty(X,A;G) \xrightarrow{j^*_\infty} \hat{H}^m_\infty(X;G) \xrightarrow{i^*_\infty} \hat{H}^m_\infty(A;G) \longrightarrow \cdots$$

Let $m > D^f_\infty(X;G)$. Note that $j^*_\infty : \hat{H}^{m-1}_\infty(X;G) \to \hat{H}^{m-1}_\infty(A;G)$ is an epimorphism. Hence,

$$d^f_\infty(X;G) \leq D^f_\infty(X;G).$$

□

Corollary 3.10. *For each metrizable space X, one has*

$$d_f(\beta X \setminus X;G) \leq D^f_\infty(X;G)$$

and

$$d^f_\infty(X;G) \leq D_f(\beta X \setminus X;G).$$

Remark 3.11. The results of this paper also hold for spaces satisfying the compact axiom of countability. Recall that a space X satisfies the compact axiom of countability if for each compact subset $B \subset X$ there exists a compact subset $B^{'} \subset X$ such that $B \subset B^{'}$ and $B^{'}$ has a countable or finite fundamental systems of neighbourhoods (see Definition 4 of [33], p.143). A space X is complete in the sense of Čech if and only if it is G_δ type set in some compact extension. Each locally metrizable spaces, complete in the seance of Čech spaces [15] and locally compact spaces satisfy the compact axiom of countability.

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