Investigation of mixed boundary value problem of the piezo-elasticity theory by the alternative potential method

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Abstract

We study a mixed boundary value problem (BVP) arising in the linear theory of piezoelasticity. The Dirichlet boundary conditions, consisting of the prescribed displacement vector and electric potential, are imposed on one portion of the boundary of the body, while the Neumann boundary conditions, namely the mechanical traction vector and the normal component of the electric displacement, are prescribed on the complementary portion of the boundary.

Employing the so-called alternative potential method, we look for a solution to the mixed BVP in the form of a linear combination of single- and double-layer potentials, with densities supported on the Dirichlet and Neumann parts of the boundary, respectively. This approach reduces the original mixed BVP to a system of boundary integral (pseudodifferential) equations that involves neither a Steklov–Poincaré type operator nor the inverse of the single-layer boundary integral operator, both of which are not available in explicit form. Moreover, within this framework, we do not need extensions of the Dirichlet or Neumann boundary data to the entire boundary. The right-hand sides of the resulting boundary integral equations coincide directly with the prescribed Dirichlet and Neumann data of the original problem. We establish the invertibility of the associated matrix integral operator in suitable Bessel potential and Besov spaces and, on this basis, prove the corresponding existence and uniqueness results for the mixed interior and exterior BVPs under consideration. Finally, we investigate the regularity properties of solutions in a closed neighbourhood of the collision curve where the different types boundary conditions meet, and we obtain almost optimal Hölder continuity results. The alternative potential method employed in this paper, based on an efficient representation of solutions as linear combinations of layer potentials, provides a robust foundation for the development of efficient numerical algorithms for mixed boundary value problems in the theory of piezoelasticity.

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1 Introduction

Piezoelectric elastic materials play a crucial role in measurement and control devices, electromechanical converters (transducers), piezoelectric stack actuators, and a wide range of related technological applications. The first linear mathematical model of an elastic medium accounting for the interaction between mechanical and electric fields was developed by W. Voigt [Vo1], who derived the corresponding system of differential equations. Subsequently, more refined models incorporating a polarization vector and its gradient were proposed by R. Toupin, R. Mindlin, L. Knopoff, S. Kaliski, and J. Petikiewicz (see [To1], [Mi1], [Mi2], [Mi3]). For an extensive collection of examples and further bibliographic references, we refer the reader to [BCNS1], [Ka], [La1], [No1], [Pa1], [Qi1], and the references therein.

In this paper we study a mixed boundary value problem (BVP) arising in the linear theory of piezoelasticity. The Dirichlet boundary conditions, consisting of the prescribed displacement vector and electric potential, are imposed on one portion of the boundary of the body, while the Neumann boundary conditions, namely the mechanical traction vector and the normal component of the electric displacement, are prescribed on the complementary portion of the boundary.

Employing the so-called alternative potential method, we look for a solution to the mixed BVP in the form of a linear combination of single- and double-layer potentials, with densities supported on the Dirichlet and Neumann parts of the boundary, respectively. This approach reduces the original mixed BVP to a system of boundary integral (pseudodifferential) equations that does not involve either a Steklov–Poincaré type operator or the inverse of the single-layer boundary integral operator, both of which are not available in explicit form. Moreover, within this framework, no extension of the Dirichlet or Neumann boundary data to the entire boundary is required. The right-hand sides of the resulting boundary integral equations coincide directly with the prescribed Dirichlet and Neumann data of the original problem. We establish the invertibility of the associated matrix integral operator in suitable Bessel potential and Besov spaces and, on this basis, prove corresponding existence and uniqueness results for the mixed BVP under consideration. Finally, we investigate the regularity properties of solutions in a neighbourhood of the collision curve where the different types boundary conditions meet, and we obtain almost optimal Hölder continuity results.

The alternative potential method employed in this work, based on an efficient representation of the solution as a linear combination of layer potentials, provides a solid foundation for the development of efficient numerical algorithms for mixed boundary value problems in the theory of piezoelasticity.

It should be noted that the differential operator arising in the theory of piezoelasticity is strongly elliptic but not formally self-adjoint. The present paper constitutes the first attempt to develop an alternative potential method for mixed boundary value problems associated with non-selfadjoint operators. For several elliptic self-adjoint operators, the alternative potential method has been studied in various function spaces (see, for example, [McLean], [NT1], [NT2], and [NMT1]).

The paper is organized as follows. In Section 2, we present the governing equations of the linear theory of piezoelasticity, introduce the associated matrix partial differential operators and generalized boundary stress operators, formulate the interior and exterior mixed boundary value problems and prove the uniqueness theorems using Green's identities. In Section 3, using the alternative potential method, we reduce the mixed BVPs to a system of boundary integral equations and show that the corresponding matrix integral operator is coercive in appropriate function spaces for smooth and Lipschitz boundary surfaces. On the basis of these results, we prove the existence theorems for solutions to the interior and exterior mixed BVPs in Sobolev-Slobodetskii, Bessel potential and Besov spaces, and establish Hölder continuity of solutions in a closed domain $\overline{\Omega}$ occupied by the piezoelastic body. Finally, in Section 4, we analyse the solvability of the mixed BVPs in the Bessel potential space $H^1_p(\Omega)$ and specify the admissible range for the parameter p.

For the convenience of the reader and to ensure the self-sufficiency of the article, we present several well-known results together with some auxiliary new results in Appendices A, B, and C, which are used extensively in the main text.

2 Basic field equations, Green's identities, and formulation of the mixed BVPs

Let a piezoelectric anisotropic elastic body occupy a bounded three-dimensional domain $\Omega = \Omega^+ \subset \mathbb{R}^3$ with the simply connected boundary $S = \partial\Omega$, which is divided into two nonempty sub-manifolds S_D and S_N, $S = \overline{S}_D \cup \overline{S}_N$. Denote by ℓ the common boundary curve of these sub-manifolds: $\ell = \partial S_D = \partial S_N$. By $n = (n_1, n_2, n_3)$ we denote the outward unit normal vector to S. Denote by Ω^- the complement of Ω to the whole space: $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}$. The symbols $\{\,\cdot\,\}^+$ and $\{\,\cdot\,\}^-$ stand for the one-sided limits (one-sided traces) on S from Ω^+ and Ω^- respectively.

By L_p, W_p^r, H_p^s, and $B_{p,q}^s$ (with $r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) we denote, respectively, the Lebesgue, Sobolev-Slobodetskii, Bessel potential, and Besov spaces of real or complex valued functions (see, e.g., [Tr1], [Tr2], [BL]). Recall that $H_2^r = W_2^r = B_{2,2}^r$, $H_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$, and $H_p^k = W_p^k$, for any $r \geq 0$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

Let us introduce also the spaces:

$$
\begin{aligned}
&\widetilde{H}_p^s(S_1) := \{f \,:\, f \in H_p^s(S),\ \operatorname{supp} f \subset \overline{S}_1\},\\
&\widetilde{B}_{p,q}^s(S_1) := \{f \,:\, f \in B_{p,q}^s(S),\ \operatorname{supp} f \subset \overline{S}_1\},\\
&H_p^s(S_1) := \{r_{S_1} f \,:\, f \in H_p^s(S)\},\\
&B_{p,q}^s(S_1) := \{r_{S_1} f \,:\, f \in B_{p,q}^s(S)\,\},
\end{aligned}
$$

where $S_1 \in \{S_D, S_N\}$ and r_{S_1} is the restriction operator onto S_1.

Throughout the paper we assume that the boundary S, the sub-manifolds S_D and S_N, and the curve ℓ are either sufficiently smooth (C^∞ say) or belong to Lipschitz classes. The results for domains with Lipschitz surfaces will be always singled out and formulated separately in fixed function spaces with $p = 2$, in contrast with the smooth cases where we consider problems in wider classes of function spaces.

Remark 2.1 *Let a function f be defined on an open proper submanifold S_1 of a closed manifold S without boundary. Let $f \in B_{p,q}^s(S_1)$ and $\widetilde{f}$ be the extension of f by zero to $S \setminus S_1$.*

If the extension preserves the space, that is, if $\widetilde{f} \in \widetilde{B}^s_{p,q}(S_1)$, then we write $f \in \widetilde{B}^s_{p,q}(S_1)$ instead of $f \in r_{S_1} \widetilde{B}^s_{p,q}(S_1)$, when it does not lead to misunderstanding.

Note that $\widetilde{B}^s_{p,q}(S_1)$ and $B^{-s}_{p',q'}(S_1)$ with $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$ are dual spaces. Similarly, $\widetilde{H}^s_p(S_1)$ and $H^{-s}_{p'}(S_1)$ are dual spaces as well (for details see [McLean], [Mur], [Tr1], [Tr2]).

Therefore, for functions $f \in B^{-s}_{p',q'}(S_1)$ and $g \in \widetilde{B}^s_{p,q}(S_1)$ (resp., $f \in H^{-s}_{p'}(S_1)$ and $g \in \widetilde{H}^s_p(S_1)$) the duality relation $\langle f, g \rangle_{S_1}$ is well defined and it generalizes the classical L_2-inner product,

$$\langle f, g \rangle_{S_1} = \int\limits_{S_1} f(x)\, \overline{g(x)}\, dS_1 \quad \textit{for} \quad f, g \in L_2(S_1),$$

where the over bar denotes complex conjugation.

2.1 Field equations

The piezoelastic field $U = (u, \varphi)^\top =: (u_1, u_2, u_3, u_4)^\top$ in Ω is described by the displacement vector $u = (u_1, u_2, u_3)^\top$ and the electric potential function $\varphi = u_4$.

In the theory of elasticity of piezoelectric bodies the entries of mechanical stress tensor $\sigma_{ij}(U)$, the components of electric displacement vector $D(U)$, and the electric field vector $E(\varphi)$, respectively, read as:

$$\sigma_{ij}(U) = c_{ijlk}\, s_{lk}(u) + e_{kij}\, \partial_k \varphi, \quad i, j = 1, 2, 3, \tag{2.1}$$

$$D_i(U) = e_{ikl}\, s_{kl}(u) - \varepsilon_{ik}\, \partial_k \varphi, \qquad i = 1, 2, 3, \tag{2.2}$$

$$E_i = -\partial_i \varphi, \qquad i = 1, 2, 3,$$

where $\partial_k = \frac{\partial}{\partial x_k}$, $\partial_x = (\partial_1, \partial_2, \partial_3)$, $s_{lk}(u) = \frac{1}{2}(\partial_l u_k + \partial_k u_l)$, $k, l = 1, 2, 3$, are elements of the mechanical strain tensor. Here and in what follows, the summation over repeated indexes is assumed from 1 to 3.

The components of the mechanical stress vector acting on a surface element with the normal $n = (n_1, n_2, n_3)$ are evaluated by the formulas

$$\sigma_{ij}\, n_i = c_{ijlk}\, n_i\, \partial_l u_k + e_{lij}\, n_i\, \partial_l \varphi \;\text{ for }\; j = 1, 2, 3,$$

while the normal component of the electric displacement vector (with opposite sign) is

$$-D_i\, n_i = -e_{ikl}\, n_i\, \partial_l u_k + \varepsilon_{il}\, n_i\, \partial_l \varphi.$$

Here c_{ijlk}, e_{kjl}, and ε_{lk} are the elastic, piezoelectric, and dielectric (permittivity) real constants, respectively, satisfying the symmetry conditions:

$$c_{ijlk} = c_{lkij} = c_{ijkl}, \quad e_{kjl} = e_{klj}, \quad \varepsilon_{lk} = \varepsilon_{kl}, \quad i,j,l,k = 1,2,3. \tag{2.3}$$

Moreover, for arbitrary real $\xi_{ij} = \xi_{ji}$ and $\xi = (\xi_1, \xi_2, \xi_j) \in \mathbb{R}^3$ the following inequalities hold:

$$c_{ijlk}\, \xi_{ij}\, \xi_{lk} \geq \delta_1\, \xi_{lk}\, \xi_{lk}, \quad \varepsilon_{lk}\, \xi_l\, \xi_k \geq \delta_2\, \xi_l\, \xi_l, \tag{2.4}$$

where δ_1 and δ_2 are some positive constants depending only on the elastic, piezoelectric and dielectric material parameters.

Due to the symmetry conditions (2.3) , with the help of inequalities (2.4), we easily derive

$$\begin{gathered} c_{rjkl}\, \zeta_{rj}\, \overline{\zeta_{kl}} \geq \delta_1\, \zeta_{kl}\, \overline{\zeta_{kl}}, \quad \varepsilon_{lk}\, \zeta_k\, \overline{\zeta_j} \geq \delta_2\, |\zeta|^2, \\ \text{for all} \quad \zeta_{kj} = \zeta_{jk} \in \mathbb{C} \quad \text{and for all} \quad \zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3. \end{gathered} \tag{2.5}$$

The mechanical static equilibrium equations and the static electric field equation, respectively, in the case of absence mass force and electric charge read as

$$\partial_i \sigma_{ij} = 0, \quad j = 1,2,3, \tag{2.6}$$

$$-\partial_i D = 0. \tag{2.7}$$

Using relations (2.1)-(2.2) from equations (2.6) and (2.7) we obtain the system of second order partial differential equations of piezoelasticity with respect to the vector $U = (u, \varphi)^\top$ (see, e.g., [No1]):

$$\begin{aligned} &c_{ijlk}\, \partial_i\, \partial_l\, u_k(x) + e_{pjq}\, \partial_p\, \partial_q\, \varphi(x) = 0, \quad j = 1,2,3, \\ &-e_{plq}\, \partial_p\, \partial_q\, u_l(x) + \varepsilon_{pq}\, \partial_p\, \partial_q\, \varphi(x) = 0. \end{aligned}$$

These equations can be rewritten in matrix form

$$A(\partial_x)\, U(x) = 0,$$

where

$$\begin{aligned} &A(\partial_x) = [A_{jk}(\partial_x)]_{4\times 4}\,, \\ &A_{jk}(\partial_x) = c_{ijlk}\, \partial_i\, \partial_l, \quad j,k = 1,2,3, \\ &A_{j4}(\partial_x) = -A_{4j}(\partial_x) = e_{pjq}\, \partial_p\, \partial_q, \quad j = 1,2,3, \\ &A_{44}(\partial_x) = \varepsilon_{pq}\, \partial_p\, \partial_q. \end{aligned}$$

Note that for arbitrary real $\xi \in \mathbb{R}^3$ and for all complex $\eta \in \mathbb{C}^4$ the following inequality holds

$$\mathrm{Re}\big(A(\xi)\,\eta\,\cdot\,\eta\big) \geq \delta_3\,|\xi|^2\,|\eta|^2,$$

where the central dot denotes the scalar product of two vectors in the complex vector space $\mathbb{C}^N$, i.e., $a \cdot b = \sum_{j=1}^{N} a_j\,\overline{b_j}$ for $a, b \in \mathbb{C}^N$, and δ_3 is a positive constant depending only on the elastic, piezoelectric, and dielectric constants. This inequality shows that $A(\partial_x)$ is a strongly elliptic but not formally self-adjoint differential operator.

Denote by $A^*(\partial_x)$ the differential operator formally adjoint to $A(\partial_x)$, $A^*(\partial_x) = A^\top(-\partial_x)$. Evidently,

$$A^*(\partial_x) = \big[A^*_{jk}(\partial_x)\big]_{4\times4} = [A_{kj}(-\partial_x)]_{4\times4} = [A_{kj}(\partial_x)]_{4\times4}\ , \tag{2.8}$$

$$\begin{aligned}
&A^*_{jk}(\partial_x) = A_{jk}(\partial_x) = c_{ijlk}\,\partial_i\,\partial_l && \text{for} \quad j,k = 1,2,3,\\
&A^*_{j4}(\partial_x) = -A_{j4}(\partial_x) = -e_{pjq}\,\partial_p\,\partial_q && \text{for} \quad j = 1,2,3,\\
&A^*_{4j}(\partial_x) = -A_{4j}(\partial_x) = e_{pjq}\,\partial_p\,\partial_q && \text{for} \quad j = 1,2,3,\\
&A^*_{44}(\partial_x) = A_{44}(\partial_x) = \varepsilon_{pq}\,\partial_p\,\partial_q.
\end{aligned}$$

Now, let us introduce the generalized boundary stress operator associated with the operator $A(\partial_x)$

$$\mathcal{T}(\partial, n) = [\,\mathcal{T}_{jk}(\partial, n)\,]_{4\times4}, \tag{2.9}$$

where

$$\begin{aligned}
&\mathcal{T}_{jk}(\partial_x, n) = c_{ijlk}\,n_i\,\partial_l && \text{for} \quad 1 \leq j,k \leq 3,\\
&\mathcal{T}_{j4}(\partial_x, n) = e_{lij}\,n_i\,\partial_l && \text{for} \quad 1 \leq j \leq 3,\\
&\mathcal{T}_{4k}(\partial_x, n) = -e_{ikl}\,n_i\,\partial_l && \text{for} \quad 1 \leq k \leq 3,\\
&\mathcal{T}_{44}(\partial_x, n) = \varepsilon_{il}\,n_i\,\partial_l.
\end{aligned} \tag{2.10}$$

The generalised stress operator applied to a vector $U = (u, \varphi)^\top$ gives a four-dimensional vector

$$\mathcal{T}(\partial_x, n)\,U = \big[\,\sigma_{i1}\,n_i, \sigma_{i2}\,n_i, \sigma_{i3}\,n_i,\ -D_i\,n_i\,\big]^\top,$$

where the first three components correspond to the mechanical stress vector, while the fourth one is the normal component of the electric displacement vector (with opposite sign).

We consider the following mixed interior boundary value problem of the piezoelasticity theory: Find a real valued vector function $U \in [W^1_p(\Omega)]^4 = [H^1_p(\Omega)]^4$ satisfying the differential equation

$$A(\partial_x)\,U(x)=0 \ \text{ in } \ \ \Omega, \tag{2.11}$$

the Dirichlet type boundary condition on S_D

$$\{U(x)\}^{+}=f(x), \quad x\in S_D, \tag{2.12}$$

and the Neumann type boundary condition on S_N

$$\{\mathcal{T}(\partial,n)U(x)\}^{+}=F(x), \quad x\in S_N, \tag{2.13}$$

where

$$f\in\big[B_{p,p}^{1-\frac{1}{p}}(S_D)\big]^4, \quad F\in\big[B_{p,p}^{-\frac{1}{p}}(S_N)\big]^4. \tag{2.14}$$

The mixed exterior boundary value problem is formulated quite similarly: Find a vector function $U\in[W^1_{p,loc}(\Omega^-)]^4=[H^1_{p,loc}(\Omega^-)]^4$ satisfying the differential equation

$$A(\partial_x)\,U(x)=0 \ \text{ in } \ \ \Omega^-, \tag{2.15}$$

the Dirichlet type boundary condition on S_D

$$\{U(x)\}^{-}=f(x), \quad x\in S_D, \tag{2.16}$$

the Neumann type boundary condition on S_N

$$\{\mathcal{T}(\partial,n)U(x)\}^{-}=F(x), \quad x\in S_N, \tag{2.17}$$

and the decay condition at infinity

$$U(x)=\mathcal{O}(|x|^{-1}) \quad \text{for} \ \ |x|\to\infty, \tag{2.18}$$

where the boundary data satisfy the conditions (2.14).

2.2 Green's formulas and uniqueness theorem

We have the following Green's formulas for smooth real valued vector functions in the domain Ω (see, e.g., [BG1], [BCNS1], [NBC1]):

$$\int_{\Omega} [\, A(\partial)U \cdot V + E(U,V)\,]\, dx = \int_{\partial\Omega} \{\mathcal{T}(\partial,n)U\}^{+} \cdot \{V\}^{+}\, dS, 2.24 \tag{2.19}$$

$$\int_{\Omega} [\, A(\partial)U \cdot V - U \cdot A^{*}(\partial)V\,]\, dx$$
$$= -\int_{\partial\Omega} \Big[\{\mathcal{T}(\partial,n)U\}^{+} \cdot \{V\}^{+} - \{U\}^{+} \cdot \{\widetilde{\mathcal{T}}(\partial,n)V\}^{+} \Big]\, dS, \tag{2.20}$$

where $U = (u_1, u_2, u_3, \varphi)^{\top}$ and $V = (v_1, v_2, v_3, \psi)^{\top}$ are real valued vector functions of the space $\big[C^2(\overline{\Omega})\big]^4$, $E\,(U,V)$ is the bilinear form associated with the density of the potential energy,

$$E\,(U,V) = c_{ijlk}\, \partial_i u_j\, \partial_l v_k + e_{pjq}\, \partial_p \varphi\, \partial_q v_j - e_{pjq}\, \partial_q u_j\, \partial_p \psi + \varepsilon_{pq}\, \partial_p \varphi\, \partial_q \psi, \tag{2.21}$$

$\widetilde{\mathcal{T}}(\partial,n)$ is the boundary differential operator associated with the adjoint operator $A^{*}(\partial_x)$,

$$\widetilde{\mathcal{T}}(\partial,n) = [\, \widetilde{\mathcal{T}}_{jk}(\partial,n)\,]_{4\times 4}, \tag{2.22}$$

with

$$\begin{aligned}
&\widetilde{\mathcal{T}}_{jk}(\partial,n) = \mathcal{T}_{jk}(\partial,n) = c_{ijlk}\, n_i\, \partial_l && \text{for} \quad 1 \le j,k \le 3,\\
&\widetilde{\mathcal{T}}_{j4}(\partial,n) = -\mathcal{T}_{j4}(\partial,n) = -e_{lij}\, n_i\, \partial_l && \text{for} \quad 1 \le j \le 3,\\
&\widetilde{\mathcal{T}}_{4k}(\partial,n) = -\mathcal{T}_{4k}(\partial,n) = e_{ikl}\, n_i\, \partial_l && \text{for} \quad 1 \le k \le 3,\\
&\widetilde{\mathcal{T}}_{44}(\partial,n) = \mathcal{T}_{44}(\partial,n) = \varepsilon_{il}\, n_i\, \partial_l.
\end{aligned} \tag{2.23}$$

The above Green formulas (2.19) and (2.20) are also valid for Lipschitz domains and for vector-functions (for details see [Ne1], [McLean])

$$U \in \big[W^1_p(\Omega)\big]^4, \quad V \in \big[W^1_q(\Omega)\big]^4,$$

satisfying the inclusions

$$A(\partial)U \in \big[L_p(\Omega)\big]^4, \quad A^{*}(\partial)V \in \big[L_q(\Omega)\big]^4, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

In this generalized case, when the integrals appearing in (2.19)-(2.20) do not exist in the usual sense, we understand them as the duality brackets between the corresponding functional spaces, e.g., between the mutually adjoint spaces $\big[B^{s}_{p,p}(S)\big]^4$ and $\big[B^{-s}_{q,q}(S)\big]^4$. It will be clear from the context when we shall refer to an ordinary integral or to a functional relation.

Remark 2.2 *For complex valued vector functions U and V the Green formulas* (2.19)-(2.20) *hold with*

$$E(U,V) = c_{ijlk}\,\partial_i u_j\,\overline{\partial_l v_k} + e_{pjq}\,\partial_p\varphi\,\overline{\partial_q v_j} - e_{pjq}\,\partial_q u_j\,\overline{\partial_p\psi} + \varepsilon_{pq}\,\partial_p\varphi\,\overline{\partial_q\psi}. \tag{2.24}$$

The following uniqueness theorem is true.

Theorem 2.3 *Let Ω be a domain with Lipschitz boundary. The mixed BVP* (2.11)-(2.14) *possesses at most one solution for $p = 2$.*

Proof. Due to the linearity of the problem it is sufficient to show that the homogeneous mixed BVP (with $f = 0$ and $F = 0$) has only the trivial solution. Let $U \in \left[H_2^1(\Omega)\right]^4$ be a solution to the homogeneous mixed BVP under consideration and write the Green identity (2.19) for U and $V = U$,

$$\int\limits_{\Omega} E(U,U)\,dx = \big\langle \{\mathcal{T}(\partial,n)U\}^+ \,,\, \{U\}^+ \big\rangle_S \tag{2.25}$$

$$= \big\langle \{\mathcal{T}(\partial,n)U\}^+ \,,\, \{U\}^+ \big\rangle_{S_D} + \big\langle \{\mathcal{T}(\partial,n)U\}^+ \,,\, \{U\}^+ \big\rangle_{S_N} = 0. \tag{2.26}$$

Since, $\{\mathcal{T}(\partial,n)U\}^+ \in \big[\widetilde{H}_2^{-\frac{1}{2}}(S_D)\big]^4$ and $\{U\}^+ \in \big[\widetilde{H}_2^{\frac{1}{2}}(S_N)\big]^4$ the duality brackets in (2.26) are well defined.

By virtue of the symmetry conditions (2.3) and inequalities (2.4) from (2.21) we have

$$\begin{aligned} E(U,U) &= c_{ijlk}\,\partial_i u_j\,\partial_l u_k + \varepsilon_{pq}\,\partial_p\varphi\,\partial_q\varphi = c_{ijlk}\,s_{ij}(u)\,s_{lk}(u) + \varepsilon_{pq}\,\partial_p\varphi\,\partial_q\varphi, \\ &\geqslant \delta_1 s_{ij}(u)\,s_{ij}(u) + \delta_2\,\partial_p\varphi\partial_p\varphi. \end{aligned} \tag{2.27}$$

In accordance with (2.26) and (2.27) we deduce $E(U,U) = 0$, implying that $s_{ij}(u) = 0$ and $\partial_p\varphi = 0$ in Ω for $i,j,p = 1,2,3$. Consequently, $u = (u_1,u_2,u_3)^\top$ is a rigid displacement vector and φ is a constant in Ω. Therefore, the Dirichlet type condition on S_D yields $u = 0$ and $\varphi = 0$ in Ω, which completes the proof. ■

3 Reduction to the boundary pseudodifferential equations and existence theorems

3.1 Reduction to the boundary pseudodifferential equations

First, we consider the interior mixed BVP (2.11)-(2.13) for $p = 2$ and look for a solution U as a linear combination of the single- and double-layer potentials with densities, respectively, supported on the Dirichlet and Neumann parts of the boundary:

$$U(x) = -V(h)(x) + W(g)(x), \quad x \in \Omega, \tag{3.1}$$

where V and W are the single- and double-layer potentials defined by relations (5.2) and (5.3) respectively (see Appendix A), and the unknown density vector functions satisfy the inclusions

$$h = (h_1, h_2, h_3, h_4)^\top \in \big[\widetilde{H}_2^{-\frac{1}{2}}(S_D)\big]^4, \quad g = (g_1, g_2, g_3, g_4)^\top \in \big[\widetilde{H}_2^{\frac{1}{2}}(S_N)\big]^4. \tag{3.2}$$

Using the properties of layer potentials described in Theorem 5.1(ii) and the mixed boundary conditions (2.12)-(2.13) along with the inclusions (3.2), we obtain the following system of integral (pseudodifferential) equations for the unknown density vector functions h and g:

$$-\mathcal{H}h + \mathcal{K}g = f \quad \text{on } S_D, \tag{3.3}$$

$$-\widetilde{\mathcal{K}}h + \mathcal{L}g = F \quad \text{on } S_N, \tag{3.4}$$

where $\mathcal{H}$, $\mathcal{K}$, $\widetilde{\mathcal{K}}$, and $\mathcal{L}$ are the boundary integral operators generated by the single- and double-layer potentials given by formulas (5.4)-(5.7) respectively, and the right-hand side vector functions satisfy the inclusions:

$$f \in \big[B_{2,2}^{\frac{1}{2}}(S_D)\big]^4 = \big[H_2^{\frac{1}{2}}(S_D)\big]^4, \quad F \in \big[B_{2,2}^{-\frac{1}{2}}(S_N)\big]^4 = \big[H_2^{-\frac{1}{2}}(S_N)\big]^4.$$

Let us introduce the matrix pseudodifferential operator generated by the left-hand side expressions of the system (3.3)-(3.4),

$$\mathcal{B} = \begin{bmatrix} -r_{S_D}\mathcal{H} & r_{S_D}\mathcal{K} \\ -r_{S_N}\widetilde{\mathcal{K}} & r_{S_N}\mathcal{L} \end{bmatrix}, \tag{3.5}$$

where r_{S_D} and r_{S_N} are restriction operators on S_D and S_N respectively.

The system (3.3)-(3.4) can be then rewritten in matrix form

$$\mathcal{B}\,\Phi = Q, \tag{3.6}$$

where

$$\Phi = \big(h, g\big)^\top \in \widetilde{\mathbb{H}}, \quad Q = \big(f, F\big)^\top \in \mathbb{H},$$

with

$$\widetilde{\mathbb{H}} = \big[\widetilde{H}_2^{-\frac{1}{2}}(S_D)\big]^4 \times \big[\widetilde{H}_2^{\frac{1}{2}}(S_N)\big]^4, \qquad \mathbb{H} = \big[H_2^{\frac{1}{2}}(S_D)\big]^4 \times \big[H_2^{-\frac{1}{2}}(S_N)\big]^4.$$

Evidently, $\widetilde{\mathbb{H}}$ and $\mathbb{H}$ are mutually adjoint spaces.

The norms in the spaces $\widetilde{\mathbb{H}}$ and $\mathbb{H}$ are defined as:

$$\begin{aligned}
||\Phi||^2_{\widetilde{\mathbb{H}}} &= ||h||^2_{[\widetilde{H}_2^{-\frac{1}{2}}(S_D)]^4} + ||g||^2_{[\widetilde{H}_2^{\frac{1}{2}}(S_N)]^4} \quad \text{for arbitrary} \quad \Phi = (h, g)^\top \in \widetilde{\mathbb{H}},\\
||Q||^2_{\mathbb{H}} &= ||f||^2_{[H_2^{\frac{1}{2}}(S_D)]^4} + ||F||^2_{[H_2^{-\frac{1}{2}}(S_N)]^4} \quad \text{for arbitrary} \quad Q = (f, F)^\top \in \mathbb{H},
\end{aligned}$$

In acordance with Theorem 5.2, we have the following mapping property for the operator $\mathcal{B}$

$$\mathcal{B} \,:\, \widetilde{\mathbb{H}} \to \mathbb{H}. \tag{3.7}$$

Our goal is to show that the operator (3.7) is invertible.

We start with the following injectivity result.

Lemma 3.1 *Equation* (3.6) *possesses at most one solution in the space* $\widetilde{\mathbb{H}}$.

Proof. It suffices to show that the homogeneous equation $\mathcal{B}\,\Phi = 0$ has only the trivial solution. Let $\Phi = (h, g)^\top \in \widetilde{\mathbb{H}}$ be a solution of this homogeneous equation and construct the vector function U by formula (3.1). Then, due to Theorem 5.1(i) in Appendix A, we have $U \in [H^1_2(\Omega)]^4$ and in view of equation $\mathcal{B}\,\Phi = 0$ the vector function U solves the homogeneous mixed BVP. Therefore, by the uniqueness Theorem 2.3 we conclude $U = 0$ in Ω.

Note that in virtue of the asymptotic properties at infinity of the fundamental matrix $\Gamma(x) = \mathcal{O}(|x|^{-1})$ and $\partial_j\Gamma(x) = \mathcal{O}(|x|^{-2})$, we have $U(x) = \mathcal{O}(|x|^{-1})$ and $\partial_j U(x) = \mathcal{O}(|x|^{-2})$ for sufficiently large $|x|$ and $j = 1, 2, 3$ (see (5.1) in Appendix A). Therefore, U belongs to the Beppo-Levi space in Ω^- (for details see, e.g., [DL1], Ch. 11, Part B),

$$U \in [BL(\Omega^-)]^4 = \left\{ Y \;:\; \frac{1}{(1+|x|^2)^{1/2}}\, Y \in [L_2(\Omega^-)]^4, \;\; \partial_j Y \in [L_2(\Omega^-)]^4, \;\; j = 1, 2, 3 \right\}.$$

The norm in Beppo-Levi space is defined by the relation

$$|| \, Y \, ||^2_{[BL(\Omega^-)]^4} := || \, (1+|x|^2)^{-1/2} Y \, ||^2_{[L_2(\Omega^-)]^4} + \sum_{k=1}^{3} || \, \partial_k Y \, ||^2_{[L_2(\Omega^-)]^4}.$$

The semi-norm defined by the last summand in the right-hand side of the previous equality is an equivalent norm in the space $[BL(\Omega^-)]^4$ (see [DL1], Ch. 11, Part B, §1, Theorem 1).

The representation (3.1) and the inclusions (3.2) imply (see Theorem 5.1(ii)):

$$\{U\}^- = \{U\}^+ = 0 \quad \text{on} \;\; S_N, \quad \{\mathcal{T}U\}^- = \{\mathcal{T}U\}^+ = 0 \quad \text{on} \;\; S_D. \tag{3.8}$$

Since $U \in [BL(\Omega^-)]^4 \cap [H^1_{2,loc}(\Omega^-)]^4$, we can write the Green formula in the exterior domain Ω^-,

$$\int_{\Omega^-} [\, A(\partial)U \, \cdot \, V + E(U,V) \,] \, dx = -\big\langle \{\mathcal{T}(\partial,n)U\}^- \, , \, \{V\}^- \big\rangle_S \, . \tag{3.9}$$

Using the relations (3.8), from (3.9) we get

$$\int_{\Omega^-} E(U,U) \, dx = - \big\langle \{\mathcal{T}(\partial,n)U\}^- \, , \, \{U\}^- \big\rangle_{S_D} - \big\langle \{\mathcal{T}(\partial,n)U\}^- \, , \, \{U\}^- \big\rangle_{S_N} = 0, \tag{3.10}$$

Based on the same reasoning as in the proof of Theorem 2.3, from (3.10) we conclude $U = 0$ in Ω^-. Now, the relations:

$$\{U\}^+ - \{U\}^- = g = 0 \quad \text{on} \;\; S, \quad \{\mathcal{T}U\}^- - \{\mathcal{T}U\}^+ = h = 0 \quad \text{on} \;\; S,$$

complete the proof. ■

Further, we prove the following auxiliary assertion which plays a crucial role in our analysis.

Lemma 3.2 *Operator* (3.7) *is coercive, i.e., there is a positive constant* δ_4*, such that for arbitrary complex valued vector function* $\Phi = (h,g)^\top \in \widetilde{\mathbb{H}}$ *the following inequality holds*

$$\begin{aligned} Re \, \big\langle \mathcal{B}\,\Phi \, , \, \Phi \big\rangle = \;\; & Re \, \Big(\langle -\mathcal{H}h \, , \, h \rangle_{S_D} + \langle \mathcal{K}g \, , \, h \rangle_{S_D} + \Big\langle -\widetilde{\mathcal{K}}h \, , \, g \Big\rangle_{S_N} + \langle \mathcal{L}g \, , \, g \rangle_{S_N} \Big) \\ & \geqslant \delta_4 \, \Big(||h||^2_{[\widetilde{H}_2^{-\frac{1}{2}}(S_D)]^4} + ||g||^2_{[\widetilde{H}_2^{\frac{1}{2}}(S_N)]^4} \Big) = \delta_4 \, ||\Phi||^2_{\widetilde{\mathbb{H}}}, \end{aligned} \tag{3.11}$$

where $\langle \, \cdot \, , \, \cdot \rangle$ *denotes the duality pairing between the mutually adjoint spaces* $\mathbb{H}$ *and* $\widetilde{\mathbb{H}}$.

Proof. For arbitrary complex valued vector function $\Phi = (h, g)^\top \in \widetilde{\mathbb{H}}$ let us consider a complex valued vector function $U = (u_1, u_2, u_3, \varphi)^\top$ constructed by formula (3.1). Due to the continuity property of the single-layer potential across the surface S_D and since g is supported on S_N, it is evident that

$$U \in [H^1_{2,loc}(\mathbb{R}^3 \setminus S_D)]^4, \quad U \in [H^1_2(\Omega^+)]^4, \quad U \in [BL(\Omega^-)]^4.$$

Therefore, we can write the Green formulas (2.19) and (3.9) for U and $V = U$. By summing these formulas and using the jump relations for U and $\mathcal{T}U$ across the boundary S, (see Theorem 5.1(ii) in Appendix A)

$$\{U\}^\pm = -\mathcal{H}h \pm \frac{1}{2}\, g + \mathcal{K}g, \quad \{\mathcal{T}U\}^\pm = \pm\frac{1}{2}\, h - \widetilde{\mathcal{K}}h + \mathcal{L}g,$$

we arrive at the equality

$$\begin{aligned}
\int\limits_{\Omega^+} E(U,U)\, dx + \int\limits_{\Omega^-} E(U,U)\, dx &= \big\langle \{\mathcal{T}U\}^+ , \{U\}^+ \big\rangle_S - \big\langle \{\mathcal{T}U\}^- , \{U\}^- \big\rangle_S \\
&= \big\langle \{\mathcal{T}U\}^+ , \{U\}^+ - \{U\}^- \big\rangle_S - \big\langle \{\mathcal{T}U\}^- - \{\mathcal{T}U\}^+ , \{U\}^- \big\rangle_S \\
&= \langle -h \, , \, \mathcal{H}h\rangle_{S_D} + \langle h, \, \mathcal{K}g\rangle_{S_D} + \langle -\widetilde{\mathcal{K}}h \, , \, g\rangle_{S_N} + \langle \mathcal{L}g \, , \, g\rangle_{S_N} \, ,
\end{aligned} \tag{3.12}$$

where in view of Remark 2.2

$$E(U,U) = c_{ijlk}\, \partial_i u_j \, \overline{\partial_l u_k} + e_{pjq}\, \partial_p \varphi \, \overline{\partial_q u_j} - e_{pjq}\, \partial_q u_j \, \overline{\partial_p \varphi} + \varepsilon_{pq}\, \partial_p \varphi \, \overline{\partial_q \varphi}. \tag{3.13}$$

Using the inequalities (2.5) we deduce

$$\begin{aligned}
\operatorname{Re} E(U,U) &= \operatorname{Re}\big(c_{ijlk}\, \partial_i u_j \, \overline{\partial_l u_k} + e_{pjq}\, \partial_p \varphi \, \overline{\partial_q u_j} - e_{pjq}\, \partial_q u_j \, \overline{\partial_p \varphi} + \varepsilon_{pq}\, \partial_p \varphi \, \overline{\partial_q \varphi}\big) \\
&= \operatorname{Re}\big(c_{ijlk}\, \partial_i u_j \, \overline{\partial_l u_k} + \varepsilon_{pq}\, \partial_p \varphi \, \overline{\partial_q \varphi}\big) \\
&\geqslant \delta_1 s_{ij}(u)\, \overline{s_{ij}(u)} + \delta_2\, \partial_p \varphi \overline{\partial_p \varphi}.
\end{aligned} \tag{3.14}$$

From the relation(3.12) we get

$$\begin{aligned}
&\operatorname{Re}\Bigg(\int\limits_{\Omega^+} E(U,U)\, dx + \int\limits_{\Omega^-} E(U,U)\, dx \Bigg) \\
&\quad = \operatorname{Re}\Big(\langle -h \, , \, \mathcal{H}h\rangle_{S_D} + \langle h, \, \mathcal{K}g\rangle_{S_D} + \langle -\widetilde{\mathcal{K}}h \, , \, g\rangle_{S_N} + \langle \mathcal{L}g \, , \, g\rangle_{S_N}\Big) \\
&\quad = \operatorname{Re}\Big(\langle -\mathcal{H}h \, , \, h\rangle_{S_D} + \langle h, \, \mathcal{K}g\rangle_{S_D} + \langle -\widetilde{\mathcal{K}}h \, , \, g\rangle_{S_N} + \langle \mathcal{L}g \, , \, g\rangle_{S_N}\Big) \\
&\quad = \operatorname{Re} \big\langle \mathcal{B}\, \Phi \, , \, \Phi \big\rangle.
\end{aligned} \tag{3.15}$$

By Theorem 3 in the reference [KO1, Section 3] along with the inequality (3.14) and the equivalence of the seminorm and the norm in Beppo-Levi spaces we have

$$\begin{aligned}
\operatorname{Re}\big\langle \mathcal{B}\,\Phi\,,\,\Phi\big\rangle &= \operatorname{Re}\Bigg(\int\limits_{\Omega^+} E(U,U)\,dx + \int\limits_{\Omega^-} E(U,U)\,dx\Bigg) = \operatorname{Re}\int\limits_{\mathbb{R}^3\setminus S_N} E(U,U)\,dx \\
&\geqslant \int\limits_{\mathbb{R}^3\setminus S_N} \big[\delta_1 s_{ij}(u)\,\overline{s_{ij}(u)} + \delta_2\,\partial_p\varphi\overline{\partial_p\varphi}\big]\,dx \\
&\geqslant a_1 \int\limits_{\mathbb{R}^3\setminus S_N} \Big[\sum_{i,j=1}^{3} |\partial_j u_i|^2 + \sum_{j=1}^{3} |\partial_j\varphi|^2\Big]\,dx \\
&\geqslant a_2\,||U||^2_{[BL(\mathbb{R}^3\setminus S_N)]^4}.
\end{aligned} \tag{3.16}$$

Here and in what follows a_k are positive constants independent of Φ.

Now, on the one hand, by the trace theorem we have

$$\begin{aligned}
||U||^2_{[BL(\mathbb{R}^3\setminus S_N)]^4} &\geqslant a_3\Big(\,||\{U\}^+||^2_{[H_2^{\frac12}(S)]^4} + ||\{U\}^-||^2_{[H^{\frac12}(S)]^4}\Big) \\
&\geqslant a_3\Big(\,||\{U\}^+ - \{U\}^-||^2_{[H_2^{\frac12}(S)]^4}\Big) \\
&= a_3\,||g||^2_{[\widetilde{H}^{\frac12}(S_N)]^4}.
\end{aligned} \tag{3.17}$$

Consequently, in view of (3.16)

$$\operatorname{Re}\big\langle \mathcal{B}\,\Phi\,,\,\Phi\big\rangle \geqslant a_4\,||g||^2_{[\widetilde{H}_2^{\frac12}(S_N)]^4} \tag{3.18}$$

On the other hand, thanks to Theorem 5.3 in Appendix A, since U is a solution of the homogeneous equation $A(\partial)U = 0$ in Ω^+ and Ω^-, we have the representation

$$U(x) = V\left([\,\mathcal{H}\,]^{-1}\{\,U\,\}^{\pm}\right)(x), \quad x \in \Omega^{\pm}.$$

Therefore,

$$\{\mathcal{T}U\}^{\pm} = \big(\mp\frac{1}{2}\,I_4 + \widetilde{\mathcal{K}}\,\big)\,\mathcal{H}^{-1}\,\{U\}^{\pm} \ \text{ on } S. \tag{3.19}$$

Using the boundedness of the integral operators involved in (3.19) (see Theorem 5.2 in Appendix A)

$$\begin{aligned}
\mp\frac{1}{2}\,I_4 + \widetilde{\mathcal{K}} &: [H_2^{-\frac12}(S)]^4 \to [H_2^{-\frac12}(S)]^4, \\
\mathcal{H}^{-1} &: [H_2^{\frac12}(S)]^4 \to [H_2^{-\frac12}(S)]^4,
\end{aligned}$$

we find

$$||\{\mathcal{T}U\}^{\pm}||_{[H_2^{-\frac{1}{2}}(S)]^4} \leqslant a_5\,||\{U\}^{\pm}||_{[H_2^{\frac{1}{2}}(S)]^4}. \tag{3.20}$$

With the help of (3.17) from (3.20) we conclude

$$||\{\mathcal{T}U\}^{\pm}||_{[H_2^{-\frac{1}{2}}(S)]^4} \leqslant a_6\,||U||_{[BL(\mathbb{R}^3\setminus S_N)]^4}.$$

Therefore,

$$\begin{aligned} ||U||_{[BL(\mathbb{R}^3\setminus S_N)]^4} &\geqslant a_7\Big(||\{\mathcal{T}U\}^{+}||_{[H_2^{-\frac{1}{2}}(S)]^4} + ||\{\mathcal{T}U\}^{-}||_{[H_2^{-\frac{1}{2}}(S)]^4}\Big) \\ &\geqslant a_7||\{\mathcal{T}U\}^{+} - \{\mathcal{T}U\}^{-}||_{[H_2^{-\frac{1}{2}}(S)]^4} = a_7||h||_{[\widetilde{H}_2^{-\frac{1}{2}}(S)]^4}, \end{aligned}$$

and by (3.16) we get

$$\mathrm{Re}\,\big\langle \mathcal{B}\,\Phi\,,\,\Phi\big\rangle \geqslant a_8\,||h||^2_{[H_2^{-\frac{1}{2}}(S)]^4}. \tag{3.21}$$

Finally, combining (3.18) and (3.21) we deduce

$$\mathrm{Re}\,\big\langle \mathcal{B}\,\Phi\,,\,\Phi\big\rangle \geqslant \delta_4\Big(||h||^2_{[H_2^{-\frac{1}{2}}(S)]^4} + ||g||^2_{[H_2^{\frac{1}{2}}(S)]^4}\Big) = \delta_4\,||\Phi||^2_{\widetilde{\mathbb{H}}},\ \ \delta_4 = \min\{a_4, a_8\} > 0,$$

which completes the proof. ■

Remark 3.3 *Lemmas 3.1 and 3.2 remain also valid when S, S_D, and S_N are Lipschitz surfaces. The proofs can be performed verbatim.*

Corollary 3.4 *Let S, S_D, and S_N be Lipschitz surfaces and $Q \in \mathbb{H}$. Then the operator* (3.7) *is isomorphism and, consequently, equation* (3.6) *is uniquely solvable in the space $\widetilde{\mathbb{H}}$.*

Proof. The proof immediately follows from Lemma 3.2 and the well-known Lax-Milgram theorem. ■

Remark 3.5 *From Corollary 3.4 it immediately follows that the adjoint operator $\mathcal{B}^*$ to the operator* (3.7) *is isomorphism as well. Moreover, using the arguments employed in the proof of Lemma 3.2, one can show that the operator $\mathcal{B}^*$ is coercive as well.*

3.2 Existence theorem for Lipschitz domains

Now we are in a position to formulate the following existence theorem in the space $[H_2^1(\Omega)]^4$ for Lipschitz domains.

Theorem 3.6 *Let Ω be a domain with Lipschitz boundary S and let S_D and S_N be a Lipschitz dissection of S. Let the boundary data in the Dirichlet and Neumann type boundary conditions satisfy the inclusions* (2.14) *with $p = 2$,*

$$f \in \left[B_{2,2}^{\frac{1}{2}}(S_D)\right]^4 = \left[H_2^{\frac{1}{2}}(S_D)\right]^4, \quad F \in \left[B_{2,2}^{-\frac{1}{2}}(S_N)\right]^4 = \left[H_2^{-\frac{1}{2}}(S_N)\right]^4.$$

Then the mixed BVP (2.11)-(2.13) *possesses a unique solution $U \in [H_2^1(\Omega)]^4 = [B_{2,2}^1(\Omega)]^4$ which can be represented by the linear combination of single- and double-layer potentials* (3.1), *where the density vector functions $h \in \left[\widetilde{H}_2^{-\frac{1}{2}}(S_D)\right]^4$ and $g \in \left[\widetilde{H}_2^{\frac{1}{2}}(S_N)\right]^4$ solve the system of boundary integral equations* (3.3)-(3.4).

Proof. The proof directly follows from representation (3.1), Lemmas 3.1 and 3.2, Remark 3.3, Corollary 3.4, and the uniqueness Theorem 2.3. ■

Remark 3.7 *It is evident that for the mixed exterior BVP* (2.15)-(2.18) *we have a similar existence theorem. First of all, note that the exterior problem has at most one solution, which follows from the decay condition* (2.18) *and the Green identity* (3.9). *Moreover, if we look for a solution U to the exterior problem* (2.15)-(2.18) *in the form* (3.1), *keeping in mind the inclusions* (3.2), *we arrive again at the uniquely solvable boundary integral equations* (3.3)-(3.4). *Therefore, the mixed exterior BVP* (2.15)-(2.18) *is uniquely solvable and the solution vector is representable in the form* (3.1) *satisfying the inclusion $U \in [BL(\Omega^-)]^4 \subset [H_{2,loc}^1(\Omega^-)]^4 = [B_{2,2,loc}^1(\Omega^-)]^4$.*

3.3 Existence theorem for smooth domains and regularity results

In what follows, we assume that the boundary S is sufficiently smooth (C^∞-smooth say) and show existence of solutions to the mixed BVPs in the Besov spaces. This gives us possibility to analyse almost optimal regularity properties of solutions near the collision curve ℓ.

Let

$$a_1 = \min\{a_1^+, a_1^-\}, \quad a_2 = \max\{a_2^+, a_2^-\}, \tag{3.22}$$

where $a_1^{\pm}$ and $a_2^{\pm}$ are defined in Appendix C (see (7.4), (7.5)). Due to (7.6) we have

$$-\frac{1}{2} < a_1 \leqslant a_2 < \frac{1}{2}.$$

It is evident that Theorems 7.1-7.5 remain valid if the numbers $a_1^{\pm}$ and $a_2^{\pm}$ are replaced by the numbers a_1 and a_2 respectively.

Theorem 3.8 *Let $S \in C^\infty$ and the Dirichlet and Neumann boundary data satisfy the conditions:*

$$f \in \left[B^s_{p,2}(S_D)\right]^4, \quad F \in \left[B^{s-1}_{p,2}(S_N)\right]^4, \quad p > 1, \tag{3.23}$$

with

$$-\tfrac{1}{2} + a_2 < s - \tfrac{1}{p} < \tfrac{1}{2} + a_1, \tag{3.24}$$

$$s \geqslant \tfrac{2}{p} - \tfrac{1}{2}, \quad s \geqslant \tfrac{1}{2}. \tag{3.25}$$

Then the system of pseudodifferential equations (3.3)-(3.4) *is uniquely and unconditionally solvable, and for the solution vector functions h and g the following inclusions hold:*

$$h \in \left[\widetilde{B}^{s-1}_{p,2}(S_D)\right]^4 \subset \left[\widetilde{H}^{-\frac{1}{2}}_2(S_D)\right]^4 = \left[\widetilde{B}^{-\frac{1}{2}}_{2,2}(S_D)\right]^4, \tag{3.26}$$

$$g \in \left[\widetilde{B}^{s}_{p,2}(S_N)\right]^4 \subset \left[\widetilde{H}^{\frac{1}{2}}_2(S_N)\right]^4 = \left[\widetilde{B}^{\frac{1}{2}}_{2,2}(S_N)\right]^4. \tag{3.27}$$

Proof. Note that if

$$1 < p, q < \infty, \;\; 1 \leqslant r \leqslant \infty, \;\; -\infty < t \leqslant s < \infty, \;\; s - \frac{2}{p} \geqslant t - \frac{2}{q},$$

then

$$H^s_p(S^*) \subset H^t_q(S^*), \quad B^s_{p,r}(S^*) \subset B^t_{q,r}(S^*), \tag{3.28}$$

where $S^* \in \{S, S_D, S_N\}$ (see [Tr1, Section 4.6]).

Therefore, if the inequalities (3.25) hold, then

$$\left[B^{s-1}_{p,2}(S_D)\right]^4 \subset \left[B^{-\frac{1}{2}}_{2,2}(S_D)\right]^4 = \left[H^{-\frac{1}{2}}_2(S_D)\right]^4, \tag{3.29}$$

$$\left[B^{s}_{p,2}(S_N)\right]^4 \subset \left[B^{\frac{1}{2}}_{2,2}(S_N)\right]^4 = \left[H^{\frac{1}{2}}_2(S_N)\right]^4, \tag{3.30}$$

and in view of (3.23) we have that the vector functions f and F satisfy also the following inclusions:

$$f \in \left[B^{\frac{1}{2}}_{2,2}(S_D)\right]^4 = \left[H^{\frac{1}{2}}_2(S_D)\right]^4, \tag{3.31}$$

$$F \in \left[B^{-\frac{1}{2}}_{2,2}(S_N)\right]^4 = \left[H^{-\frac{1}{2}}_2(S_N)\right]^4. \tag{3.32}$$

Due to Corollary 3.4 the system of integral equations (3.3)-(3.4) with the right-hand side vectors f and F satisfying the inclusions (3.31) and (3.32), is uniquely solvable in the space $\widetilde{\mathbb{H}}$, i.e.,

$$h \in \big[\widetilde{H}_2^{-\frac{1}{2}}(S_D)\big]^4 = \big[\widetilde{B}_{2,2}^{-\frac{1}{2}}(S_D)\big]^4, \quad g \in \big[\widetilde{H}_2^{\frac{1}{2}}(S_N)\big]^4 = \big[\widetilde{B}_{2,2}^{\frac{1}{2}}(S_N)\big]^4.$$

Let us construct the vector functions $U^{(1)}$ and $U^{(2)}$ using the solutions h and g as densities:

$$U^{(1)}(x) = -V(h)(x) + W(g)(x), \quad x \in \Omega^+, \quad U^{(1)} \in [H_2^1(\Omega^+)]^4, \tag{3.33}$$

$$U^{(2)}(x) = -V(h)(x) + W(g)(x), \quad x \in \Omega^-, \quad U^{(2)} \in [BL(\Omega^-)]^4 \subset [H_{2,loc}^1(\Omega^-)]^4. \tag{3.34}$$

Now, on the one hand, since $(h,g)^\top$ is a solution pair of the system of integral equations (3.3)-(3.4), in accordance with Theorem 3.6 and Remark 3.7, it is evident that $U^{(1)}$ and $U^{(2)}$ defined by (3.33) and (3.34) are unique solutions, respectively, to the mixed interior BVP (2.11)-(2.13) and mixed exterior BVP (2.15)-(2.18) with the same Dirichlet and Neumann data, f and F. On the other hand, the mixed interior and exterior BVPs with the same boundary data f and F satisfying inclusions (3.23), have unique solutions belonging to the spaces $[B_{p,2}^{s+\frac{1}{p}}(\Omega^+)]^4$ and $[B_{p,2,loc}^{s+\frac{1}{p}}(\Omega^-)]^4$ in virtue of Theorems 7.4 and 7.5 in Appendix C if conditions(3.23)-(3.24) are satisfied. Therefore, due to the uniqueness of solutions to the interior and exterior mixed BVPs the vector functions defined by (3.33) and (3.34) satisfy also the inclusions

$$U^{(1)} \in [B_{p,2}^{s+\frac{1}{p}}(\Omega^+)]^4, \quad U^{(2)} \in [B_{p,2,loc}^{s+\frac{1}{p}}(\Omega^-)]^4. \tag{3.35}$$

From (3.33), (3.34), and (3.35) we find

$$\{U^{(1)}\}^+ - \{U^{(2)}\}^- = g \in [\widetilde{B}_{p,2}^{s}(S_N)]^4, \quad \{\mathcal{T}U^{(1)}\}^+ - \{\mathcal{T}U^{(2)}\}^- = h \in [\widetilde{B}_{p,2}^{s-1}(S_D)]^4,$$

which along with (3.28), (3.29), and (3.30) complete the proof. ■

Remark 3.9 *The arguments applied in the proof of Theorem 3.8 imply that if conditions* (3.23)-(3.25) *hold, then the following operator is invertible*

$$\mathcal{B} \, : \, [\widetilde{B}_{p,2}^{s-1}(S_D)]^4 \times [\widetilde{B}_{p,2}^{s}(S_N)]^4 \to [B_{p,2}^{s}(S_D)]^4 \times [B_{p,2}^{s-1}(S_N)]^4,$$

where $\mathcal{B}$ is defined in (3.5)*.*

Now, we can prove the following assertion about existence of solutions to mixed BVPs and their regularity properties.

Theorem 3.10 *Let conditions* (3.23)-(3.25) *be satisfied.*

(i) *Then the interior and exterior mixed BVPs have unique solutions* $U^{(1)}$ *in* Ω^+ *and* $U^{(2)}$ *in* Ω^-*, respectively, belonging to the following spaces:*

$$U^{(1)} \in \left[H_2^1(\Omega^+)\right]^4 \cap \left[B_{p,2}^{s+\frac{1}{p}}(\Omega^+)\right]^4, \tag{3.36}$$

$$U^{(2)} \in \left[BL(\Omega^-)\right]^4 \cap \left[H_{2,loc}^1(\Omega^-)\right]^4 \cap \left[B_{p,2,loc}^{s+\frac{1}{p}}(\Omega^-)\right]^4. \tag{3.37}$$

The solutions are representable in the form of linear combination of the single- and double-layer potentials (3.1), $U^{(1)} = -V(h) + W(g)$ *in* Ω^+ *and* $U^{(2)} = -V(h) + W(g)$ *in* Ω^-*, where the density vector functions* h *and* g *solve the system of integral equations* (3.3)-(3.4) *and belong to the spaces* (3.26)-(3.27).

(ii) *If, in addition to* (3.23)-(3.25),

$$p \geqslant 2, \quad a_1 \geqslant 0, \quad \frac{1}{2} \leqslant s < \frac{1}{2} + \frac{1}{p} + a_1, \tag{3.38}$$

then the following inclusions hold:

$$U^{(1)} = -V(h) + W(g) \in \left[H_2^1(\Omega^+)\right]^4 \cap \left[B_{p,2}^{s+\frac{1}{p}}(\Omega^+)\right]^4 \subset \left[C^t(\overline{\Omega^+})\right]^4, \tag{3.39}$$

$$U^{(2)} = -V(h) + W(g) \in \left[BL(\Omega^-)\right]^4 \cap \left[H_{2,loc}^1(\Omega^-)\right]^4 \cap \left[B_{p,2,loc}^{s+\frac{1}{p}}(\Omega^-)\right]^4 \subset \left[C^t(\overline{\Omega^-})\right]^4, \tag{3.40}$$

$$g \in \left[\widetilde{B}_{p,2}^s(S_N)\right]^4 \subset \left[C^t(S)\right]^4, \quad r_{S_N} g = 0, \tag{3.41}$$

where

$$t = \frac{1}{2} + a_1 - \frac{1}{p} > 0. \tag{3.42}$$

(iii) *If, in addition to* (3.24)-(3.25),

$$-\frac{1}{4} < a_1 < 0, \quad 4 < p < -\frac{1}{a_1}, \quad 0 < \frac{1}{2} - \frac{2}{p} \leqslant s - \frac{2}{p} < \frac{1}{2} - \frac{1}{p} + a_1, \tag{3.43}$$

then the following inclusions hold:

$$U^{(1)} = -V(h) + W(g) \in \left[H_2^1(\Omega^+)\right]^4 \cap \left[B_{p,2}^{s+\frac{1}{p}}(\Omega^+)\right]^4 \subset \left[C^t(\overline{\Omega^+})\right]^4, \tag{3.44}$$

$$U^{(2)} = -V(h) + W(g) \in \left[BL(\Omega^-)\right]^4 \cap \left[H_{2,loc}^1(\Omega^-)\right]^4 \cap \left[B_{p,2,loc}^{s+\frac{1}{p}}(\Omega^-)\right]^4 \subset \left[C^t(\overline{\Omega^-})\right]^4, \tag{3.45}$$

$$g \in \left[\widetilde{B}_{p,2}^s(S_N)\right]^4 \subset \left[C^t(S)\right]^4, \quad r_{S_N} g = 0, \tag{3.46}$$

where

$$0 < \frac{1}{2} - \frac{2}{p} < t = s - \frac{2}{p} < \frac{1}{2} - \frac{1}{p} + a_1. \tag{3.47}$$

Proof. The first part of the theorem directly follows from Theorem 3.8, Remark 3.9, and mapping and jump properties of the single- and double-layer potentials (see Theorem 5.1 in Appendix A).

To prove the second and the third parts of the theorem, let us recall that (see [Tr1, Section 4.6])

$$B^r_{p,q}(S) \subset \mathcal{C}^\tau(S), \quad B^s_{p,q}(\Omega) \subset \mathcal{C}^t(\overline{\Omega}),$$
$$1 < p < \infty, \;\; 1 \leqslant q \leqslant \infty, \;\; \tau \geqslant 0, \;\; r > \tau + \frac{2}{p}, \;\; t \geqslant 0, \;\; s > t + \frac{3}{p}, \tag{3.48}$$

where $\mathcal{C}^t(\overline{\Omega})$ and $\mathcal{C}^\tau(S)$ with $t > 0$ and $\tau > 0$ denote the Zygmund spaces, which for non-integers t and τ coincide with the Hölder spaces with smoothness indices $t > 0$ and $\tau > 0$. Note that if t and τ are not integers, then in (3.48) we can take $\tau = r - \frac{2}{p}$ and $t = s - \frac{3}{p}$. Therefore, from the inclusions $g \in \big[\widetilde{B}^s_{p,2}(S_N)\big]^4$, $U^{(1)} \in \big[B^{s+\frac{1}{p}}_{p,2}(\Omega^+)\big]^4$, $U^{(2)} \in \big[B^{s+\frac{1}{p}}_{p,2,loC}(\Omega^-)\big]^4$, and inequalities (3.38) and (3.43), we obtain the regularity results (3.39)-(3.42) and (3.44)-(3.47).

■

4 Existence results in the spaces $[H^1_p(\Omega)]^4$ and $[H^1_{p,loc}(\Omega^-)]^4$

Finally, we consider the mixed interior and exterior BVPs, respectively, in the spaces $[H^1_p(\Omega)]^4 = [W^1_p(\Omega)]^4$ and $[H^1_{p,loc}(\Omega^-)]^4 = [W^1_{p,loc}(\Omega^-)]^4$ assuming that the Dirichlet and Neumann boundary data in both cases are the same and satisfy the conditions:

$$f \in \big[B^{1-\frac{1}{p}}_{p,p}(S_D)\big]^4, \quad F \in \big[B^{-\frac{1}{p}}_{p,p}(S_N)\big]^4, \quad p > 1. \tag{4.1}$$

Using the above described approach based on the representation (3.1) we again arrive at the system of integral equations (3.3)-(3.4) with right-hand sides satisfying inclusions (4.1).

Introduce the notation (cf. (3.22))

$$a_1^* = \min\{a^+_{1*}, a^-_{1*}\}, \quad a_2^* = \max\{a^+_{2*}, a^-_{2*}\}, \tag{4.2}$$

where $a^\pm_{1*}$ and $a^\pm_{2*}$ are defined by (7.12) and (7.13) in Appendix C (see Remark 7.6). Due to (7.14) we have

$$-\frac{1}{2} < a_1^* \leqslant a_2^* < \frac{1}{2}.$$

It is evident that Theorems 7.1-7.5 and their counterpart theorems for the "adjoint" operators and mixed BVPs remain valid if the numbers $a_1^{\pm}$, $a_2^{\pm}$ and $a_{1*}^{\pm}$, $a_{2*}^{\pm}$ are replaced by the numbers a_1, a_2 and a_1^*, a_2^* respectively (see (3.22) and (4.2)).

Now we prove the following assertion.

Theorem 4.1 *Let inclusions* (4.1) *hold with*

$$\frac{4}{3+2a_1^*} < p < \frac{4}{1-2a_1}\,,$$

where a_1 *and* a_1^* *are defined in* (3.22) *and* (4.2) *respectively.*

(i) *The system of integral equations* (3.3)-(3.4) *is unconditionally uniquely solvable and solution vector functions satisfy the inclusions:*

$$h \in \big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(S_D)\big]^4, \quad g \in \big[\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)\big]^4.$$

(ii) *The operator*

$$\mathcal{B}\,:\,\big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(S_D)\big]^4 \times \big[\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)\big]^4 \to \big[B_{p,p}^{1-\frac{1}{p}}(S_D)\big]^4 \times \big[B_{p,p}^{-\frac{1}{p}}(S_N)\big]^4 \tag{4.3}$$

is invertible.

(iii) *The mixed interior and exterior BVPs possess unique solutions,* $U^{(1)}$ *and* $U^{(2)}$*, which are representable by formula* (3.1) *and*

$$U^{(1)} \in \big[W_p^1(\Omega)\big]^4 = \big[H_p^1(\Omega)\big]^4, \quad U^{(2)} \in \big[W_{p,loc}^1(\Omega^-)\big]^4 = \big[H_{p,loc}^1(\Omega^-)\big]^4.$$

Proof. It is evident that due to Theorem 5.2 the operator $\mathcal{B}$ defined in (3.5) has the mapping property (4.3) for arbitrary $p > 1$.

It can be easily checked that

$$\frac{4}{3+2a_1^*} < 2 < \frac{4}{1-2a_1}\,.$$

For $p = 2$ the theorem is true even for Lipschitz domains due to Corollary 3.4, Theorem 3.6, and Remark 3.7.

Now, let us consider the case

$$2 < p < \frac{4}{1-2a_1}. \tag{4.4}$$

Then the inequalities (3.24)-(3.25) with $s = 1 - \frac{1}{p}$ are satisfied if

$$\frac{1}{p} > \frac{1-2a_1}{4}, \quad \frac{1}{p} < \frac{3-2a_2}{4}, \quad \frac{1}{p} < \frac{1}{2}, \tag{4.5}$$

which is equivalent to the relation

$$\frac{1-2a_1}{4} < \frac{1}{p} < \frac{1}{2}, \tag{4.6}$$

since the third inequality in (4.5) implies the second one. Evidently, (4.6) is the same as (4.4).

Note that the following inclusions are valid (see [Tr1, Section 4.6]):

$$\begin{aligned} B_{p,p}^{1-\frac{1}{p}}(S_D) \subset B_{p,\infty}^{1-\frac{1}{p}}(S_D) \subset B_{p,1}^{1-\frac{1}{p}-\varepsilon}(S_D) \subset B_{p,2}^{1-\frac{1}{p}-\varepsilon}(S_D) \subset \\ \subset B_{2,2}^{\frac{1}{2}}(S_D) = H_2^{\frac{1}{2}}(S_D), \ \ p > 2, \end{aligned} \tag{4.7}$$

$$\begin{aligned} B_{p,p}^{-\frac{1}{p}}(S_N) \subset B_{p,\infty}^{-\frac{1}{p}}(S_N) \subset B_{p,1}^{-\frac{1}{p}-\varepsilon}(S_N) \subset B_{p,2}^{-\frac{1}{p}-\varepsilon}(S_N) \subset \\ \subset B_{2,2}^{-\frac{1}{2}}(S_N) = H_2^{-\frac{1}{2}}(S_N), \ \ p > 2, \end{aligned} \tag{4.8}$$

where ε is an arbitrarily small positive number. One can take $0 < \varepsilon < \frac{p-2}{2p}$.

Therefore, if $f \in \big[B_{p,p}^{1-\frac{1}{p}}(S_D)\big]^4$ and $F \in \big[B_{p,p}^{-\frac{1}{p}}(S_N)\big]^4$ with p satisfying (4.4), then in view of embedding relations (4.7)-(4.8) and Corollary 3.4 the system of pseudodifferential equations (3.3)-(3.4) is uniquely solvable in the space $\widetilde{\mathbb{H}}$ and for the solution $(h,\, g)^\top$ we have the inclusions:

$$h \in \big[\widetilde{B}_{2,2}^{-\frac{1}{2}}(S_D)\big]^4 = \big[\widetilde{H}_2^{-\frac{1}{2}}(S_D)\big]^4, \quad g \in \big[\widetilde{B}_{2,2}^{\frac{1}{2}}(S_N)\big]^4 = \big[\widetilde{H}_2^{\frac{1}{2}}(S_N)\big]^4.$$

Therefore, on the one hand, for the unique solutions $U^{(1)}$ and $U^{(2)}$ to the corresponding interior and exterior mixed BVPs with the boundary data (4.1), there hold the representations and inclusions (3.33)-(3.34).

On the other hand, for $2 < p < \frac{4}{1-2a_1}$ and $s = 1 - \frac{1}{p}$ the inequality $\frac{1}{p} - \frac{1}{2} < s < \frac{1}{p} + \frac{1}{2}$ is satisfied, and due to Theorems 7.4-7.5 in the Appendix C, the interior and exterior mixed BVPs are uniquely solvable in appropriate Bessel potential and Besov spaces. Consequently, in accordance with the uniqueness theorems we have the inclusions

$$U^{(1)} \in \big[H_p^1(\Omega)\big]^4 \bigcap \big[B_{p,p}^1(\Omega)\big]^4 = \big[H_p^1(\Omega)\big]^4 = \big[W_p^1(\Omega)\big]^4, \tag{4.9}$$

$$U^{(2)} \in \big[H_{p,loc}^1(\Omega^-)\big]^4 \bigcap \big[B_{p,p,loc}^1(\Omega^-)\big]^4 = \big[H_{p,loc}^1(\Omega^-)\big]^4 = \big[W_{p,loc}^1(\Omega^-)\big]^4. \tag{4.10}$$

Here we employed the embedding relations

$$B^s_{p,\min\{2,p\}}(\Omega) \subset H^s_p(\Omega) \subset B^s_{p,\max\{2,p\}}(\Omega), \quad s \in \mathbb{R}, \quad p > 2,$$

implying

$$H^1_p(\Omega) = W^1_p(\Omega) \subset B^1_{p,p}(\Omega), \quad H^1_{p,loc}(\Omega^-) = W^1_{p,loc}(\Omega^-) \subset B^1_{p,p,loc}(\Omega^-).$$

Note that the traces on S of elements from both spaces $H^1_p(\Omega) = W^1_p(\Omega)$ and $B^1_{p,p}(\Omega)$ belong to the Besov space $B^{1-\frac{1}{p}}_{p,p}(S) = W^{1-\frac{1}{p}}_p(S)$ assuming $p > 1$.

Due to the well-posedness of the mixed BVPs under consideration (see Appendix C) we have the following norm estimates

$$\begin{aligned} &\|\{U^{(1)}\}^+\|_{[B^{1-\frac{1}{p}}_{p,p}(S)]^4} + \|\{\mathcal{T}(\partial_x, n)U^{(1)}\}^+\|_{[B^{-\frac{1}{p}}_{p,p}(S)]^4} \\ &\qquad \leqslant C_1 \Big(\|f\|_{\left[B^{1-\frac{1}{p}}_{p,p}(S_D)\right]^4} + \|F\|_{[B^{-\frac{1}{p}}_{p,p}(S_N)]^4} \Big), \end{aligned} \tag{4.11}$$

$$\begin{aligned} &\|\{U^{(2)}\}^-\|_{[B^{1-\frac{1}{p}}_{p,p}(S)]^4} + \|\{\mathcal{T}(\partial_x, n)U^{(2)}\}^-\|_{[B^{-\frac{1}{p}}_{p,p}(S)]^4} \\ &\qquad \leqslant C_2 \Big(\|f\|_{[B^{1-\frac{1}{p}}_{p,p}(S_D)]^4} + \|F\|_{[B^{-\frac{1}{p}}_{p,p}(S_N)]^4} \Big), \end{aligned} \tag{4.12}$$

with some positive constants C_1 and C_2 .

Therefore, with the help of relations (4.7)-(4.8), inclusions (4.9)-(4.10), and representation formulas of solutions (3.33)-(3.34) we have

$$\begin{aligned} h &= \big\{\mathcal{T}U^{(1)}\big\}^- - \big\{\mathcal{T}U^{(1)}\big\}^+ \in \big[\widetilde{H}^{-\frac{1}{2}}_2(S_D)\big]^4 \bigcap \big[\widetilde{B}^{-\frac{1}{p}}_{p,p}(S_D)\big]^4 = \big[\widetilde{B}^{-\frac{1}{p}}_{p,p}(S_D)\big]^4, \\ g &= \big\{U^{(1)}\big\}^+ - \big\{U^{(1)}\big\}^- \in \big[\widetilde{H}^{\frac{1}{2}}_2(S_N)\big]^4 \bigcap \big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)\big]^4 = \big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)\big]^4, \end{aligned}$$

and in view of (4.11) and (4.12) we deduce

$$\|h\|_{[B^{-\frac{1}{p}}_{p,p}(S)]^4} + \|g\|_{[B^{1-\frac{1}{p}}_{p,p}(S)]^4} \leqslant C_3 \left(\|f\|_{[B^{1-\frac{1}{p}}_{p,p}(S_D)]^4} + \|F\|_{[B^{-\frac{1}{p}}_{p,p}(S_N)]^4} \right),$$

where C_3 is a positive constant.

This yields that the operator (4.3) is invertible for p satisfying (4.4). Thus, the theorem is true if p satisfies condition (4.4).

Now, let us consider the case

$$\frac{4}{3 + 2a_1^*} < p < 2. \tag{4.13}$$

Here we will use the well known fact that a linear bounded operator $\mathbf{T} : B_1 \to B_2$ between reflexive Banach spaces B_1 and B_2 is invertible if and only if its adjoint operator $\mathbf{T}^* : B_2^* \to B_1^*$ is invertible. Here B_j^* is the adjoint space to B_j, $j = 1, 2$ (see, e.g., [KaAk, Ch. XII, Theorem 4, Theorem 5]).

We recall that all the Besov and Bessel potential spaces involved in our analysis are reflexive Banach spaces.

The adjoint operator $\mathcal{B}^*$ to the operator $\mathcal{B}$ has the following structure (see, (3.5))

$$\mathcal{B}^* := \begin{bmatrix} -r_{S_D}\mathcal{H}^* & -r_{S_D}\widetilde{\mathcal{K}}^* \\ r_{S_N}\mathcal{K}^* & r_{S_N}\mathcal{L}^* \end{bmatrix}_{6\times 6}.$$

In accordance with Remark 5.4 in Appendix A we have the following mapping property for the adjoint operator to the operator (4.3)

$$\mathcal{B}^* : \big[\widetilde{B}_{q,q}^{-\frac{1}{q}}(S_D)\big]^4 \times \big[\widetilde{B}_{q,q}^{1-\frac{1}{q}}(S_N)\big]^4 \to \big[B_{q,q}^{1-\frac{1}{q}}(S_D)\big]^4 \times \big[B_{q,q}^{-\frac{1}{q}}(S_N)\big]^4 \tag{4.14}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Note that if p satisfies condition (4.13), then $q = \frac{p}{p-1}$ satisfies the inequality

$$2 < q < \frac{4}{1 - 2a_1^*}\,. \tag{4.15}$$

Consider the interior and exterior mixed BVPs in $\Omega = \Omega^+$ and Ω^- for the adjoint operator $A^*(\partial_x)$ in the spaces $[H_q^1(\Omega)]^4$ and $[H_{q,loc}^1(\Omega^-)]^4$ with the same Dirichlet and Neumann data f^* and F^* on S_D and S_N, respectively, satisfying the conditions

$$f^* \in \big[B_{q,q}^{1-\frac{1}{q}}(S_D)\big]^4, \quad F^* \in \big[B_{q,q}^{-\frac{1}{q}}(S_N)\big]^4, \quad q > 1. \tag{4.16}$$

We look for solutions to these mixed BVPs again in the form of linear combination of the single- and double-layer potentials (cf. (3.1)),

$$U^*(x) = -V^*(h^*)(x) - W^*(g^*)(x), \quad x \in \Omega^\pm,$$

assuming initially that the density vectors h^* and g^* belong to the following spaces:

$$h^* = (h_1^*, h_2^*, h_3^*, h_4^*)^\top \in \big[\widetilde{H}_2^{-\frac{1}{2}}(S_D)\big]^4, \qquad g^* = (g_1^*, g_2^*, g_3^*, g_4^*)^\top \in \big[\widetilde{H}_2^{\frac{1}{2}}(S_N)\big]^4.$$

Then both the interior and the exterior mixed BVPs under consideration are reduced to the same integral equations:

$$-\mathcal{H}^* h^* - \widetilde{\mathcal{K}}^* g^* = f^* \quad \text{on} \quad S_D, \tag{4.17}$$

$$\mathcal{K}^* h^* + \mathcal{L}^* g^* = -F^* \quad \text{on} \quad S_N, \tag{4.18}$$

i.e.,

$$\mathcal{B}^* \Phi^* = G^*,$$

with $\Phi^* = \big(h^*,\, g^*\big)^\top$ and $G^* = \big(f^*,\, -F^*\big)^\top$.

Due to Remark 3.5, the adjoint operator (4.14) is isomorphism for $q = 2$.

Further, by the word for word arguments employed above in the case $2 < p < \frac{4}{1-2a_1}$, one can prove that the system (4.17)-(4.18) with the right hand side vectors f^* and F^* satisfying inclusions (4.16) is unconditionally uniquely solvable, and

$$(h^*, g^*) \in \big[\widetilde{B}_{q,q}^{-\frac{1}{q}}(S_D)\big]^4 \times \big[\widetilde{B}_{q,q}^{1-\frac{1}{q}}(S_N)\big]^4$$

if q satisfies condition (4.15).

Moreover, there is a constant $C_3^* > 0$, such that

$$\|h^*\|_{\big[B_{q,q}^{-\frac{1}{q}}(S)\big]^4} + \|g^*\|_{\big[B_{q,q}^{1-\frac{1}{q}}(S)\big]^4} \leqslant C_3^* \Big(\|f^*\|_{\big[B_{q,q}^{1-\frac{1}{q}}(S_D)\big]^4} + \|F^*\|_{\big[B_{q,q}^{-\frac{1}{q}}(S_N)\big]^4} \Big).$$

Consequently, the operator $\mathcal{B}^*$ in (4.14) is invertible if q satisfies condition (4.15). Therefore, the operator (4.3) with p satisfying the relation $\frac{1}{p} + \frac{1}{q} = 1$, as the adjoint operator to (4.14), is invertible under the same condition (4.15). In turn, condition (4.15) for the parameter $p = \frac{q}{q-1}$ implies the inequality (4.13).

So, the operator (4.3) is invertible under the condition (4.13).

Thus, the items (i), (ii), and (iii) of the theorem hold true for the values of the parameter p satisfying (4.13). This completes the proof. ■

5 Appendix A: Layer potentials and their mapping properties

Denote by $\Gamma(\cdot) = \big[\,\Gamma_{kj}(\cdot)\,\big]_{4\times 4}$ the fundamental matrix of the operator $A(\partial_x)$ (for details, see [Jo1], [BG1], [McLean], [BCN2]),

$$\Gamma(x) = \mathcal{F}^{-1}_{\xi\to x}\big[A^{-1}(-i\xi)\big] = -\frac{1}{8\pi^2\,|x|}\int\limits_0^{2\pi}[A(\Lambda\,\eta)]^{-1}\,d\theta, \tag{5.1}$$

where $\mathcal{F}^{-1}$ is the generalized inverse Fourier transform operator, $\Lambda = [\Lambda_{kj}]_{3\times 3}$ is an orthogonal matrix possessing the property $\Lambda^\top x = (0,0,|x|)^\top$, and $\eta = (\cos\theta, \sin\theta, 0)^\top$. The entries of the fundamental matrix $\Gamma(x)$ are real valued even functions having singularities of type $\mathcal{O}(|x|^{-1})$ in a neighbourhood of the origin and at infinity decay as $\mathcal{O}(|x|^{-1})$. Note that

$$\begin{aligned}
&A(\partial_x)\Gamma(x-y) = \delta(x-y)\,I_4, \qquad \Gamma(x-y) = \Gamma(y-x),\\
&\Gamma_{kj}(x-y) = \Gamma_{jk}(x-y) \ \text{ for } \ j,k=1,2,3, \quad \Gamma_{4j}(x-y) = -\Gamma_{j4}(x-y) \ \text{ for } \ j=1,2,3,
\end{aligned}$$

where I_4 stands for the 4×4 unit matrix and $\delta(\cdot)$ denotes Dirac's delta function.

Let us introduce the single- and double-layer potentials:

$$V(h)(x) = \int\limits_S \Gamma(x-y)\,h(y)\,d_yS, \quad x\in\mathbb{R}^3\setminus S, \tag{5.2}$$

$$W(g)(x) = \int\limits_S \big[\,\widetilde{\mathcal{T}}(\partial_y, n(y))\Gamma^\top(x-y)\,\big]^\top\,g(y)\,d_yS, \quad x\in\mathbb{R}^3\setminus S, \tag{5.3}$$

where $h = (h_1,h_2,h_3,h_4)^\top$ and $g = (g_1,g_2,g_3,g_4)^\top$ are density vector functions defined on $S = \partial\Omega$ and the boundary operator $\widetilde{\mathcal{T}}(\partial, n)$ is defined by (2.22) and (2.23).

For the readers convenience, here we collect some results concerning the layer potentials and the corresponding boundary integral operators generated by them. The proofs of the theorems can be found in the references [BG1], [BCD1], [BCNS1], [BCN2], [CW1], [DNS1], [NCS1], [Se1].

Theorem 5.1 *Let $S\in C^\infty$, $1<p<\infty$, $1\le t\le\infty$, and $s\in\mathbb{R}$.*

(i) *The operators*

$$\begin{aligned}
&V : \big[\,B^{s-1}_{p,p}(S)\,\big]^4 \to \big[\,H^{s+\frac{1}{p}}_p(\Omega)\,\big]^4, \qquad V : \big[\,B^{s-1}_{p,t}(S)\,\big]^4 \to \big[\,B^{s+\frac{1}{p}}_{p,t}(\Omega)\,\big]^4,\\
&W : \big[\,B^{s}_{p,p}(S)\,\big]^4 \to \big[\,H^{s+\frac{1}{p}}_p(\Omega)\,\big]^4, \qquad W : \big[\,B^{s}_{p,t}(S)\,\big]^4 \to \big[\,B^{s+\frac{1}{p}}_{p,t}(\Omega)\,\big]^4,
\end{aligned}$$

are continuous.

(ii) *For $h\in\big[B^{s-1}_{p,t}(S)\big]^4$ and $g\in\big[B^{s}_{p,t}(S)\big]^4$ with $1<p<\infty$, $1\le t\le\infty$, and $s>0$, the* *following jump relations hold:*

$$\begin{aligned}
&\big\{V(h)\big\}^+ = \big\{V(h)\big\}^- =: \mathcal{H}\,h \quad on \quad S,\\
&\big\{\mathcal{T}(\partial,n)V(h)\big\}^\pm = \big[\,\mp\,\tfrac{1}{2}\,I_4 + \widetilde{\mathcal{K}}\,\big]\,h \quad on \quad S,\\
&\big\{W(g)\big\}^\pm = \big[\,\pm\,\tfrac{1}{2}\,I_4 + \mathcal{K}\,\big]\,g \quad on \quad S,\\
&\big\{\,\mathcal{T}(\partial,n)W(g)\,\big\}^+ = \big\{\,\mathcal{T}(\partial,n)W(g)\,\big\}^- =: \mathcal{L}\,g \quad on \quad S,
\end{aligned}$$

where $\mathcal{T}(\partial,n)$ is the generalised boundary stress operator given by (2.9)-(2.10) *and $\mathcal{H}$, $\widetilde{\mathcal{K}}$, $\mathcal{K}$, and $\mathcal{L}$ are the boundary integral (pseudodifferential) operators generated by the single- and double-layer potentials:*

$$\mathcal{H}(h)(x) := \int\limits_S \Gamma(x-y)\;h(y)\,d_yS, \quad x\in S, \tag{5.4}$$

$$\widetilde{\mathcal{K}}(h)(x) := \int\limits_S \big[\,\mathcal{T}(\partial_x,n(x))\Gamma(x-y)\,\big]\;h(y)\,d_yS, \quad x\in S, \tag{5.5}$$

$$\mathcal{K}(g)(x) := \int\limits_S \big[\,\widetilde{\mathcal{T}}(\partial_y,n(y))\Gamma^\top(x-y)\,\big]^\top\;g(y)\,d_yS, \quad x\in S, \tag{5.6}$$

$$\mathcal{L}(g)(x) := \big\{\,\mathcal{T}(\partial_x,n(x))W(g)(x)\,\big\}^\pm, \quad x\in S. \tag{5.7}$$

The boundary operators $\mathcal{H}$ and $\mathcal{L}$ are strongly elliptic, but not formally self-adjoint pseudodifferential operators of order -1 and 1 respectively. Their principal homogeneous symbol matrices are strongly elliptic. The operators $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ are Calderon-Zygmund type singular integral operators (pseudodifferential operators of zero order).

Theorem 5.2 *Let $S\in C^\infty$, $1<p<\infty$, $1\le t\le\infty$, $s\in\mathbb{R}$. The operators*

$$\begin{aligned}
&\mathcal{H}:\big[\,H_p^{s-1}(S)\,\big]^4 \to \big[\,H_p^{s}(S)\,\big]^4, &&\mathcal{H}:\big[\,B_{p,t}^{s-1}(S)\,\big]^4 \to \big[\,B_{p,t}^{s}(S)\,\big]^4,\\
&\mathcal{K},\widetilde{\mathcal{K}}:\big[\,H_p^{s}(S)\,\big]^4 \to \big[\,H_p^{s}(S)\,\big]^4, &&\mathcal{K},\widetilde{\mathcal{K}}:\big[\,B_{p,t}^{s}(S)\,\big]^4 \to \big[\,B_{p,t}^{s}(S)\,\big]^4,\\
&\mathcal{L}:\big[\,H_p^{s}(S)\,\big]^4 \to \big[\,H_p^{s-1}(S)\,\big]^4, &&\mathcal{L}:\big[\,B_{p,t}^{s}(S)\,\big]^4 \to \big[\,B_{p,t}^{s-1}(S)\,\big]^4,
\end{aligned}$$

are continuous.

The following operator equalities hold in appropriate function spaces:

$$\mathcal{K}\,\mathcal{H} = \mathcal{H}\,\widetilde{\mathcal{K}}, \quad \mathcal{L}\,\mathcal{H} = -\tfrac{1}{4}I_4 + \widetilde{\mathcal{K}}^2, \quad \mathcal{H}\,\mathcal{L} = -\tfrac{1}{4}I_4 + \mathcal{K}^2. \tag{5.8}$$

Theorem 5.3 *There is a positive constant c_1 such that*

$$\operatorname{Re}\langle -\mathcal{H}h\,,\,h\rangle_S \geq c_1\,||\,h\,||^2_{\left[\,H_2^{-1/2}(S)\,\right]^4} \tag{5.9}$$

for any complex-valued vector function $h \in \left[\,H_2^{-1/2}(S)\,\right]^4$.

The operators

$$\mathcal{H} : \left[\,H_p^{s-1}(S)\,\right]^4 \to \left[\,H_p^{s}(S)\,\right]^4, \quad \mathcal{H} : \left[\,B_{p,t}^{s-1}(S)\,\right]^4 \to \left[\,B_{p,t}^{s}(S)\,\right]^4,$$

are invertible for all $1 < p < \infty$, $1 \leq t \leq \infty$, and $s \in \mathbb{R}$.

Let $U \in [\,W_p^1(\Omega)\,]^4 = [\,H_p^1(\Omega)\,]^4$ with $1 < p < \infty$ be a solution to the homogeneous equation $A(\partial_x)\,U = 0$ in Ω. Then U can be represented in the form of the single-layer potential

$$U(x) = V\left([\,\mathcal{H}\,]^{-1}\{\,U\,\}^+\right)(x), \quad x \in \Omega. \tag{5.10}$$

The same representation holds also for a solution $U \in [\,B_{p,t}^1(\Omega)\,]^4$ with $1 < p < \infty$ and $1 \leq t \leq \infty$.

Similar representation is also valid in the domain Ω^-,

$$U(x) = V\left([\,\mathcal{H}\,]^{-1}\{\,U\,\}^-\right)(x), \quad x \in \Omega^-, \tag{5.11}$$

and the vector function defined by (5.11) belongs to the Beppo-Levi space in Ω^-.

If Ω is a Lipschitz domain, then some of the above formulated mapping properties of the layer potentials and the corresponding boundary integral operators still hold true (cf. [Ag1], [Ag2], [Co1], [DoM], [Gao], [McLean], [MeMi], [Ne1], [Ver]). In particular, in the case of Lipschitz domains Theorems 5.1, 5.2, and 5.3 are true with $p = t = 2$, and $s = -1/2$, and the boundary integral operators $\mathcal{H}$, $\widetilde{\mathcal{K}}$, $\mathcal{K}$, and $\mathcal{L}$ have the following continuous mapping properties:

$$\begin{aligned}
\mathcal{H} &: \left[\,H_2^{-\frac{1}{2}}(S)\,\right]^4 \to \left[\,H_2^{\frac{1}{2}}(S)\,\right]^4, \\
\widetilde{\mathcal{K}} &: \left[\,H_2^{-\frac{1}{2}}(S)\,\right]^4 \to \left[\,H_2^{-\frac{1}{2}}(S)\,\right]^4, \\
\mathcal{K} &: \left[\,H_2^{\frac{1}{2}}(S)\,\right]^4 \to \left[\,H_2^{\frac{1}{2}}(S)\,\right]^4, \\
\mathcal{L} &: \left[\,H_2^{\frac{1}{2}}(S)\,\right]^4 \to \left[\,H_2^{-\frac{1}{2}}(S)\,\right]^4.
\end{aligned} \tag{5.12}$$

Moreover, the coercivity property (5.9) for the operator $\mathcal{H}$ remains valid, and hence the operator in (5.12) is invertible for Lipschitz surfaces. Therefore, for a solution $U \in [\,W_2^1(\Omega)\,]^4 =$

$[\,H^1_2(\Omega)\,]^4$ to the corresponding homogeneous differential equation $A(\partial_x)U=0$, the integral representation formulas (5.10) and (5.11) are valid. Note that the operator equalities (5.8) also remain valid in the corresponding function spaces.

Remark 5.4 *It is evident that $\Gamma^*(x)=\Gamma^\top(-x)=\Gamma^\top(x)$ is a fundamental matrix of the formally adjoint differential operator $A^*(\partial)=A^\top(-\partial)=A^\top(\partial)$ (see (2.8)).*
All the above formulated mapping properties in the cases of smooth and Lipschitz domains are also valid for the corresponding "adjoint" single- and double-layer potentials constructed by the fundamental matrix $\Gamma^*(x-y)=\Gamma^\top(x-y)$:

$$V^*(h)(x)=\int\limits_S \Gamma^*(x-y)\,h(y)\,d_yS,\quad x\in\mathbb{R}^3\setminus S,$$
$$W^*(g)(x)=\int\limits_S \big[\,\mathcal{T}(\partial_y,n(y))[\Gamma^*(x-y)]^\top\,\big]^\top\,g(y)\,d_yS,\quad x\in\mathbb{R}^3\setminus S,$$

and for the boundary integral operators generated by them:

$$\mathcal{H}^*(h)(x):=\int\limits_S \Gamma^*(x-y)\,h(y)\,d_yS,\quad x\in S, \tag{5.13}$$
$$\mathcal{K}^*(h)(x):=\int\limits_S \big[\,\widetilde{\mathcal{T}}(\partial_x,n(x))\Gamma^*(x-y)\,\big]\,h(y)\,d_yS,\quad x\in S, \tag{5.14}$$
$$\widetilde{\mathcal{K}}^*(g)(x):=\int\limits_S \big[\,\mathcal{T}(\partial_y,n(y))[\Gamma^*(x-y)]^\top\,\big]^\top\,g(y)\,d_yS,\quad x\in S,$$
$$\mathcal{L}^*(g)(x):=\big\{\,\widetilde{\mathcal{T}}(\partial_x,n(x))W^*(g)(x)\,\big\}^\pm,\quad x\in S.$$

One can easily check that the operators $\mathcal{H}^*$, $\mathcal{K}^*$, $\widetilde{\mathcal{K}}^*$, and $\mathcal{L}^*$ are formally adjoint operators to $\mathcal{H}$, $\mathcal{K}$, $\widetilde{\mathcal{K}}$, and $\mathcal{L}$ respectively.
We have the following jump relations for the "adjoint" layer potentials:

$$\begin{aligned}
&\big\{V^*(h)\big\}^+=\big\{V^*(h)\big\}^-=:\mathcal{H}^*h\quad \textit{on}\quad S,\\
&\big\{\widetilde{\mathcal{T}}(\partial,n)V^*(h)\big\}^\pm=\big[\mp\tfrac12\,I_4+\mathcal{K}^*\big]h\quad \textit{on}\quad S,\\
&\big\{W^*(g)\big\}^\pm=\big[\pm\tfrac12\,I_4+\widetilde{\mathcal{K}}^*\big]g\quad \textit{on}\quad S,\\
&\big\{\,\widetilde{\mathcal{T}}(\partial,n)W^*(g)\,\big\}^+=\big\{\,\widetilde{\mathcal{T}}(\partial,n)W^*(g)\,\big\}^-=:\mathcal{L}^*g\quad \textit{on}\quad S.
\end{aligned}$$

Further, the operators $\mathcal{H}^*$ and $\mathcal{L}^*$ have the same mapping properties as the operators $\mathcal{H}$ and $\mathcal{L}$ respectively, while $\mathcal{K}^*$ and $\widetilde{\mathcal{K}}^*$ have the same mapping properties as the operators $\widetilde{\mathcal{K}}$ and $\mathcal{K}$ respectively (see Theorems 5.1, 5.2, and 5.3). Moreover, any solution $U^*\in[\,W^1_p(\Omega)\,]^4$ with

$1 < p < \infty$ to the homogeneous equation $A^*(\partial_x)\,U^* = 0$ in Ω can be represented in the form of the single-layer potential

$$U^*(x) = V^*\left([\,\mathcal{H}^*\,]^{-1}\{\,U^*\,\}^+\right)(x), \quad x \in \Omega, \tag{5.15}$$

where $\{\,U^*\,\}^+$ is the trace of U^* on $S = \partial\Omega$.

6 Appendix B: Pseudodifferential equations on manifolds with boundary

Let $\overline{\mathcal{M}} \in C^\infty$ be a compact, n-dimensional, nonintersecting manifold with boundary $\partial\mathcal{M} \in C^\infty$ and let $\mathcal{A}$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\mathfrak{S}(\mathcal{A}; x, \xi)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system ($x \in \overline{\mathcal{M}},\ \xi \in \mathbb{R}^n \setminus \{0\}$).

Let $\lambda_1(x), \cdots, \lambda_N(x)$ be the eigenvalues of the matrix

$$[\,\mathfrak{S}(\mathcal{A}; x, 0, \cdots, 0, +1)\,]^{-1}[\,\mathfrak{S}(\mathcal{A}; x, 0, \cdots, 0, -1)\,], \quad x \in \partial\overline{\mathcal{M}},$$

and introduce the notation

$$\delta_j(x) = \mathrm{Re}\big[\,(2\pi\, i)^{-1} \ln \lambda_j(x)\,\big],\ j = 1, \cdots, N.$$

Here the branch in the logarithmic function $\ln \tau$ is chosen with regard to the inequality $-\pi < \arg \tau \le \pi,\ j = 1, \cdots, N$. Due to the strong ellipticity of $\mathcal{A}$ we have the strong inequality $-1/2 < \delta_j(x) < 1/2$ for $x \in \overline{\mathcal{M}}$.

Fredholm properties of strongly elliptic pseudodifferential operators are described by the following assertion (see, e.g., [Esk1], [Grb1], [Sh1]).

Theorem 6.1 *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, and let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator of order $\nu \in \mathbb{R}$, i.e., there is a positive constant c_0 such that*

$$\mathrm{Re}\big[\,\mathfrak{S}(\mathcal{A}; x, \xi)\zeta \cdot \zeta\,\big] \ge c_0\,|\zeta|^2$$

for $x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\zeta \in \mathbb{C}^m$.

Then the operators

$$\mathcal{A} \ : \ [\widetilde{H}^s_p(\mathcal{M})]^N \to [H^{s-\nu}_p(\mathcal{M})]^N, \tag{6.1}$$

$$\mathcal{A} \ : \ [\widetilde{B}^s_{p,q}(\mathcal{M})]^N \to [B^{s-\nu}_{p,q}(\mathcal{M})]^N, \tag{6.2}$$

are Fredholm operators with index zero if

$$\frac{1}{p} - 1 + \sup_{x\in\partial\mathcal{M},\,1\le j\le N} \delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{x\in\partial\mathcal{M},\,1\le j\le N} \delta_j(x). \tag{6.3}$$

Moreover, the null-spaces and indices of the operators (6.1) *and* (6.2) *are the same* (*for all values of the parameter* $q \in [1, +\infty]$) *provided* p *and* s *satisfy the inequality* (6.3).

Remark 6.2 *It is evident, that if the principal homogeneous symbol matrix* $\mathfrak{S}(\mathcal{A}; x, \xi)$ *is positive definite, then all eigenvalues* λ_j *are positive numbers and all the numbers* δ_j *equal to zero. Therefore, the condition* (6.3) *simplifies and takes the form*

$$\frac{1}{p} - 1 + < s - \frac{\nu}{2} < \frac{1}{p}.$$

7 Appendix C: Qualitative properties of mixed BVP of piezoelasticity

Here we present the results obtained in [BCN2] about qualitative properties of solutions to the interior and exterior mixed BVPs. These results are obtained by a different method which is based on the representation of solutions by formulas (5.10)-(5.11) leading to the pseudodifferential equations with the Steklov-Poincaré type operators containing the inverse single layer boundary operator which is not available in explicit form (for details see [BCN2, Chapter 5]).

The Steklov-Poincaré type operators for the interior and exterior domains are defined, respectively, by the relations:

$$\mathcal{A}^+ = (-2^{-1}I_4 + \widetilde{\mathcal{K}})\,\mathcal{H}^{-1}, \quad \mathcal{A}^- = -(2^{-1}I_4 + \widetilde{\mathcal{K}})\,\mathcal{H}^{-1}. \tag{7.1}$$

They are pseudodifferential operators of order 1 in view of Theorems 5.2 and 5.3.

If $S \in C^\infty$, $1 < p < \infty$, $1 \le t \le \infty$, and $s \in \mathbb{R}$, then the operators are continuous in the following setting:

$$\mathcal{A}^+ \ : \ \big[B^s_{p,t}(S)\big]^4 \to \big[B^{s-1}_{p,t}(S)\big]^4, \tag{7.2}$$

$$\mathcal{A}^- \ : \ \big[B^s_{p,t}(S)\big]^4 \to \big[B^{s-1}_{p,t}(S)\big]^4. \tag{7.3}$$

Using the Green formulas (2.25) and (3.9) along with (2.27) for the vector function $U = V(\mathcal{H}^{-1}g)$ with $g \in [H_2^{\frac{1}{2}}(S)]^4$, we get

$$\int\limits_{\Omega^\pm} E(U,U)\,dx = \big\langle \mathcal{A}^\pm g\,,\,g\big\rangle_S,$$

implying that the principal homogeneous symbol matrices of the Steklov-Poincaré type operators $\mathcal{A}^\pm$,

$$\mathfrak{S}(\mathcal{A}^\pm;x,\xi_1,\xi_2) = \mathfrak{S}(-2^{-1}I_4 \pm \widetilde{\mathcal{K}};x,\xi_1,\xi_2)\;\mathfrak{S}(\mathcal{H}^{-1};x,\xi_1,\xi_2),\ \ x\in S,\ \ (\xi_1,\xi_2)\in\mathbb{R}^2,$$

are strongly elliptic symbols. Consequently, the operators (7.2) and (7.3) are strongly elliptic pseudodifferential operators with zero index.

Denote the eigenvalues of the matrices

$$M^\pm := [\,\mathfrak{S}(\mathcal{A}^\pm;x,0,+1)\,]^{-1}\,\mathfrak{S}(\mathcal{A}^\pm;x,0,-1),\quad x\in\ell,$$

by $\lambda_j^\pm(x)$, $j=1,2,3,4$, and put

$$\delta_j^\pm(x) = \mathrm{Re}\big[\,(2\pi\, i)^{-1}\ln\lambda_j^\pm(x)\,\big] = \tfrac{1}{2\pi}\arg\lambda_j^\pm(x) \in \Big(-\tfrac12,\tfrac12\Big),\ j=1,2,3,4, \tag{7.4}$$

$$a_1^\pm = \inf_{x\in\ell,\,1\le j\le 4}\delta_j^\pm(x),\quad a_2^\pm = \sup_{x\in\ell,\,1\le j\le 4}\delta_j^\pm(x). \tag{7.5}$$

The last inclusion in (7.4) follows from the strong ellipticity property of the Steklov-Poincaré type operators. Evidently,

$$-\frac12 < a_1^+ \leqslant a_2^+ < \frac12,\quad -\frac12 < a_1^- \leqslant a_2^- < \frac12. \tag{7.6}$$

In accordance with Theorem 6.1 and Remark 6.2 in Appendix B, we have the following assertions (cf., Subsection 5.7 in [BCN2]).

Theorem 7.1 *The operators*

$$r_{S_N}\,\mathcal{A}^\pm \quad : \quad [\widetilde{H}^s_p(S_N)]^4 \to [H^{s-1}_p(S_N)]^4,$$

$$r_{S_N}\,\mathcal{A}^\pm \quad : \quad [\widetilde{B}^s_{p,t}(S_N)]^4 \to [B^{s-1}_{p,t}(S_N)]^4,\quad t\ge 1,$$

are invertible if

$$-\frac{1}{2}+a_2^{\pm} < s-\frac{1}{p} < \frac{1}{2}+a_1^{\pm}\,.$$

Theorem 7.2 *Let*

$$f^{(D)} \in \big[B_{p,p}^{1-\frac{1}{p}}(S_D)\big]^4, \qquad F^{(N)} \in \big[B_{p,p}^{-\frac{1}{p}}(S_N)\big]^4, \tag{7.7}$$

and let

$$\frac{4}{3-2a_2^{+}} < p < \frac{4}{1-2a_1^{+}}\,. \tag{7.8}$$

Then the mixed interior boundary value problem (2.11)-(2.13) *has a unique solution* $U \in [W_p^1(\Omega^+)]^4$ *which is representable in the form of single-layer potential,*

$$U = V\big(\mathcal{H}^{-1}\,(f^{(e)}+\widetilde{f})\big),$$

where $f^{(e)} \in \big[B_{p,p}^{1-1/p}(S)\big]^4$ *is a fixed extension of the vector function* $f^{(D)} \in \big[B_{p,p}^{1-1/p}(S_D)\big]^4$ *from* S_D *onto* S *preserving the functional space and* $\widetilde{f} \in \big[\widetilde{B}_{p,p}^{1-1/p}(S_N)\big]^4$ *is defined by the uniquely solvable pseudodifferential equation*

$$r_{_{S_N}}\,\mathcal{A}^+\,\widetilde{f} = F^{(0)} \quad \text{on} \quad S_N$$

with

$$F^{(0)} := F^{(N)} - r_{_{S_N}}\,\mathcal{A}^+\,f^{(e)} \in \big[B_{p,p}^{-1/p}(S_N)\big]^4.$$

Theorem 7.3 *Let conditions* (7.7) *be fulfilled and*

$$\frac{4}{3-2a_2^{-}} < p < \frac{4}{1-2a_1^{-}}. \tag{7.9}$$

Then the mixed exterior BVP (2.15)-(2.18) *has a unique solution* $U \in [W_{p,loc}^1(\Omega^-)]^4$ *which is representable in the form of single layer-potential,*

$$U = -V\big(\mathcal{H}^{-1}\,(f^{(e)}+\widetilde{f})\big),$$

where $f^{(e)} \in \big[B_{p,p}^{1-1/p}(S)\big]^4$ *is a fixed extension of the vector function* $f^{(D)} \in \big[B_{p,p}^{1-1/p}(S_D)\big]^4$ *from* S_D *onto* S *preserving the functional space and* $\widetilde{f} \in \big[\widetilde{B}_{p,p}^{1-1/p}(S_N)\big]^4$ *is defined by the uniquely solvable pseudodifferential equation*

$$r_{_{S_N}}\,\mathcal{A}^-\,\widetilde{f} = F^{(0)} \quad \text{on} \quad S_N$$

with

$$F^{(0)} := F^{(N)} - r_{_{S_N}} \mathcal{A}^- f^{(e)} \in \big[B^{-1/p}_{p,p}(S_N) \big]^4.$$

Theorem 7.4 *Let inclusions* (7.7) *and inequality* (7.8) *hold and*

$$-\frac{1}{2} + a_2^+ < s - \frac{1}{r} < \frac{1}{2} + a_1^+, \quad 1 < r < \infty.$$

Let $U \in [W^1_p(\Omega^+)]^4$ *be a unique solution to the mixed interior BVP* (2.11)-(2.13). *Then the following statements are true:*

(i) *If*

$$f^{(D)} \in [B^s_{r,r}(S_D)]^4, \qquad F^{(N)} \in [B^{s-1}_{r,r}(S_N)]^4,$$

then $U \in \big[H_r^{s+\frac{1}{r}}(\Omega^+) \big]^4$;

(ii) *If*

$$f^{(D)} \in [B^s_{r,t}(S_D)]^4, \qquad F^{(N)} \in [B^{s-1}_{r,t}(S_N)]^4, \quad 1 \le t \le \infty,$$

then

$$U \in \big[B^{s+\frac{1}{r}}_{r,t}(\Omega^+) \big]^4;$$

(iii) *If* $\alpha > 0$ *and*

$$f^{(D)} \in [C^{\,\alpha}(\overline{S_D})]^4, \qquad F^{(N)} \in [B^{\alpha-1}_{\infty,\infty}(S_N)]^4,$$

then

$$U \in \bigcap_{\beta < \kappa} [\, C^{\,\beta}(\,\overline{\Omega^+}\,)\,]^4,$$

where $0 < \kappa = \min\{\alpha,\ \frac{1}{2} + a_1^+\}$.

Theorem 7.5 *Let inclusions* (7.7) *and inequalities* (7.9) *hold, and*

$$-\frac{1}{2} + a_2^- < s - \frac{1}{r} < \frac{1}{2} + a_1^-, \quad 1 < r < \infty. \tag{7.10}$$

Let $U \in [W^1_{p,loc}(\Omega^-)]^4$ *be a unique solution to the mixed exterior BVP* (2.15)-(2.18). *Then the following statements are true:*

(i) *If*

$$f^{(D)} \in [B^s_{r,r}(S_D)]^4, \qquad F^{(N)} \in [B^{s-1}_{r,r}(S_N)]^4,$$

then $U \in \big[H_{r,loc}^{s+\frac{1}{r}}(\Omega^-) \big]^4$;

(ii) *If*

$$f^{(D)} \in [B^s_{r,t}(S_D)]^4, \qquad F^{(N)} \in [B^{s-1}_{r,t}(S_N)]^4, \quad 1 \le t \le \infty,$$

then

$$U \in \big[\, B^{s+\frac{1}{r}}_{r,t,loc}(\Omega^-)\,\big]^4;$$

(iii) *If* $\alpha > 0$ *and*

$$f^{(D)} \in [C^{\,\alpha}(\overline{S_D})]^4, \qquad F^{(N)} \in [B^{\alpha-1}_{\infty,\,\infty}(S_N)]^4,$$

then

$$U \in \bigcap_{\beta<\kappa} [\, C^{\,\beta}(\,\overline{\Omega^-}\,)\,]^4,$$

where $0 < \kappa = \min\{\alpha,\ \frac{1}{2} + a_1^-\}$.

Remark 7.6 *Similar theorems are valid also for the mixed interior and exterior BVPs for the adjoint operator* $A^*(\partial)$*. In particular, the interior mixed BVP for the operator* $A^*(\partial)$ *is formulated as follows: Find a vector function* $U^* \in \big[W^1_q(\Omega)\big]^4 = \big[H^1_q(\Omega)\big]^4$, $q > 1$, *satisfying the differential equation*

$$A^*(\partial_x)\, U^*(x) = 0 \ \text{ in } \ \ \Omega,$$

the Dirichlet type boundary condition on S_D

$$\{U^*(x)\}^+ = f^*(x), \quad x \in S_D,$$

and the Neumann type boundary condition on S_N

$$\{\widetilde{\mathcal{T}}(\partial,n)U^*(x)\}^+ = F^*(x), \quad x \in S_N,$$

where

$$f^* \in \big[B^{1-\frac{1}{q}}_{q,q}(S_D)\big]^4, \quad F^* \in \big[B^{-\frac{1}{q}}_{q,q}(S_N)\big]^4.$$

The exterior mixed BVP is formulated analogously with additional decay condition at infinity similar to (2.18).

The Steklov-Poincaré type operators associated with the adjoint differential operator $A^*(\partial)$ *for the interior and exterior domains are defined, respectively, by the relation:*

$$\mathcal{A}^+_* = (-2^{-1}I_4 + \mathcal{K}^*)\,[\mathcal{H}^*]^{-1}, \quad \mathcal{A}^-_* = -(2^{-1}I_4 + \mathcal{K}^*)\,[\mathcal{H}^*]^{-1}, \tag{7.11}$$

where the operators $\mathcal{H}^*$ *and* $\mathcal{K}^*$ *are defined by* (5.13) *and* (5.14)*. These operators are strongly elliptic pseudodifferential operators of order* -1*.*

Denote the eigenvalues of the matrices

$$M_*^{\pm} := [\,\mathfrak{S}(\mathcal{A}_*^{\pm}; x, 0, +1)\,]^{-1}\,\mathfrak{S}(\mathcal{A}_*^{\pm}; x, 0, -1), \quad x \in \ell,$$

by $\lambda_{j*}^{\pm}(x)$*,* $j = 1, 2, 3, 4$*, and put*

$$\delta_{j*}^{\pm}(x) = \mathrm{Re}\big[\,(2\pi\, i)^{-1} \ln \lambda_{j*}^{\pm}(x)\,\big] = \tfrac{1}{2\pi} \arg \lambda_{j*}^{\pm}(x) \in \Big(-\tfrac{1}{2}, \tfrac{1}{2}\Big),\ j = 1, 2, 3, 4, \tag{7.12}$$

$$a_{1*}^{\pm} = \inf_{x\in\ell,\, 1\le j\le 4} \delta_{j*}^{\pm}(x), \quad a_{2*}^{\pm} = \sup_{x\in\ell,\, 1\le j\le 4} \delta_{j*}^{\pm}(x). \tag{7.13}$$

Evidently,

$$-\frac{1}{2} < a_{1*}^{+} \leqslant a_{2*}^{+} < \frac{1}{2}, \quad -\frac{1}{2} < a_{1*}^{-} \leqslant a_{2*}^{-} < \frac{1}{2}. \tag{7.14}$$

The Steklov-Poincaré type operators (7.11) *have the same mapping properties as their counterpart operators* (7.1) *described in Theorem 7.1 with* $a_{1*}^{\pm}$ *and* $a_{2*}^{\pm}$ *for* $a_1^{\pm}$ *and* $a_2^{\pm}$ *respectively. Therefore, assertions similar to Theorems 7.2, 7.3, 7.4, and 7.5 with* $a_{1*}^{\pm}$ *and* $a_{2*}^{\pm}$ *for* $a_1^{\pm}$ *and* $a_2^{\pm}$*, respectively, hold also true for the above formulated interior and exterior mixed BVPs for the adjoint operator* $A^*(\partial)$*.*

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References

[Ag1] M. S. Agranovich, *Elliptic operators on closed manifolds.* Partial Differential Equations VI, Encyclopedia Math. Sciences, 63, 1-130. Springer-Verlag, Berlin, 1994.

[Ag2] M. S. Agranovich, *Spectral properties of potential type operators for a certain class of strongly elliptic systems on smooth and Lipschitz surfaces.* Trans. Mosc. Math. Soc., **62**, 2001, 1–47.

[BL] J. Bergh, J. Löfström, *Interpolation spaces, an introduction.* Springer-Verlag, Berlin-Heidelberg, 1976.

[BCD1] T. Buchukuri, O. Chkadua, and R. Duduchava, *Crack-type boundary value problems of electro–elasticity*. Operator Theory: Advances and Application **147**(2004), 189–212.

[BCN2] T. Buchukuri, O. Chkadua, D. Natroshvili, *Mathematical problems of generalized thermo-electro-magneto-elasticity theory*. Memoirs on Differential Equations and Mathematical Physics, **68** (2016), 1–166.

[BCNS1] T. Buchukuri, O. Chkadua, D. Natroshvili, A.-M. Sändig, *Solvability and regularity results to boundary-transmission problems for metallic and piezoelectric elastic materials*. Math. Nachr., 282, No. 8 (2009), 1079-1110.

[BG1] T. Buchukuri, and T.Gegelia, *Some dynamical problems of the theory of electro–elasticity*. Memoirs on Differential Equations and Mathematical Physics, **10**(1997), No. 1, 1-53.

[Co1] M. Costabel, *Boundary integral operators on Lipschitz domains: elementary results*. SIAM J. Math. Anal., **19** (1988), 613–626.

[CW1] M. Costabel, and W.L. Wendland, *Strong ellipticity of boundary integral operators*. J. Reine Angew. Math., **372** (1986), 34–63.

[DL1] R. Dautray, and J.L. Lions, *Mathematical analysis and numerical methods for science and technology, Vol. 4. Integral equations and numerical methods*. Springer Verlag, Berlin, Heidelberg, 1990.

[DNS1] R. Duduchava, D. Natroshvili, and E. Shargorodsky, *Basic boundary value problems of thermoelasticity for anisotropic bodies with cuts* I, II. Georgian Math. J., **2** (1995), No. 2, 123–140, No. 3, 259–276.

[Esk1] G. Eskin, *Boundary value problems for elliptic pseudodifferential equations*. Transl. of Mathem. Monographs, Amer. Math. Soc. **52**, Providence, Rhode Island, 1981.

[Gao] W. Gao, *Layer potentials and boundary value problems for elliptic systems in Lipschitz domains*. Journal of Functional Analysis, **95** (1991), 377–399.

[Grb1] G. Grubb, *Pseudodifferential boundary problems in L_p spaces*. Comm. P. D. E., **15**(1990), No. 3, 289–340.

[Jo1] F. John, *Plane waves and spherical means* (*applied to partial differential equations*). Interscience Publishers, London, 1955.

[Ka] M. Kamlah, *Ferroelectric and ferroelastic piezoceramics - modelling of electromechanical hysteresis phenomena*. Continuum Mech. Thermodyn., **13**(2001), 219 – 268.

[KaAk] L.V. Kantorovich, G.P. Akilov, *Functional analysis*, 2nd eddition. Pergamon Press, Oxford-New York-Pars, 1982.

[KO1] V.A. Kondratiev and O.A. Oleinik, *Boundary value problems for the system of elasticity theory in unbounded domains. Korn's inequalities.* Uspekhi Mat. Nauk **43**, 5(263) (1988), 55–98 .

[KGBB1] V.D. Kupradze, T.G. Gegelia, M.O. Basheleishvili, and T.V. Burchuladze,, *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity.* North-Holland Publ. Comp., Amsterdam, 1979.

[La1] S. Lang, *Guide to the literature of piezoelectricity and pyroelectricity*, **24**. Ferroelectrics, 322(1) (2005), 115-210.

[McLean] W. McLean, *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, Cambridge, 2000.

[MeMi] O. Mendez and M. Mitrea, *The Banach envelopes of Besov and Triebel-Lizorkin spaces and applications to partial differential equations.* The Journal of Fourier Analysis and Applications, **6**, Issue 5, 2000, 503–531.

[Mi1] R.D. Mindlin, *On the equations of motion of piezoelectric crystals.* In: J.R.M. Radok(ed): Problems of Continuum Mechanics. SIAM, Philadelphia, Pennsylvania, 1961, 282-290.

[Mi2] R.D. Mindlin, *Polarization gradient in elastic dielectrics.* Int. J. Solids Structures, **4**(6) (1968), 637-642.

[Mi3] R.D. Mindlin, *Elasticity, piezoelasticity and crystal lattice dynamics.* J. Elasticity, **2**, No. 4 (1972), 217-282.

[MT1] M. Mitrea and M. Taylor, *Potential theory on Lipschitz domains: Sobolev–Besov space results and the Poisson problem*. Journal of Functional Analysis, **176** (2000), 1–79.

[DoM] D. Mitrea, *The transmission problem for multilayered anisotropic elastic bodies with rough interfaces*. Numerical Functional Analysis and Optimisation, **20** (9-10) (1999), 929–956.

[Mur] T. Muramatu, *On Besov spaces and Sobolev spaces of generalized functions defined on a general region*. Publ. Res. Inst. Math. Sci. Kyoto Univ., **9** (1974), 325–396.

[NBC1] D. Natroshvili, T. Buchukuri, O. Chkadua, *Mathematical modelling and analysis of interaction problems for piezoelectric composites*. Rendiconti Academia Nazionale delle Scienze detta dei XL, Memorie di Matematica e Applicazioni 124° (2006), Vol. XXX, fasc. 1, 159-190.

[NCS1] D. Natroshvili, O. Chkadua, and E. Shargorodsky, *Mixed type boundary value problems of the anisotropic elasticity*. Tr. In-ta Prikl. Mat. Tbilisi Univ., **39**(1990), 133-181.

[NT1] D. Natroshvili, T. Tsertsvadze, *On an alternative approach for mixed boundary value problems for the Laplace equation*. Georgian Mathematical Journal, vol. 29, no. 6 (2022), 883-895.

[NT2] D. Natroshvili, T. Tsertsvadze, *On an alternative approach for mixed boundary value problems for the Lamé system*. J. Elast., **153** (2023), 399–422.

[NMT1] D. Natroshvili, M. Mrevlishvili, Z. Tediashvili, *An alternative potential method for mixed steady-state elastic oscillation problems*. Math. Meth. Appl. Sci., **47**, Issue 16, 2024, 12423–12442.

[Ne1] J. Nečas, *Méthodes Directes en Théorie des Équations Élliptiques*. Masson Éditeur, Paris, 1967. (English translation: J. Nečas, , Direct methods in the theory of elliptic equations. Springer-Verlag, Berlin, Heidelberg, 2012.)

[No1] W. Nowacki, *Efecty electromagnetyczne w stalych cialach odksztalcalnych*. Warszawa, Panstwowe Widawnictwo Naukowe, 1983 (Russian translation: Electromagnetic effects in solids, Moscow, Mir, 1986).

[Pa1] Parkus, H., *Magneto-thermoelasticity.* Springer-Verlag, Wien-New York, 1972.

[Qi1] Q.H. Qin, *Fracture mechanics of piezoelastic materials.* WIT Press, 2001.

[Se1] R.T. Seeley, *Singular integrals and boundary value problems.* Amer. J. Math., **88,** 4 (1966), 781-809.

[Sh1] E. Shargorodsky,, *An L_p-analogue of the Vishik-Eskin theory.* Memoirs on Differential Equations and Mathematical Physics, **2**(1994), 41-148.

[To1] R.A. Toupin, *The elastic dielectrics*, J. Rat. Mech. Analysis **5**, 849–915 (1956).

[Tr1] H. Triebel, *Interpolation theory, function spaces, differential operators.* North Holland, Amsterdam, 1978.

[Tr2] H. Triebel, *Theory of function spaces.* Birkhäuser Verlag, Basel-Boston-Stuttgart, 1983.

[Ver] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains.* J. Functional Anal., **59**(3), 1984, 572–611.

[Vo1] W. Voigt, *Lehrbuch der Kristall-Physik.* Leipzig, Teubner, 1910.