

FRANKLIN SYSTEM IN VARIABLE EXPONENT SPACES

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ABSTRACT. In this paper, we study the properties of the Franklin system in variable exponent Lebesgue spaces $L^{p(\cdot)}([0, 1])$. The main result of this paper is that the Franklin system is an unconditional basis in $L^{p(\cdot)}([0, 1])$, $1 < p_- \leq p_+ < \infty$ if and only if the exponent function $p(\cdot)$ belongs to the Muckenhoupt class \mathcal{A} . We establish this equivalence using characterizations based on the Paley function and associated operators.

The classical Franklin system [11], consisting of piecewise linear continuous functions with dyadic knots, is a fundamental complete orthonormal basis in $C[0, 1]$. Its properties are often studied using the exponential estimates established by Ciesielski [6]. It is well known that the Franklin system is an unconditional basis in $L^p[0, 1]$ for $1 < p < \infty$ [5], as well as a basis in H^p for $1/2 \leq p \leq 1$ (unconditional for $1/2 < p \leq 1$ [14, 13]), and VMO [7, 14]. Higher-order spline generalizations [12, 8] and their applications in signal and image processing have also been extensively investigated (see, e.g., [4]).

In this paper, we extend these investigations to variable exponent Lebesgue spaces on $I := [0, 1]$, where the constant exponent p is replaced by a measurable function $p(\cdot)$. These spaces serve as a crucial framework in modern harmonic analysis and partial differential equations (see, e.g., [10]).

Our main result establishes that the Franklin system forms an unconditional basis in these variable exponent Lebesgue spaces, provided the variable exponent satisfies the Muckenhoupt condition \mathcal{A} . The study of such orthogonal systems often relies on the boundedness of the Hardy-Littlewood maximal operator M , which is non-trivial in the variable setting since standard sufficient criteria, like local log-Hölder continuity, are not strictly necessary.

The Muckenhoupt condition \mathcal{A} is a powerful and versatile tool in harmonic analysis. For instance, interesting summability results and properties on Vilenkin groups under the condition \mathcal{A} were obtained in [2]. Also, quite surprising regularity results for the minimizers of double-phase functionals were established based solely on the class \mathcal{A} in the recent work [3].

Let $I := [0, 1]$. A measurable function $p : I \rightarrow [1, \infty]$ is called a variable exponent on I . We denote the set of all such variable exponents by $\mathcal{P}(I)$. For a given $p \in \mathcal{P}(I)$, we define

$$p_- := \operatorname{ess\,inf}_{x \in I} p(x) \quad \text{and} \quad p_+ := \operatorname{ess\,sup}_{x \in I} p(x).$$

The conjugate exponent $p' \in \mathcal{P}(I)$ is defined pointwise by the relation $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

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For a measurable function $f : I \rightarrow \mathbb{R}$, we define the modular

$$\rho_{p(\cdot)}(f) := \int_I |f(x)|^{p(x)} dx,$$

where we use the standard convention $t^\infty := 0$ for $t \in [0, 1]$ and $t^\infty := \infty$ for $t > 1$. The variable exponent Lebesgue space $L^{p(\cdot)}(I)$ is defined as the set of all measurable functions f such that $\rho_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 0$. It becomes a Banach space when equipped with the Luxemburg norm

$$\|f\|_{p(\cdot)} := \inf \{ \lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1 \}.$$

If the exponent p is constant, $L^{p(\cdot)}(I)$ simply coincides with the classical Lebesgue space. Moreover, the generalized Hölder inequality holds in these spaces: for any $f \in L^{p(\cdot)}(I)$ and $g \in L^{p'(\cdot)}(I)$, we have

$$\int_I |f(x)g(x)| dx \leq 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

In the context of harmonic analysis tools, the variable analogue of the classical Muckenhoupt condition plays a pivotal role. For a variable exponent $p \in \mathcal{P}(I)$, we say that p satisfies the \mathcal{A} condition (and write $p \in \mathcal{A}$) if

$$(0.1) \quad [p]_{\mathcal{A}} := \sup_{J \subset I} \frac{1}{|J|} \|\mathbb{1}_J\|_{p(\cdot)} \|\mathbb{1}_J\|_{p'(\cdot)} < \infty,$$

where the supremum is taken over all subintervals $J \subset I$, $|J|$ denotes the Lebesgue measure of J , and $\mathbb{1}_J$ is the indicator function of J . This condition is obviously stable under duality, meaning $[p]_{\mathcal{A}} = [p']_{\mathcal{A}}$. Note that in case $1 < p_- \leq p_+ \leq \infty$, the condition is necessary and sufficient condition for boundedness Hardy-Littlewood maximal operator in $L^{p(\cdot)}([0, 1])$ (see [1], [9], [10]).

The Franklin system $\mathcal{F} = \{f_n : n = 0, 1, \dots\}$ is obtained from the standard Schauder system via the Gram-Schmidt orthonormalization process. Thus, the first two Franklin functions are given by the formulas

$$f_0(t) = 1, \quad f_1(t) = \sqrt{3}(2t - 1), \quad \text{for } t \in [0, 1],$$

and, for $n = 2^m + r$ with $m = 0, 1, \dots$ and $r = 1, \dots, 2^m$, each f_n is both continuous on $[0, 1]$ and linear on each dyadic interval $[t_{i-1}^n, t_i^n]$, where

$$t_i^n = i/2^{m+1}, \quad i = 0, \dots, 2r, \quad \text{and} \quad t_i^n = (i - r)/2^m, \quad i = 2r + 1, \dots, 2^m + r.$$

A fundamental role in the study of the unconditional basis property of \mathcal{F} is played by the Paley function

$$P(f) = P(f, x) = \left\{ \sum_{n=0}^{\infty} [\langle f, f_n \rangle f_n(x)]^2 \right\}^{1/2}, \quad f \in L^1(I),$$

and the function

$$T_\varepsilon(f) = T_\varepsilon(f, x) = \sum_{n=0}^{\infty} \varepsilon_n \langle f, f_n \rangle f_n(x), \quad f \in L^1(I),$$

where $\varepsilon_n = \pm 1$ for $n = 0, 1, \dots$ is a sequence of "signs".

The main results of this article are the following theorems:

Theorem 0.1. *Let $1 < p_- \leq p_+ < \infty$. The following statements are equivalent:*

- (1) $p(\cdot) \in \mathcal{A}$;

- (2) The Franklin system forms an unconditional basis in $L^{p(\cdot)}(I)$;
 (3) There exist constants $C_1, C_2 > 0$ such that

$$C_1 \|f\|_{p(\cdot)} \leq \|P(f)\|_{p(\cdot)} \leq C_2 \|f\|_{p(\cdot)}$$

for every $f \in L^{p(\cdot)}(I)$;

- (4) There exist constants $C_1, C_2 > 0$ such that

$$C_1 \|f\|_{p(\cdot)} \leq \|T_\varepsilon(f)\|_{p(\cdot)} \leq C_2 \|f\|_{p(\cdot)}$$

for every $f \in L^{p(\cdot)}(I)$ and every sequence $\varepsilon = \{\varepsilon_n\}_{n=0}^\infty$ $\varepsilon_n = \pm 1$.

Theorem 0.2. *Let $1 \leq p_- \leq p_+ < \infty$. The following statements are equivalent:*

- (1) $p(\cdot) \in \mathcal{A}$;
 (2) The Franklin system forms a basis in $L^{p(\cdot)}(I)$.

REFERENCES

- [1] D. Adamadze, L. Diening, and T. Kopaliani, Maximal operator on variable exponent spaces. *J. Math. Anal. Appl.*, 554.2 (2026).
- [2] D. Adamadze and T. Kopaliani, Vilenkin-Fourier series in variable Lebesgue spaces. *Math. Inequal. Appl.*, 26 (2023), 4, 977–993.
- [3] D. Adamadze, L. Diening, T. Kopaliani, and J. Ok, Double phase meets Muckenhoupt, *arXiv:2601.20736* (2026).
- [4] Z. Averbuch, P. Neittaanmäki, V.A. Zheludev, *Spline and Spline Wavelet Methods with Applications to Signal and Image Processing. Volume I: Periodic Splines* (Springer, Berlin, 2014).
- [5] S. V. Bochkarev, Some inequalities for the Franklin series. *Anal. Math.*, 1 (1975), 249–257.
- [6] Z. Ciesielski, Properties of the orthonormal Franklin system. *Studia Math.*, 23 (1963), 141–157.
- [7] Z. Ciesielski, Properties of the orthonormal Franklin system II. *Studia Math.*, 27 (1966), 289–323.
- [8] Z. Ciesielski and A. Kamont, Survey on the orthogonal Franklin system. In: *Approximation Theory, DARBA, Sofia, 2002*, 84–132.
- [9] D. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue Spaces. Foundations and Harmonic Analysis*, Birkhäuser, Basel (2013).
- [10] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev spaces with variable exponents. Lecture Notes in Mathematics*, 2017. Springer, Heidelberg (2011).
- [11] Ph. Franklin, A set of continuous orthogonal function. *Math. Ann.*, 100 (1928), 522–528.
- [12] G. G. Gevorkyan and A. Kamont, Unconditionality of general Franklin systems in $L^p[0, 1]$, $1 < p < \infty$. *Studia Math.*, 164(2):161–204, 2004.
- [13] P. Sjölin and J. O. Strömberg, Basis properties of Hardy spaces. *Ark. Mat.*, 21 (1983), 111–125.
- [14] P. Wojtaszczyk, The Franklin system is an unconditional basis in H^1 . *Ark. Mat.*, 20 (1982), 293–300.

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