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HARDY–LITTLEWOOD INEQUALITIES FOR FRACTIONAL INTEGRALS IN CLASSICAL MORREY SPACES

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Abstract. We present two-weight inequalities with power-type weights for fractional integrals in classical Morrey spaces defined on the semi-axis. The derived results involve one-sided and two-sided potentials. We treat both Adams and Spanne type results.

1. Introduction

In our note, we present two-weight inequalities with power weights for fractional integrals in classical Morrey spaces defined on the semi-axis. We consider one-sided (Riemann–Liouville and Weyl integral transforms) and two-sided fractional integral operators. We cover both Adams and Spanne type estimates. Such type inequalities play an important role, for example, in the theory of differential and integral equations (see, e.g., [10]).

The classical Morrey spaces $L^{p,\lambda}$ were introduced by C. Morrey in 1938 (see [9]) to investigate local behavior and regularity of solutions to second order elliptic partial differential equations (see, e.g., [3, 4]). There exists two types of the boundedness of fractional integral operators in Morrey spaces: S. Spanne (unpublished) and D. R. Adams [1] type theorems.

For mapping properties of integral operators involving fractional integrals in Morrey spaces, we refer, e.g., to the monographs [7, 11] and references cited therein.

The weighted inequalities with power weights for fractional integrals in the classical Lebesgue spaces were established by Hardy and Littlewood [5] in the one-dimensional case (see also the monograph [12]). Later, similar results for the higher dimensional Riesz potentials were established by Stein and Weiss [13].

2. Preliminaries

For a measurable function $f : \mathbb{R}_+ \to \mathbb{R}$, the Morrey norm is defined as follows:

$$\|f\|_{M_p^\lambda(\mathbb{R}_+)} = \|f\|_{M_p^\lambda} := \sup_{I\subset\mathbb{R}_+} \left(|I|^{-\lambda} \int\limits_I |f(x)|^p dx \right)^{\frac{1}{p}},$$

where $0 \le \lambda < 1$, $1 \le p < \infty$, and the supremum is taken over all bounded intervals $I \subset \mathbb{R}_+$. If $\lambda = 0$, then $M_p^\lambda(\mathbb{R}_+)$ coincides with the Lebesgue space $L^p(\mathbb{R}_+)$, whose norm is given by

$$\|f\|_{L^p(\mathbb{R}_+)} := \left(\int\limits_{\mathbb{R}_+} |f(x)|^p dx \right)^{1/p}.$$

Denote by I_γ, R_γ, and W_α the fractional integral operators defined on $\mathbb{R}_+$ and given by the formulas

$$I_\gamma f(x) = \int\limits_0^\infty \frac{f(t)}{|x-t|^{1-\gamma}} dt, \quad R_\gamma f(x) = \int\limits_0^x \frac{f(t)}{(x-t)^{1-\gamma}} dt, \quad W_\gamma f(x) = \int\limits_x^\infty \frac{f(t)}{(t-x)^{1-\gamma}} dt,$$

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where $x \in \mathbb{R}_+$ and $0 < \gamma < 1$. It is easy to check that the following pointwise estimates hold for non-negative locally integrable f:

(i)

$$R_\gamma f(x) + W_\gamma f(x) = I_\gamma f(x),$$

(ii)

$$I_\gamma f(x) \geq \max\left\{W_\gamma f(x),\, R_\gamma f(x)\right\},\ f \geq 0.$$

Let us recall, see Theorems A and B, the classical Hardy–Littlewood inequalities for the operators I_γ, R_γ, and W_γ, where throughout we set $p' := \frac{p}{p-1}$:

Theorem A ([5]). *Let $0 < \gamma < 1$, $1 < p \leq q < \infty$, $\alpha < \frac{1}{p'}$, $\beta < \frac{1}{q}$, $\alpha + \beta \geq 0$, and $\frac{1}{p} - \frac{1}{q} = \gamma - \alpha - \beta$. Then there is a positive constant C such that for all f, $\|x^\alpha f(x)\|_{L^p(\mathbb{R}_+)} < \infty$,*

$$\left\|x^{-\beta} I_\gamma f(x)\right\|_{L^q(\mathbb{R}_+)} \leq C \left\|x^\alpha f(x)\right\|_{L^p(\mathbb{R}_+)}.$$

Theorem B (see, e.g., [12, Theo. 5.4]). *Let $1 < p < \infty$, $0 < \gamma < \alpha + \beta + \frac{1}{p}$, $0 \leq \alpha + \beta \leq \gamma$, and $\frac{1}{p} - \frac{1}{q} = \gamma - \alpha - \beta$. Then the following statements are true:*

(i) if $\alpha < \frac{1}{p'}$, then the following inequality holds for all f, $\|x^\alpha f(x)\|_{L^p(\mathbb{R}_+)} < \infty$,

$$\left\|x^{-\beta} W_\gamma f(x)\right\|_{L^q(\mathbb{R}_+)} \leq C \left\|x^\alpha f(x)\right\|_{L^p(\mathbb{R}_+)};$$

(ii) if $\beta < \frac{1}{q}$, then for all f, $\|x^\alpha f(x)\|_{L^p(\mathbb{R}_+)<\infty}$,

$$\left\|x^{-\beta} R_\alpha f(x)\right\|_{L^q(\mathbb{R}_+)} \leq C \left\|x^\alpha f(x)\right\|_{L^p(\mathbb{R}_+)}$$

for the positive constant C.

The next statement is Adams type inequality for fractional integrals which is well-known for fractional integrals, but we are interested in operator norm estimate.

In [6] the following Adams type Stein–Weiss inequality was proved on homogeneous groups but we formulate it for the Riesz potential

$$I_\gamma^{(n)} f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}} dy, \quad 0 < \gamma < n, \quad x \in \mathbb{R}^n,$$

defined on $\mathbb{R}^n$.

Theorem C. *Let $0 < \gamma < n$, $0 \leq \alpha + \beta \leq \gamma < n$, $1 < p < \frac{n}{\gamma - \alpha - \beta}$, $\frac{1}{p} - \frac{1}{q} = \frac{\gamma - \alpha - \beta}{n - \lambda}$, $\beta < \frac{n-\lambda}{q}$, $\alpha < \frac{n}{p'}$, $0 < \lambda < n - (\gamma - \alpha - \beta)p$. Then the following weighted inequality holds for f, $\left\||x|^\alpha f(x)\right\|_{M_p^\lambda(\mathbb{R}^n)} < \infty$,*

$$\left\| |x|^{-\beta} I_\gamma^{(n)} f(x)\right\|_{M_q^\lambda(\mathbb{R}^n)} \leq C \left\||x|^\alpha f(x)\right\|_{M_p^\lambda(\mathbb{R}^n)}.$$

Remark. Taking $\alpha = \beta = \lambda = 0$ in Theorem C, we have the classical Hardy–Littlewood–Sobolev inequality.

3. Main Results

Now we formulate the main statements of our paper; their proofs will appear in a forthcoming publication.

Adams-type results.

Theorem 3.1. *Let $\gamma \in (0,1)$ and let $\alpha, \beta \in \mathbb{R}$. If $0 \leq \alpha + \beta \leq \gamma < 1$, $1 < p < \frac{1}{\gamma - \alpha - \beta}$, $\frac{1}{q} = \frac{1}{p} - \frac{\gamma - \alpha - \beta}{1 - \lambda}$, $0 < \lambda < 1 - (\gamma - \alpha - \beta)p$, and $\beta < \frac{1-\lambda}{q}$. Then there is a positive constant C such that the weighted inequality holds for all f, $\|x^\alpha f(x)\|_{M_p^\lambda} < \infty$:*

$$\left\|x^{-\beta} R_\gamma f(x)\right\|_{M_q^\lambda} \leq C \left\|x^\alpha f(x)\right\|_{M_p^\lambda}.$$

Theorem 3.2. *Let* $0<\gamma<1$, $0\le\alpha+\beta\le\gamma<1$, $1<p<\frac{1}{\gamma-\alpha-\beta}$, $\frac{1}{p}-\frac{1}{q}=\frac{\gamma-\alpha-\beta}{1-\lambda}$, $\alpha<\frac{1}{p'}$, *and* $0<\lambda<1-(\gamma-\alpha-\beta)p$*. Then the following inequality holds for all* f, $\|x^{\alpha}f(x)\|_{M_p^{\lambda}(\mathbb{R}_+)}<\infty$*:*

$$\left\|x^{-\beta}W_{\gamma}f(x)\right\|_{M_q^{\lambda}(\mathbb{R}_+)}\le C\left\|x^{\alpha}f(x)\right\|_{M_p^{\lambda}(\mathbb{R}_+)}$$

with some positive constant C*.*

By using Theorems 3.1 and 3.2 and the relation $I_{\alpha}f(x)=W_{\gamma}f(x)+R_{\gamma}f(x)$, we get the next statement:

Theorem 3.3. *Let* $\gamma\in(0,1), 0\le\alpha+\beta\le\gamma<1$, $1<p<\frac{1}{\gamma-\alpha-\beta}$, $\frac{1}{p}-\frac{1}{q}=\frac{\gamma-\alpha-\beta}{1-\lambda}$, $0<\lambda<1-(\gamma-\alpha-\beta)p$, $\alpha<\frac{1}{p'}$, *and* $\beta<\frac{1-\lambda}{q}$*. Then the following estimate holds for all* f, $\|x^{\alpha}f(x)\|_{M_p^{\lambda}}<\infty$*:*

$$\left\|x^{-\beta}I_{\gamma}f(x)\right\|_{M_q^{\lambda}(\mathbb{R}+)}\le C\left\|x^{\alpha}f(x)\right\|_{M_p^{\lambda}(\mathbb{R}+)}$$

with some positive constant C*.*

Spanne type Hardy–Littlewood Inequalities. Now we discuss Spanne type results for fractional integrals in the classical Morrey spaces.

Theorem 3.4. *Let* $0<\gamma<1$, $1<p\le q<\infty$, $\alpha<1/p'$, $0<\beta<\frac{1}{q}$, $\gamma\ge\alpha+\beta\ge0$, $\frac{1}{p}-\frac{1}{q}=\gamma-\alpha-\beta$, $0<\lambda<1$, $0<\mu<1$, *and* $\frac{\lambda}{p}=\frac{\mu}{q}$*. Then there is a positive constant* C *such that for all* f, $\|x^{\alpha}f(x)\|_{M_p^{\lambda}}<\infty$*, the inequality*

$$\left\|x^{-\beta}I_{\gamma}f(x)\right\|_{M_q^{\mu}(\mathbb{R}_+)}\le C\left\|x^{\alpha}f(x)\right\|_{M_p^{\lambda}(\mathbb{R}_+)}$$

holds.

Theorem 3.5. *Let* $1<p\le q<\infty$, $\alpha<\frac{1}{p'}$, $\beta>0$, $\alpha+\beta\le\gamma<\alpha+\beta+\frac{1}{p'}$, $\frac{1}{p}-\frac{1}{q}=\gamma-\alpha-\beta$, $0<\lambda<1$, $0<\mu<1$, *and* $\frac{\lambda}{p}=\frac{\mu}{q}$*. Then there is a constant* C *such that for all* f, $\|x^{\alpha}f(x)\|_{M^{p,\lambda}(\mathbb{R}_+)}<\infty$*, the inequality*

$$\left\|x^{-\beta}W_{\gamma}f(x)\right\|_{M_q^{\mu}(\mathbb{R}_+)}\le C\left\|x^{\alpha}f(x)\right\|_{M_p^{\lambda}(\mathbb{R}_+)}$$

holds.

Theorem 3.6. *Let* $1<p\le q<\infty$, $0<\beta<\frac{1}{q}$, $0<\gamma<\alpha+\beta+\frac{1}{p'}$, $\frac{1}{p}-\frac{1}{q}=\gamma-\alpha-\beta$, $0<\lambda,\mu<1$, *and* $\frac{\lambda}{p}=\frac{\mu}{q}$*. Then the following inequality holds for all* f, $\|x^{\alpha}f(x)\|_{M^{p,\lambda}(\mathbb{R}_+)}<\infty$,

$$\left\|x^{-\beta}R_{\gamma}f(x)\right\|_{M_q^{\mu}(\mathbb{R}_+)}\le C\left\|x^{\alpha}f(x)\right\|_{M_p^{\lambda}(\mathbb{R}_+)}$$

for some positive constant C*.*

We can introduce a Morrey norm in a different way: we say that $f\in M^{p,s}(\mathbb{R}_+)$ if

$$\|f\|_{M^{p,s}(\mathbb{R}_+)}=\sup_{I\subset\mathbb{R}_+}|I|^{\frac{1}{s}-\frac{1}{p}}\left(\int\limits_I|f(x)|^p dx\right)^{\frac{1}{p}},\ 1<p<s<\infty.$$

It is convenient to formulate Theorems 3.4–3.6 in terms of $M^{p,s}(\mathbb{R}_+)$ norms.

Theorem 3.7. *Let* $0<\gamma<1$, $1<p\le q<\infty$, $p\le s<\infty$, *and* $q\le r<\infty$*. Suppose that* $\alpha<\frac{1}{p'}$, $0<\beta<\frac{1}{q}$, $0<\alpha+\beta\le\gamma$, *and* $\frac{1}{s}-\frac{1}{r}=\frac{1}{p}-\frac{1}{q}=\gamma-\alpha-\beta$*. Then there is a positive constant* $C=C(p,q,s,\gamma,\alpha,\beta)$ *such that for all* f, $\|f(x)x^{\alpha}\|_{M^{p,s}(\mathbb{R}_+)}<\infty$,

$$\left\|x^{-\beta}I_{\gamma}f(x)\right\|_{M^{q,r}(\mathbb{R}_+)}\le C\left\|x^{\alpha}f(x)\right\|_{M^{p,s}(\mathbb{R}_+)}.$$

Theorem 3.8. *Let* $0<\gamma<1, 1<p\le q<\infty$, $p\le s<\infty$ *and* $q\le r<\infty$*. Suppose that* $\alpha<\frac{1}{p'}$, $0<\gamma<\alpha+\beta+\frac{1}{p'}$, *and* $\frac{1}{s}-\frac{1}{r}=\frac{1}{p}-\frac{1}{q}=\gamma-\alpha-\beta$*. Then there is a positive constant* $C=C(p,q,s,\gamma,\alpha,\beta)$ *such that the inequality*

$$\left\|x^{-\beta}W_{\gamma}f(x)\right\|_{M^{q,r}(\mathbb{R}_+)}\le C\left\|x^{\alpha}f(x)\right\|_{M^{p,s}(\mathbb{R}_+)}$$

holds for all f, $\|x^{\alpha}f(x)\|_{M^{p,s}(\mathbb{R}_+)}<\infty$*.*

Theorem 3.9. *Let* $0 < \gamma < 1$, $1 < p \le q < \infty$, $p \le s < \infty$, *and* $q \le r < \infty$. *Suppose that* $0 \le \beta < \frac{1}{q}$, $0 < \gamma < \alpha + \beta + \frac{1}{p'}$, *and* $\frac{1}{s} - \frac{1}{r} = \frac{1}{p} - \frac{1}{q} = \gamma - \alpha - \beta$. *Then there is a positive constant* $C = C(p, q, s, \gamma, \alpha, \beta)$ *such that the the inequality*

$$\left\| x^{-\beta} R_\gamma f(x) \right\|_{M^{q,r}(\mathbb{R}_+)} \le C \left\| x^\alpha f(x) \right\|_{M^{p,s}(\mathbb{R}_+)}$$

holds for all f, $\| x^\alpha f(x) \|_{M^{p,s}(\mathbb{R}_+)} < \infty$.

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