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ON UNIVERSAL FUNCTIONS WITH RESPECT TO THE CLASSICAL SYSTEMS

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Abstract. This article considers the problem of the existence of functions that are universal, in a certain sense, for various function classes with respect to the trigonometric system and the Walsh system, as well as universal triads for these systems.

1. Introduction

We need some standard notation.

Let $|E|$ be the Lebesgue measure of a measurable set $E \subseteq [a,b]$ ($[a,b] = [-\pi,\pi]$ or $[0,1]$) .

Let $L^p(E)$, $p>0$ is the class of all measurable functions f on E with finite integral $\int_E |f(x)|^p\,dx$ and

$L^0(E)-$class of all almost everywhere finite, Lebesgue measurable functions on E, with almost everywhere (a.e.) convergence. We denote by $M(E)$ the class of all Lebesgue measurable functions on E with convergence understood in the almost everywhere (a.e.) sense.

The sequence of functions $\{f_k(x)\}_{k=1}^{\infty} \subset M(E)$ is said to converge to a function to $f \in M(E)$ in $M(E)$, if $\{f_k(x)\}_{k=1}^{\infty}$ converges to $f\,(x)$ almost everywhere on E.

The sequence of functions $\{f_k\}_{k=1}^{\infty} \subset L^p(E)$ is said to converge to a function $f \in L^p(E)$ in $L^p(E)$, if it convergens to f in the $L^p(E)$ metric, that is

$$\lim_{k\to\infty} \int\limits_E |f_k(x) - f(x)|^p\,dx = 0.$$

Let

$$a_k(f) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos kx dx, \quad b_k(f) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin kx dx, \;\; k \geq 0.$$

the Fourier coefficients of function $f \in L^1[-\pi,\pi]$ in the trigonometric system and let

$$S_N(x,f) = \frac{a_0(f)}{2} + \sum_{k=1}^{N-1} a_k(f)\cos kx + b_k(f)\sin kx,$$

$N-$ th partial sum of the function f of Fourier series.

Let $\Phi := \{\varphi_k(x)\}_{k=0}^{\infty}$ be an orthonormal system on $[a,b]$ and let $f \in L^1[a,b]$.

We denote by $c_k(f)$ the Fourier coefficients of the function $f \in L^1[a,b]$ in the system Φ, that is

$$c_k(f) = \int\limits_a^b f(x)\varphi_k(x)dx, \qquad S_n(x,f) = \sum_{k=0}^{n-1} c_k(f)\varphi_k(x)$$

Below S will denote one of the spaces $M(E)$ or $L^p(E)$, $p \geq 0$.

This article continues the author's studies on establishing the existence and describing the structure of functions that are universal, in a certain sense, for various function classes in the trigonometric and Walsh systems, as well as on universal triads for these systems (see [1–15]).

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Definition 1. A series $\sum_{k=0}^{\infty} f_k(x)$ ($f_k \subset S, k = 0, 1, 2, ..$) is called universal in S (universal **in the usual sense),** if for each function $f \in S$ there exists an increasing sequence of natural numbers $\{ N_m\}_{m=1}^{\infty} \nearrow$, such that the corresponding subsequence of partial sums $\{ \sum_{k=0}^{N_m} f_k(x) \}_{m=1}^{\infty}$ converges to f in S .

The first example is due to Birkhoff [16] (1929), who proved the existence of an entire function $f(z)$ with the property that, for any entire function $g(z)$ and every $r > 0$, there exists a subsequence of the natural numbers $\{n_k\}_{k=1}^{\infty}$ such that $\{f(z + n_k)\}_{k=1}^{\infty}$ converges to $g(z)$ uniformly on the disc $\{z \in C : |z| \le r\}$.

In 1935, Marcinkiewicz [17] proved that for any (converging to zero) sequence $h_n \to 0$ there exists a continuous function $F(x)$ on $[0, 1]$ having the property: for every measurable function $f(x)$ on $[0, 1]$, there is an increasing subsequence $n_k \nearrow^{\infty}$of natural numbers such that

$$\frac{F(x + h_{n_k}) - F(x)}{h_{n_k}} \to g(x),$$

as $k \to \infty$ almost everywhere on $[0, 1]$ (see also [18]).

In 1987, Grosse-Erdman [22] proved the existence of an infinitely differentiable function $h(x)$ with universal Taylor expansion . Namely, there exists an infinitely differentiable function $h(x)$ on $(-\infty,\infty)$ with $h(0) = 0$, such that it‘s the Taylor series ($\sum_{k=1}^{\infty} \frac{h^{(k)}(0)}{k!} x^k$) at $x_0 = 0$ is locally uniformly universal in $C(-\infty, \infty)$, the space of all continuous functions on $(-\infty, \infty)$.That is for every function $f(x) \in C(-\infty, \infty)$ with $f(0) = 0$ and any number $r > 0$ there exists an increasing subsequence $\{m_q\} \nearrow \infty$ of natural numbers such that the corresponding subsequence of partial sums

$$\sum_{k=1}^{m_q} \frac{h^{(k)}(0)}{k!} x^k$$

converges to $f(x)$ uniformly on the interval $(-r, r)$.

The notion of a universal series in $M[-\pi, \pi]$ both in the trigonometric system and in general orthonormal systems, is due to Menshov [23] and Talalyan [24] (see also [25-29]).

In this direction, important results were obtained by these authors and their students.

The first construction of a universal trigonometric series in the class of all measurable functions, in the sense of almost everywhere convergence, was given by Menshov. He proved the following theorem.

Theorem 1.1. *There is a trigonometric series*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx, \quad |a_k| + |b_k| \to 0 \quad as \quad k \to \infty$$

with the following property. For every measurable function $f(x)$ on $[-\pi, \pi]$ there is an increasing sequence $\{m_q\}$ of natural numbers such that the subsequence of partial sums

$$\frac{a_0}{2} + \sum_{k=1}^{m_q} a_k \cos kx + b_k \sin kx,$$

converges to $f(x)$ as $q \to \infty$ almost everywhere on $[-\pi, \pi]$ (that is this series is universal in $M[-\pi, \pi]$.)

Remark 1. As noted above, there is an infinitely differentiable function with universal Taylor series, **but there is no function** $U \in L^1[-\pi, \pi]$, whose Fourier series in the trigonometric system is universal in $M[-\pi, \pi]$.

Otherwise, if there were a function $U \in L^1[-\pi, \pi]$, whose Fourier series in the trigonometric system is universal in $M[-\pi, \pi]$, then for the function $f(x) = 2U(x)$ one could find a growing subsequence of natural numbers $\{N_m\} \nearrow \infty$ such that

$$S_{N_m}(x, U) = \frac{a_0(U)}{2} + \sum_{k=1}^{N_m - 1} a_k(U) \cos kx + b_k(U) \sin kx$$

converges to $2U(x)$ as $q \to \infty$ almost everywhere on $[-\pi, \pi]$.

On the other hand, from the **well-known theorem of Kolmogorov [30](the Fourier series in the trigonometric system of any integrable function converges to it in** $L^p[-\pi,\pi], p \in (0,1)$, it follows that $\lim_{m\to\infty} \int_{-\pi}^{\pi} |S_{N_m}(x,U) - U(x)|^p \, dx = 0$), and therefore for some subsequence $\{N_{m_q}\}_{q=1}^{\infty}$ of sequence $\{N_m\}_{m=1}^{\infty}$ we have that

$$S_{N_{m_q}}(x,U) = \frac{a_0(U)}{2} + \sum_{k=1}^{N_{m_q}-1} a_k(U)\cos x + b_k(U)\sin kx$$

converges to $U(x)$ as $q \to \infty$ almost everywhere on $[-\pi, \pi]$.

Hence we see that $U(x) = 2U(x)$ almost everywhere on $[-\pi, \pi]$.

This contradiction shows that there does not exist a function $U \in L^1[-\pi,\pi]$, whose Fourier series is universal in the trigonometric system for the class $M[-\pi,\pi]$.

Remark 2. A similar analysis shows that there does not exist an integrable function U, whose Fourier series, in the (Walsh, Haar, Franklin, and Vilenkin) system , is universal in $M[0,1]$.

The above considerations suggest the following question, the answer to which is not yet known.

Question 1. Do there exist an orthonormal system $\{\varphi_k(x)\}_{k=1}^{\infty}$ of bounded functions and a function $U \in L^1[0,1]$ whose Fourier series in the system $\{\varphi_k(x)\}_{k=1}^{\infty}$ is universal in $M[0,1]$?

Despite the fact that, as Menshov proved, there exists a trigonometric series that is universal in $M[-\pi,\pi]$ and (as noted above) there is no function $U \in L^1[-\pi,\pi]$ whose Fourier series in the trigonometric system is universal in $M[-\pi,\pi]$, we have nevertheless succeeded in constructing an integrable function U and proving that, after an appropriate choice of signs $\{\delta_k; \delta_k = \pm 1\}_{k=0}^{\infty}$ for its Fourier coefficients, the resulting series $\frac{a_0(U)}{2} + \sum_{k=1}^{\infty} \delta_k(a_k(U)\cos kx + b_k(U)\sin kx)$ is universal in $M[-\pi,\pi]$.

Before formulating the main result of the paper, we provide the necessary definitions.

2. Definitions and auxiliary statements

Let $\Phi := \{\varphi_k(x)\}_{k=0}^{\infty}$ be an orthonormal system on $[a,b]$.

Definition 2. We say that a function $U \in L^1[a,b]$ and a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0,1,2,...\}$ form **universal pairs:** $(\mathbf{U}, \delta)$ for the space S with respect to this system $\{\varphi_k(x)$ in sense of universal series, if the series $\sum_{k=0}^{\infty} \delta_k c_k(f)\varphi_k(x)$ is universal in S.

Definition 3. We say that a function $U \in L^1[a,b]$ and a measurable set $E \subset [a,b]$ form a **universal pairs:** $(\mathbf{U}, \mathbf{E})$ with respect to this system $\{\varphi_k(x)\}_{k=0}^{\infty}$ in sense of modification, if for each function $f \in L^1[a,b]$ there exists a function $g \in L^1[a,b]$ that coincides with f on E, and such that

$$|c_k(g)| = |c_k(U)| \ \ , k = 0,1,2,..$$

Definition 4. We say that a function $U \in L^1[a,b]$, a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0,1,2,...\}$ and a measurable set $E \subset [a,b]$ form a **universal triads:** $(\mathbf{U}, \delta, \mathbf{E})$ for the space S with respect to this system $\{\varphi_k(x)\}_{k=0}^{\infty}$ if the series $\sum_{k=0}^{\infty} \delta_k c_k(f)\varphi_k(x)$ is universal in S and for each function $f \in L^1[a,b]$ there exists a function $g \in L^1[a,b]$ that coincides with f on E, and such that

$$|c_k(g)| = |c_k(U)| \ \ , k = 1,2,..$$

Definition 5. We say that a function $U \in L^1[a,b]$ is

1) **universal for** the space S with respect to a system $\{\varphi_k(x)\}_{k=0}^{\infty}$, if the Fourier series of the function U with respect to this system is universal in S,

2) **conditional universal** for the space S with respect to a system $\{\varphi_k(x)\}_{k=0}^{\infty}$ if there exists a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0,1,2,...\}$ such that the series $\sum_{k=0}^{\infty} \delta_k c_k(U)\varphi_k(x)$ is universal in S,

3) **quasiuniversal** for the space S with respect to a system $\{\varphi_k(x)\}$, if there exists a sequence of signs $\delta=\{\delta_k=\pm 1, k=0,1,2,...\}$, with $d_\Lambda(\Omega^+)=1$, such that the series $\sum_{k=0}^{\infty}\delta_k c_k(U)\varphi_k(x)$ is universal in S, where

$$d_\Lambda(\Omega^+):=\lim_{n\to\infty}\sup\frac{\#(\Omega^+\cap[0,n))}{\#(\Lambda\cap[0,n))}=\lim_{n\to\infty}\sup\frac{\#(k\in[0,n),\delta_k=1)}{\#(spec(U)\cap[0,n))}$$

is the upper density of the subset $\Omega^+=\{k\in N,\delta_k=1\}$ with respect to the set $\Lambda=spec(U)=\{k\in N, c_k(U)\neq 0\}$,

and $\#(*)$ is cardinilaty of finite set, where $\#(E)$ is the number of elements of a finite set E.

4) **almost universal** for the space S with respect to a system $\{\varphi_k(x)\}$, if there exists a sequence of signs $\delta=\{\delta_k=\pm 1, k=0,1,2,...\}$, with $\rho_\Lambda(\Omega)=0$, such that the series $\sum_{k=0}^{\infty}\delta_k c_k(U)\varphi_k(x)$ is universal in S, where

$$\rho_\Lambda(\Omega):=\lim_{n\to\infty}\frac{\#(\Omega\cap[0,n))}{\#(\Lambda\cap[0,n))}=\lim_{n\to\infty}\frac{\#(k\in[0,n),\delta_k=-1)}{\#(spec(U)\cap[0,n))}$$

is the density of the subset $\Omega=\{k\in N,\delta_k=-1\}$ with respect to the set $\Lambda=spec(U)=\{k\in N, c_k(U)\neq 0\}$,

5) **universal in sense of signs**, if its Fourier series $\sum_{k=0}^{\infty}c_k(U)\varphi_k(x)$ in the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ is universal in S **in sense of signs**: that is for each function $f\in S$ one can find a sequence of signs $\delta=\{\delta_k=\pm 1, k=0,1,2,...\}$ such that the series $\sum_{k=1}^{\infty}\delta_k c_k(U)\varphi_k(x)$ converges to f in S,

6) **universal in sense of rearrangements**, if its Fourier series $\sum_{k=0}^{\infty}c_k(U)\varphi_k(x)$ in the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ is universal in S in sense of rearrangements: that is for each function $f\in S$ one can find $\{\sigma(k)\}_{k=1}^{\infty}$ some permutation of the natural numbers, such that the series $\sum_{k=1}^{\infty}f_{\sigma(k)}(x)$ converges to f in S .

The Walsh system, an extension of the Rademacher system, may be obtained in the following manner. Let r be the periodic function, of least period 1, defined on $[0,1)$ by

$$r(x)=\chi_{[0,1/2)}(x)-\chi_{[1/2,1)}(x),$$

where $\chi_E(x)-$ characteristic function of set E .

The Rademacher system, $R=r_n : n=0,1,...$, is defined by the conditions

$$r_n(x)=r(2^n x),\ \forall x\in R, n=0,1,...,$$

and, in the ordering employed by Payley (see [31], [32]), the n-th element of the Walsh system $\{W_n(x)\}_{n=0}^{\infty}$ is given by

$$W_n(x)=\prod_{k=0}^{\infty}(r_k(x))^{\theta_k(n)},$$

where $\sum_{k=0}^{\infty}\theta_k(n)2^k$ is the unique binary expansion of n, with each $\theta_k(n)$ either 0 or 1.

Note that (see [33]) the Walsh system is a basis for all $L^p[0,1]$ $p\in(1,\infty)$, i.e., every function $f(x)$ is $L^p[0,1]$ $p\in(1,\infty)$ uniquely representable by the series $\sum_{k=0}^{\infty}b_kW_k(x)$ in the Walsh system by norm $L^p[0,1]$ convergent to $f(x)$. It easy to see that for all $k\in\mathbb{N}\cup\{0\}$

$$b_k=c_k(f)=\int_0^1 f(x)W_k(x)dx.$$

We will use the following properties:

for all $k\in\mathbb{N}\cup\{0\}$

$$\sum_{j=0}^{2^k-1}W_j(x)=\begin{cases}2^k\ , & \text{if } x\in[0,\frac{1}{2^k}),\\ 0\ , & \text{if } x\in[\frac{1}{2^k},1)\end{cases}$$

and therefore

$$\int_0^1 \left| \sum_{j=0}^{2^k-1} V_j(x) \right| dx = 1, \quad \forall\ k \in \mathbb{N} \cup \{0\}.$$

We also use (in the proof of Theorem 3.7) the following lemma previously proved in [4].

Lemma 1. *Let numbers* $n_0 \in N, (N_0 = 2^{n_0}),\ 0 < p_1 < p_2 < 1, \varepsilon \in (0,1)$ *and polynomial* $W(x)$ *in the Walsh system* $\{W_k(x)\}_{k=0}^{\infty}$ *are given. Then there exist polynomials* $U(x)$ *and* $B(x)$ *in the Walsh system of the following form*

$$U(x) = \sum_{k=2^{n_0}}^{2^n-1} b_k W_k(x)\ , \quad B(x) = \sum_{k=N_0}^{N-1} \varepsilon_k b_k W_k(x),\ N_0 = 2^{n_0},\ N = 2^n,$$

which satisfy the following conditions:

1) $$0 < b_{k+1} \le b_k < \varepsilon, \quad \varepsilon_k = \pm 1,\ \forall k \in [2^{n_0}, 2^n) = [N_0, N)$$

2) $$B(x) = W(x)\ \forall x \in G,\ \ |G| \ge 1 - \varepsilon - 2^{-n_0}$$

3) $$U(x)\chi_{[2^{-n_0},1)}(x) = 0,$$

4) $$\max_{m \in [N_0, N)} \int_0^1 \left| \sum_{k=N_0}^{m} \varepsilon_k b_k W_k(x) \right|^p dx < 4 \int_0^1 |W(x)|^p dx, \quad \forall p \in (p_1, p_2)\,,$$

5) $$\max_{m \in [N_0, N)} \int_0^1 \left| \sum_{k=N_0}^{m} \varepsilon_k b_k W_k(x) \right| dx < 5 \int_0^1 |W(x)| dx,$$

6) $$\int_0^1 |W(x) - B(x)|^p dx < \varepsilon \quad \forall p \in (p_1, p_2)\,,$$

7) $$\max_{m \in [N_0, N)} \int_0^1 \left| \sum_{k=N_0}^{m} b_k W_k(x) \right| dx < \varepsilon.$$

3. Theorems

In recent years, several results have been obtained concerning both the existence and the structural characterization of functions (so-called universal functions) whose Fourier series, with respect to a given classical system, are universal in a certain sense for various classes of functions.

In particular, in [2], [3], [11], and [15], the following results were obtained.

Theorem 3.1. *Let* $\{W_k(x)\}_{k=0}^{\infty}$ *- Walsh system. There exists an integrable function* U *with monotonically decreasing Fourier–Walsh coefficients, which is universal in the sense of signs with respect to the Walsh system for the spaces* $L^p[0,1]$ *for all* $p \in (0,1)$*. In addition, the Fourier–Walsh series of the function* U *converges to it by norm* $L^1[0,1]$.

Theorem 3.2. *There exist an integrable function* $U \in L^1[-\pi,\pi]$ *and a sequence of signs* $\delta = \{\delta_k = \pm 1, k = 0, 1, 2, ...\}$ *, which are form* ***universal pairs:*** $(\mathbf{U}, \delta)$ *for space* $L^p[-\pi,\pi], p \in (0,1)$ *with respect to the trigonometric system .*

Theorem 3.3. *For any $p \in (0,1)$ there exists an integrable function U that is both conditional universal and universal in the sense of signs with respect to the trigonometric system for the class $L^p[-\pi,\pi]$.*

Theorem 3.4. *There exists an integrable function U, whose the Fourier series in the Walsh system converges to it by norm $L^1[0,1]$ with monotonically decreasing Fourier–Walsh coefficients, which is universal in the sense of signs with respect to the Walsh system for the space $L^0[0,1]$. In addition, the Fourier–Walsh series of the function U converges to it by norm $L^1[0,1]$.*

Remark 3. It should be noted that there is no integrable function U, that is universal in the sense of signs with respect to the Walsh system for the space $M[0,1]$, in the case of convergence almost everywhere. However, there exists an integrable function U with monotonically decreasing Fourier–Walsh coefficients that is universal in the sense of signs with respect to the Walsh system for the space $M[0,1]$ in the case of convergence in measure; that is, for each function , one can find a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0, 1, 2, ...\}$, such that the series $\sum_{k=0}^{\infty} \delta_k c_k(U) W_k(x)$ converges to f in measure.

The following theorems are true.

Theorem 3.5. *There exist an integrable function $U \in L^1[-\pi,\pi]$ and a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0, 1, 2, ...\}$, and a measurable set $E \subset [a,b]$ form* ***universal triads:*** $(\mathbf{U}, \delta, \mathbf{E})$ *for space $L^0[-\pi,\pi]$. with respect to the trigonometric system.*

Theorem 3.6. *There exist an integrable function $U \in L^1[0,1]$ and a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0, 1, 2, ...\}$, and a measurable set $E \subset [a,b]$ form* ***universal triads:*** $(\mathbf{U}, \delta, \mathbf{E})$ *for space $L^0[0,1]$) with respect to the Walsh system.*

Moreover the following statement holds.

Theorem 3.7. *For any $\varepsilon \in (0,1)$ there exists a function $U \in L^1[0,1]$, with $sup(U) \subset [0,\varepsilon]$, which has the following properties:*

a) the Fourier coefficients of the function U in the Walsh system are positive and monotonically decreasing

b) the Fourier–Walsh series of the function U converges to it by norm $L^1[0,1)$.

c) the function U is conditional universal for the class $L^0[0,1]$ with respect to the Walsh system

d) for any $\delta \in (0,1)$ exists a measurable set $E \subset [0,1]$ with $|E| > 1-\delta$, so that for each function $g \in L^1[0,1]$ one can find such a function $f \in L^1[0,1]$ coinciding with $g(x)$ on E, and such that

$$|c_k(f)| = c_k(U)\ , k = 0, 1, 2, ..$$

e) the Fourier–Walsh series of the corrected function f converges to it by norm $L^1[0,1]$.

Remark 4. The method used to prove Theorem 7 provides a novel approach to constructing universal series in the Walsh system: any measurable, almost everywhere finite function can be transformed, by modifying its values on a set of arbitrarily small measure into a function such that, after choosing appropriate signs for the terms of its Fourier–Walsh series, the resulting series becomes universal in $M[0,1]$.

Remark 5. It should be noted (as also follows from the above) that the existence of universal functions and universal triads depends on the type (or sense) of universality, the system, and the space S. Therefore, questions in this direction are quite broad.

The following questions arise, the answers to which are not yet known:

Question 2. Do there exist an orthonormal system $\{\varphi_k(x)\}_{k=1}^{\infty}$ of bounded functions and a function $U \in L^1[0,1]$ that is universal for the space $L^p[0,1]$ for some $p \in [0,1)$ (or at least for the space $M[0,1]$) with respect to the system $\{\varphi_k(x)\}_{k=1}^{\infty}$?

Question 3. Are Theorems 3.4 and 3.6 true for the trigonometric system ?

Question 4. Are theorems 3.1-3.7 true for the Vilenkin system?

Question 5. Is it possible to construct a function that is universal for the space $L^0[0,1]$ with respect to the Walsh system in the sense of rearrangements?.

Question 6. Does there exist a function $U \in L^1[-\pi,\pi]$ that is universal for the class $L^p[-\pi,\pi]$, with respect to the trigonometric system, in the sense of permutations?

Question 7. Are theorems 3.1-3.7 true system for the Franklin (for the Haar system)?

Question 8. Are theorems 3.1-3.6 true for spherical harmonics?

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