Transactions of A. Razmadze Mathematical Institute

Summability (C, α) of series of generalized spherical functions

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Abstract

For the function given on the surface of a sphere, generalized Laplace operators are defined, and by means of these operators, sufficient conditions are established for the (C,α)-summability of the series obtained through iterated application of the Laplace operator to the series of generalized spherical functions.

Key words and phrases: Generalized spherical function; Generalized Laplace operators; (C,α) summability.

2020 Mathematical Subject classification: 42B05, 40A05

1 Introduction

Gelfand and Shapiro considered the system of generalized spherical functions and the expansion of a vector function, given on the surface of a sphere, into a Fourier series with respect to this system of generalized spherical functions (see [1]). Subsequently, the problems of convergence of this series (see [2]) and the questions of its summability by various methods (see [3], [4]) were investigated.

The paper discusses the means of the series, the series (C,α) obtained from the generalized spherical function series by several "differentiations", that is, by repeated application of the Laplace operator. The properties of these means are established, and sufficient conditions are proved for the summability of the series after multiple differentiations.

2 Notations and definitions

Generalized spherical functions are defined as follows (see [1])

$$T^l_m(\varphi_1,\vartheta,\varphi_2)=e^{-im\varphi_2}P^l_{mn}(\cos\vartheta)e^{-im\varphi_1},\quad m,n=-l,-l+1,\dots,l,$$

where

$$P^l_{mn}(x)=\frac{(-1)^{l-m}i^{n-m}}{2^l(n-m)!}\sqrt{\frac{(l-m)!(l+n)!}{(l+m)!(l-n)!}}(1-x)^{-\frac{n-m}{2}}(1+x)^{-\frac{n+m}{2}}\frac{d^{l-n}}{dx^{l-n}}\left[(1-x)^{l-n}(1+x)^{l+n}\right],$$

and $\varphi_1,\vartheta,\varphi_2$ - are Euler angles with ranges $0\le\varphi_1\le 2\pi$, $0\le\varphi_2\le 2\pi$, $0\le\vartheta\le\pi$.

Define the operator:

$$\tilde{\Delta}=\frac{\partial^2}{\partial\vartheta^2}+\operatorname{ctg}\vartheta\frac{\partial}{\partial\vartheta}+\frac{1}{\sin^2\vartheta}\left(\frac{\partial^2}{\partial\varphi_1^2}-2\cos\vartheta\frac{\partial^2}{\partial\varphi_1\partial\varphi_2}+\frac{\partial^2}{\partial\varphi_2^2}\right),$$

then (see [1])

$$\tilde{\Delta}T^l_{mn}(\varphi_1,\vartheta,\varphi_2)=-l(l+1)T^l_{mn}(\varphi_1,\vartheta,\varphi_2).$$

Thus, we have

$$\Delta T^l_{mn}(\frac{\pi}{2}-\varphi,\vartheta,0) = -l(l+1)T^l_{mn}\left(\frac{\pi}{2}-\varphi,\vartheta,0\right),$$

where Δ is the angular part of the Laplace operator:

$$\Delta = \frac{\partial^2}{\partial\vartheta^2} + \mathrm{ctg}\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}.$$

If we accept that $\Delta^r = \Delta(\Delta^{r-1})$, $r = 2, 3, \dots$, then

$$\Delta^r T^l_{mn}\left(\frac{\pi}{2}-\varphi,\vartheta,0\right) = [-l(l+1)]^r T^l_{mn}\left(\frac{\pi}{2}-\varphi,\vartheta,0\right)$$

Let S^2 be the surface of the sphere with unit radius in R^3. Assume that on the surface S^2 a vector function is given, $V(\vartheta,\varphi) = V(V_x, V_y, V_z)$, where ϑ and φ are spherical coordinates. The components of this function in spherical coordinates are as follows (see [1])

$$V_r = V_x\sin\vartheta\cos\varphi + V_y\sin\vartheta\sin\varphi + V_z\cos\vartheta$$

$$V_\vartheta = V_x\cos\vartheta\cos\varphi + V_y\cos\vartheta\sin\varphi - V_z\sin\vartheta$$

$$V_\varphi = -V_x\sin\varphi + V_y\cos\varphi.$$

We define the following combinations of the components of the function $V(\vartheta,\varphi)$ (see [1])

$$V_0 = V_r, \quad V_{-1} = -\frac{1}{\sqrt{2}}(V_\varphi + iV_\vartheta), \quad V_1 = \frac{1}{\sqrt{2}}(V_\varphi - iV_\vartheta)$$

We then consider the Fourier series of the function $V_m(\vartheta,\varphi) \in L(S^2)$ $(m = 0, \pm 1)$ function (see [1], [2])

$$S(V_m;\vartheta,\varphi) = \sum_{l=|m|}^{\infty}\sum_{n=-l}^{l} C^l_{mn}T^l_{mn}\left(\frac{\pi}{2}-\varphi,\vartheta,0\right), \quad (m = 0, \pm 1)$$

where

$$C^l_{mn} = \frac{2l+1}{4\pi}\int_0^\pi\int_0^{2\pi} V_m(\vartheta',\varphi')\overline{T^l_{mn}(\frac{\pi}{2}-\varphi',\vartheta',0)}\sin\vartheta' d\vartheta' d\varphi'. \tag{2.1}$$

Let $S^{(r)}(V_m;\vartheta,\varphi)$ denote the result of the r-times action of the operator Δ on the series $S(V_m;\vartheta,\varphi)$. Then

$$S^{(r)}(V_m;\vartheta,\varphi) = \sum_{l=|m|}^{\infty}\sum_{n=-l}^{l} C^l_{mn}[-l(l+1)]^r T^l_{mn}\left(\frac{\pi}{2}-\varphi,\vartheta,0\right). \tag{2.2}$$

If in the series (2.2), we substitute the values of the coefficients $\mathrm{C}^l_{mn}(obtained)fromequality(2.1)andappl$ $\sum_{l=|m|}^{\infty}[-l(l+1)]^r I^l_m(\vartheta,\varphi) = \sum_{l=0}^{\infty}[-l(l+1)]^r I^l_m(\vartheta,\varphi)$, (2.3) where

$$I^l_m(\vartheta,\varphi) = \frac{(-1)^m(2l+1)}{4\pi}\int_0^\pi\int_0^{2\pi} V_m(\vartheta',\varphi')e^{-im(\varphi_1+\varphi_2)}P^l_{mm}(\cos\gamma)\sin\vartheta' d\vartheta' d\varphi', \quad m = 0, \pm 1,$$

$$\cos\gamma = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\varphi - \varphi'), \quad 0 \le \vartheta \le \pi, \quad 0 \le \varphi \le 2\pi,$$

$$P^l_{mm}(\mu) = \frac{(-1)^{l-m}}{2^l(l-m)!}(1+\mu)^{-m}\frac{d^{l-m}}{d\mu^{l-m}}[(1-\mu)^{l-m}(1+\mu)^{l+m}],$$

$$\mathrm{tg}\varphi_1 = \frac{\sin(\varphi-\varphi')\sin\vartheta'}{-\cos\vartheta\sin\vartheta'\cos(\varphi-\varphi')+\cos\vartheta'\sin\vartheta},$$

$$\mathrm{tg}\varphi_2 = \frac{\sin(\varphi-\varphi')\sin\vartheta}{-\cos\vartheta'\sin\vartheta\cos(\varphi-\varphi')+\cos\vartheta\sin\vartheta'}.$$

Let us consider (C,α) $(\alpha > -1)$ of the series (2.3):

$$\sigma_n^{r,\alpha}(V_m;\vartheta,\varphi) = \frac{(-1)^m}{4\pi}\int_0^\pi\int_0^{2\pi} V_m(\vartheta',\varphi')e^{-im(\varphi_1+\varphi_2)}K_{n,m}^{r,\alpha}(\cos\gamma)\sin\vartheta' d\vartheta' d\varphi', \tag{2.4}$$

where

$$K_{n,m}^{r,\alpha}(\cos\gamma) = \frac{1}{A_n^\alpha}\sum_{l=0}^{n} A_{n-l}^\alpha(2l+1)[-l(l+1)]^r P_{mm}^l(\cos\gamma). \tag{2.5}$$

If the identity (see [2])

$$P_{1,1}^l(\mu) = P_{-1,-1}^l(\mu) = \frac{1}{2l+1}\frac{d}{d\mu}(P_{l+1}(\mu)-P_{l-1}(\mu)) - \frac{1}{2l+1}\cdot\frac{1}{1+\mu}(P_{l+1}(\mu)-P_{l-1}(\mu)),$$

is taken into account, and noting that ([5], p. 37)

$$\frac{d}{d\mu}(P_{l+1}(\mu)-P_{l-1}(\mu)) = (2l+1)P_l(\mu),$$

we obtain

$$P_{1,1}^l(\mu) = P_l(\mu) - \int_{-1}^{\mu} P_l(t)dt.$$

Thus, by virtue of equality (2.5), when $m=\pm 1$, we have

$$K_{n,m}^{r,\alpha}(\cos\gamma) = K_n^{r,\alpha}(\cos\gamma) - \Phi_n^{r,\alpha}(\cos\gamma), \tag{2.6}$$

Here

$$K_n^{r,\alpha}(\cos\gamma) = \frac{1}{A_n^\alpha}\sum_{l=0}^{n} A_{n-l}^\alpha[-l(l+1)]^r(2l+1)P_l(\cos\gamma),$$

$$\Phi_n^{r,\alpha}(\cos\gamma) = \frac{1}{1+\cos\gamma}\int_\gamma^\pi K_n^{r,\alpha}(\cos t)\sin t dt.$$

Let us assume the spherical coordinates (ϑ,φ) of point P, and (ϑ',ϕ') of point Q. Then equality (2.4) can be written as

$$\sigma_n^{r,\alpha}(V_m;P) = \frac{(-1)^m}{4\pi}\iint_{S^2} V_m(Q)e^{-im(\varphi_1+\varphi_2)}K_{n,m}^{r,\alpha}[(P,Q)]dS_Q,$$

and if $P(\vartheta,\varphi)$ is the north pole of the sphere S^2, then (2.4) reduces to

$$\sigma_n^{r,\alpha}(V_m;P) = \frac{(-1)^m}{4\pi}\int_0^\pi\int_0^{2\pi} V_m(\gamma,\overline{\varphi})e^{-im(\overline{\varphi_1}+\overline{\varphi_2})}K_{n,m}^{r,\alpha}(\cos\gamma)\sin\gamma d\gamma d\overline{\varphi}.$$

Let $C(P,h)$ denote the circumference on the surface S^2, whose center is located at point P and whose radius is $2\sin\frac{h}{2}$. It is clear that

$$C(P,h) = \{Q : Q\in S^2, (P,Q)=\cos h\}.$$

By the symbol $\omega(P,h)$ we designate the spherical segment

$$\omega(P,h) = \{Q : Q \in S^2, (P,Q) \geq \cos h\}.$$

We say that the function $V_m(Q)$ at point P ($m = 0, \pm 1$) has an operator of order r, denoted D^r, if there exist numbers a_r^m, $r = 0, 1, \ldots$, such that

$$\frac{1}{2\pi \sin h} \int_{(P,Q)=\cos h} V_m(Q) e^{-im(\varphi_1+\varphi_2)} dS_Q = \sum_{k=0}^{r} \frac{a_k^m}{(k!)^2} \left(\sin \frac{h}{2}\right)^{2k} + o\left(\sin^{2r} \frac{h}{2}\right). \quad (2.7)$$

Let us denote the value of this operator at point P by the symbol $D^r V_m(P)$. The values of the operator D^r can be calculated from the expansion (2.7) in the following way:

$$\left.\begin{array}{l} D^0 V_m(P) = V_m(P) = a_0^m, \quad DV_m(P) = a_1^m, \\ D(D + (m+1)(m+2))V_m(P) = a_2^m, \quad D(D + (m+1)(m+2))(D + (m+2)(m+3))V_m(P) = a_3^m, \\ \ldots (D(D + (m+1)(m+2))(D + (m+2)(m+3)) \ldots (D + (m+k-2)(m+k-1))V_m(P) = a_k^m, \end{array}\right\} \quad (2.8)$$

$$k = 2, 3, \ldots r; \quad D^i D^j = D^{i+j}, \quad i, j \geq 0, \quad i + j \leq r.$$

Let us now define a more generalized operator, in comparison with $\tilde{D}$. Let $V_m(Q)$, with ($m = 0, \pm 1$), be a function defined on the surface of sphere S^2, integrable in an arbitrary spherical neighborhood of any point $P \in S^2$. We say that at point P the function $V_m(Q)$ admits an operator of order r, denoted $\tilde{D}^r V_m(P)$, if there exists a following representation:

$$\frac{1}{|\omega(P,h)|} \iint_{\omega(P,h)} V_m(Q) e^{-im(\varphi_1+\varphi_2)} dS_Q = \sum_{\mu=0}^{r} \frac{b_\mu}{\mu!(\mu+1)!} \left(\sin \frac{h}{2}\right)^{2\mu} + o(1 - \cos h)^r, \quad r = 0, 1, \ldots \quad (2.9)$$

The operator $\tilde{D}^r V_m(P)$ defined by equality (2.9) is related to the coefficients b_μ, $\mu = 0, 1, \ldots, r$ through equalities similar to those in (2.8).

3 Proof of key results

The following lemmas hold.

Lemma 1. *If $-1 \leq x \leq 1$, $m \geq 0$, then for $P_{mm}^n(x)$ polynomial we have*

$$P_{mm}^n(x) = \left(\frac{1+x}{2}\right)^m \times \quad (3.1)$$

$$\times \left\{1 + \sum_{\nu=1}^{\infty} \frac{[-n(n+1) + m(m+1)] \ldots [-n(n+1) + (m+\nu-1)(m+\nu)]}{(\nu!)^2} \left(\frac{1-x}{2}\right)^\nu \right\}$$

Lemma 2. *If $0 < \vartheta < \pi$, then $P_{mn}^n(\cos\vartheta)$, $n \geq m$, and for the polynomials there exists a Dirichlet-Meller type integral presentation, given by*

$$P_{mm}^n(\cos\vartheta) = \frac{(-1)^m \sqrt{2}}{\pi(1+\cos\vartheta)^m} \int_0^\vartheta \frac{\left(\sqrt{2} \sin \frac{\vartheta}{2} + (\cos\vartheta - \cos t)^{1/2}\right)^{2m}}{(\cos\vartheta - \cos t)^{1/2}} \sin\left(n + \frac{1}{2}\right) t \, dt. \quad (3.2)$$

Lemma 3. *If from equation $D^k V_m(P) = 0$, for $k = 0, 1, \dots, r$, and $m = 0, \pm 1$, it follows that $\Delta^r S(V_m; P)$ is (C, α) summable to zero, then from the equality $D^r V_m(P) = S$, it follows that $\Delta^r S(V_m; P)$ is (C, α) summable for number S.*

Proof. Let us suppose that $D^r V_m(P) = S$. That is, at point P, there exist $D^0 V_m, DV_m, \dots, D^r V_m$ numbers. Let us consider the finite sum of polynomials $P^l_{mm}([P, Q])$

$$T_m(Q) = \sum_{l=0}^{r} C_l P^l_{mm}([P, Q])$$

We select the coefficients C_l in such a way that $D^k T_m(P) = D^k V_m(P)$, $k = 0, \dots, r$.

If we take into account the expansion (3.1) and the definition of the operator D^k, we can easily verify that $D^k P^l_{mm}(P) = \Delta^k P^l_{mm}(P)$. From this, we conclude that $D^k T(P) = \Delta^k T(P)$, $k = 0, 1, \dots, r$. That is, the coefficients C_1 are selected so that $\Delta^k T(P) = D^k V_m(P)$, holds for $k = 0, 1, \dots, r$.

The determinant of the resulting system with respect to the unknowns $C_0, C_1, \dots, C_r$ is not equal to zero; thus, the coefficients would be selected so that $\Delta^k T(P) = D^k V_m(P)$, $k = 0, 1, \dots, r$. The determinant of the resulting system with respect to the unknowns is not equal to zero, thus $C_0, C_1, \dots, C_r$ exist.

Let $F_m(Q) = V_m(Q) - T_m(Q)$. Then $D^k F_m(P) = 0$ holds for $k = 0, 1, \dots, r$.

By the condition of the lemma, $\Delta^r S(F_m; P)$ is (C, α)-summable to zero. As a consequence

$$\Delta^r S(F_m; P) = \Delta^r S(V_m; P) - \Delta^r S(T_m; P) = \Delta^r S(V_m; P) - S,$$

The lemma is proved. □

Lemma 4. *If from the equalities $\tilde{D}^k V_m(P) = 0$, $k = 0, 1, \dots, r$, $m = 0, \pm 1$ it follows that $\Delta^r S(V_m; P)$ is (C, α)-summable to zero, then from the equality $\tilde{D}^r V_m(P) = S$ it follows that $\Delta^r S(V_m; P)$ is (C, α) summable to the number S.*

Proof. Lemma 4 is proved similarly to Lemma 3. □

Lemma 5. *There exists a positive c such that*

$$\int_0^{1/n} |K^{r,\alpha}_{n,m}(\cos\gamma)| (\sin\gamma)^{2r+1} d\gamma < c, \quad m = 0, \pm 1, \quad r = 0, 1, \dots, \quad n = 1, 2, \dots$$

Proof. If $0 \le \gamma \le \pi$, the polynomial $P^l_{mm}(\cos\gamma)$ satisfies (see [6]):

$$|P^l_{mm}(\cos\gamma)| \le \frac{1}{2}(1 + 2^{|m|}) \le \frac{3}{2},$$

Thus,

$$|K^{r,\alpha}_{n,m}(\cos\gamma)| \le \frac{3}{2A^\alpha_n} \sum_{l=0}^{n} A^\alpha_{n-l} (2l+1)[l(l+1)]^r.$$

Obviously, there exists a positive constant c such that $(2l+1)[l(l+1)]^r \le cn^{2r+1}$. Thus,

$$|K^{r,\alpha}_{n,m}(\cos\gamma)| \le \frac{cn^{2r+1}}{A^\alpha_n} \sum_{l=0}^{n} A^\alpha_{n-l} = cn^{2r+1} \frac{A^{\alpha+1}_n}{A^\alpha_n} \le cn^{2r+2}$$

Hence, we have

$$\int_0^{1/n} |K_{n,m}^{r,\alpha}(\cos\gamma)| \sin^{2r+1}\gamma d\gamma \le cn^{2r+2}\int_0^{1/n} \sin^{2r+1}\gamma d\gamma \le cn^{2r+2}\int_0^{1/n} \gamma^{2r+1} d\gamma = c.$$

The lemma is proved. □

Lemma 6. *Let $\frac{1}{n} \le \delta \le \frac{\pi}{2}$, $n \in N$ and $\alpha > 2r+1$. Then there exists a positive constant c such that*

$$K_{n,m}^{r,\alpha}(\cos\delta) = o(1) \tag{3.3}$$

and

$$\int_{1/n}^{\delta} |K_{n,m}^{r,\alpha}(\cos\gamma)| \sin^{2r+1}\gamma d\gamma < c, \quad r = 0, 1, \dots, \quad m = 0, \pm 1.$$

Theorem 1. *Let $V_m(\vartheta,\varphi)$, $m = 0, \pm 1$, be an integrable function on the surface of the sphere S^2. If in any spherical neighborhood of the point (ϑ,φ) we have $V_m(\vartheta,\varphi) = 0$, then the series $S^{(r)}(V_m;\vartheta,\varphi)$ is (C,α)-summable to zero for $\alpha > 2r+1$. If $\delta > 0$ and $\alpha > 2r+1$, then*

$$\lim_{n\to\infty} \iint\limits_{S^2\setminus\omega(P,\delta)} V_m(\vartheta',\varphi')e^{im(\varphi_1+\varphi_2)}K_{n,m}^{r,\alpha}(\cos\gamma)\sin\vartheta' d\vartheta' d\varphi' = 0.$$

Proof. It is known that (see [7]) that if $P_j(z)$ is an odd polynomial of degree j, and $0 < \delta < \frac{\pi}{2}$, then the series

$$\sum_{l=0}^{\infty} P_j(2l+1)\sin(2l+1)\gamma$$

is uniformly (C,α)-summable to zero, where $\alpha > j$ and $\delta \le \gamma \le \pi - \delta$. Thus, the series

$$\sum_{l=0}^{\infty}[-l(l+1)]^r(2l+1)\sin(2l+1)\gamma = \frac{(-1)^r}{4}\sum_{l=0}^{\infty}[(2l+1)^2-1]^r(2l+1)\sin(2l+1)\gamma$$

is uniformly (C,α)-summable to zero, where $\alpha > 2j+1$ and $\delta \le \gamma \le \pi - \delta$. That is, an $n \to \infty$ and $\frac{\delta}{2} \le \gamma \le \frac{\pi}{2}$, the summability is uniform with respect to γ

$$\Phi_{n,m}^{r,\alpha}(\cos\gamma) = \sum_{l=0}^{\infty}[-l(l+1)]^r(2l+1)A_{n-l}^{\alpha}\sin(2l+1)\gamma = o(n^\alpha). \tag{3.4}$$

Let's show that if $\alpha > 2r+1$ and $\delta \le \gamma \le \pi$, then the series is uniformly (C,α)-summable to zero with respect to γ:

$$T_{n,m}^{r,\alpha}(\cos\gamma) = \sum_{l=0}^{\infty}[-l(l+1)]^r(2l+1)A_{n-l}^{\alpha}P_{mm}^{l}(\cos\gamma) = o(n^\alpha).$$

For this, first of all, let us prove that if $\delta \le \gamma \le \pi$, then

$$\int_{\gamma}^{\pi} \frac{dt}{(\cos\gamma - \cos t)^{1/2}} < c. \tag{3.5}$$

Let us assume that $\delta < \frac{\pi}{4}$. Since $\cos\gamma - \cos t = 2\sin\frac{t-\gamma}{2}\sin\frac{t+\gamma}{2}$ and $\sin t \geq \frac{2t}{\pi}$, for $0 \leq t \leq \frac{\pi}{2}$, it follows that

$$\int_\gamma^\pi \frac{dt}{(\cos\gamma - \cos t)^{1/2}} \leq \pi^{1/2}\int_\gamma^\pi (t-\gamma)^{-1/2}\left(\sin\frac{t+\gamma}{2}\right)^{-1/2} dt.$$

Let us assume that $\delta \leq \gamma \leq \frac{\pi}{2}$. Then $\sin\frac{\gamma+t}{2} \geq \sin\gamma \geq \sin\delta$. Thus,

$$\int_\gamma^\pi \frac{dt}{(\cos\gamma - \cos t)^{1/2}} \leq \frac{\pi^{1/2}}{(\sin\delta)^{1/2}}\int_\gamma^\pi (t-\gamma)^{-1/2}dt \leq c.$$

This completes the proof of (3.5)

For the functions $P^l_{m,m}(\cos\gamma)$, from the integral presentation (3.2) and noting that $P^l_{nm} = P^l_{-m,-m}$, we obtain that for $m = 0, 1,,$

$$\begin{aligned}
T^{r,\alpha}_{n,m}(\cos\gamma) &= -\frac{\sqrt{2}}{\pi}(1+\cos\gamma)^m \int_\gamma^\pi \sum_{l=0}^n [-l(l+1)]^r(2l+1)A^\alpha_{n-l}(\cos\gamma - \cos t)^{-1/2}\times \\
&\quad\times\left[\sqrt{2}\sin\frac{t}{2} + (\cos\gamma - \cos t)^{1/2}\right]^{-2m}\sin\left(l+\frac{1}{2}\right)t\,dt \\
&= -\frac{2\sqrt{2}}{\pi}(1+\cos\gamma)^m\times \\
&\quad\times\int_{\gamma/2}^{\pi/2}\sum_{l=0}^n[-l(l+1)]^r(2l+1)A^\alpha_{n-l}(\cos\gamma - \cos 2t)^{-\frac{1}{2}}\times \\
&\quad\times[\sqrt{2}\sin t + (\cos\gamma - \cos 2t)^{1/2}]^{-2m}\sin(2l+1)t\,dt
\end{aligned}$$

If we take into account $(1+\cos\gamma)^m \leq 2$ and $[\sqrt{2}\sin t + (\cos\gamma - \cos 2t)^{1/2}]^{-2m} \leq \frac{1}{\sin^2 t} \leq \frac{1}{\sin^2\frac{\delta}{2}}$, then from the last equality we have

$$\begin{aligned}
T^{r,\alpha}_{n,m}(\cos\gamma) &< c\left\{\max_{\gamma\in\left[\frac{\delta}{2},\frac{\pi}{2}\right]}|\Phi^{r,\alpha}_{n,m}(\cos\gamma)|\right\}\int_{\gamma/2}^{\pi/2}\frac{dt}{(\cos\gamma-\cos t)^{1/2}} \\
&= c\left\{\max_{\gamma\in[\frac{\delta}{2},\frac{\pi}{2}]}|\Phi^{r,\alpha}_{n,m}(\cos\gamma)|\right\}\int_\gamma^\pi\frac{dt}{(\cos\gamma-\cos t)^{1/2}}.
\end{aligned}$$

From the preceding equality, we obtain the acceleration from of (3.5).

$$T^{r,\alpha}_{n,m}(\cos\gamma) < c\max_{\gamma\in[\frac{\delta}{2},\frac{\pi}{2}]}|\Phi^{r,\alpha}_{n,m}(\cos\gamma)|. \tag{3.6}$$

From (3.6), and taking (3.4), into account, we deduce

$$T^{r,\alpha}_{n,m}(\cos\gamma) = o(n^\alpha), \quad \delta \leq \gamma \leq \pi. \tag{3.7}$$

Let us now begin to proof the theorem. For the accelerated from of (3.7), and for an arbitrary number $\varepsilon > 0$, there exists N such that for all $n > N$

$$|K^{r,\alpha}_{n,m}(\cos\gamma)| < \varepsilon. \tag{3.8}$$

Thus, when $n > N$ we obtain

$$\left|\iint_{S^2\setminus\omega(P,\delta)} V_m(\gamma,\bar{\varphi}) e^{-im(\varphi_1+\varphi_2)} K^{r,\alpha}_{n,m}(\cos\gamma)\sin\gamma\, d\gamma d\bar{\varphi}\right| <$$
$$< \varepsilon \iint_{S^2\setminus\omega(P,\delta)} |V_m(\gamma,\bar{\varphi})|\sin\gamma\, d\gamma d\bar{\varphi} \le \varepsilon \iint_{S^2} |V_m(P)| dS_Q < c\varepsilon.$$

□

Theorem 2. *Let us say that $V_m(\mathcal{P})$, $m = 0, \pm 1$, is an integrable function defined on S^2. If for nonnegative number r, there exists $D^r V_m(\mathcal{P})$, then $\Delta^r S(V_m;\mathcal{P})$ is (C,α)-summable to $D^r V_m(\mathcal{P})$ when $\alpha > 2r+1$.*

Proof. Without loss of generality, let us assume that the point P is the north pole of the sphere S^2, and the origin of coordinates is located at the center of S^2 sphere. Then $\sigma^{r,\alpha}_{n,m}(V_m;\mathcal{P})$ would be written down as

$$\sigma^{r,\alpha}_{n,m}(V_m;\mathcal{P}) = \frac{(-1)^m}{4\pi}\int_0^\pi \left\{\int_{C(P,\gamma)} V_m(Q) e^{-im(\tilde{\varphi}_1+\tilde{\varphi}_2)} dS_Q\right\} K^{r,\alpha}_{n,m}(\cos\gamma) d\gamma.$$

For the accelerated from of Lemma 3, it is possible to assume $D^k V_m(\mathcal{P}) = 0$, $k = 0, 1, \ldots, r$. Thus, for any number $\varepsilon > 0$, there exists $\delta > 0$ such that, when $\gamma \le \delta$ we have

$$\left|\int_{C(\mathcal{P},\gamma)} V_m(Q) e^{-im(\tilde{\varphi}_1+\tilde{\varphi}_2)} dS_Q\right| < c\varepsilon(\sin\gamma)^{2r+1}. \tag{3.9}$$

Let us represent $\sigma^{r,\alpha}_{n,m}(V_m;\mathcal{P})$ as the sum of two summands in the following way:

$$\sigma^{r,\alpha}_{n,m}(V_m;\mathcal{P}) = \frac{(-1)^m}{4\pi}\int_0^\delta \left\{\int_{C(\mathcal{P},\gamma)} V_m(Q) e^{-im(\overline{\varphi}_1+\overline{\varphi}_2)} dS_Q\right\} K^{r,\alpha}_{n,m}(\cos\gamma) d\gamma$$
$$+ \iint_{S^2\setminus\omega(\mathcal{P},\delta)} V_m(Q) e^{-im(\overline{\varphi_1}+\overline{\varphi_2})} K^{r,\alpha}_{n,m}(\cos\gamma)\sin\gamma d\gamma d\overline{\varphi} = I_1 + I_2.$$

By Theorem 1 $\lim\limits_{n\to\infty} I_2 = 0$.

By virtue of (3.9)

$$|I_1| < c\varepsilon \int_0^\delta |K^{r,\alpha}_{n,m}(\cos\gamma)|(\sin\gamma)^{2r+1} d\gamma.$$

And by virtue of Lemma 5 and Lemma 6

$$\int_0^\delta |K^{r,a}_{n,m}(\cos\gamma)|(\sin\gamma)^{2r+1} d\gamma = \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^\delta < c,$$

thus $|I_1| < c\varepsilon$. The theorem is proved. □

Lemma 7. *If r is a nonnegative integer $0 < \delta < \frac{\pi}{2}$ and $\alpha > 2r+2$ hold, then there exists a constant $c > 0$ such that*

$$\int_0^\delta \left|\frac{d}{d\gamma} K^{r,\alpha}_{n,m}(\cos\gamma)\right| \sin^{2r+2}\gamma d\gamma < c, \quad m = 0, 1.$$

Proof. Let us first consider the case $m=0$.Then $K^{r,\alpha}_{n,m}(\cos\gamma)=K^{r,\alpha}_{n}(\cos\gamma)$. By partial integration, we obtain

$$\int_0^\delta \frac{d}{d\gamma}K^{r,\alpha}_n(\cos\gamma)\sin^{2r+2}\gamma d\gamma=\sin^{2r+2}\delta K^{r,\alpha}_n(\cos\delta)-(2r+2)\int_0^\delta K^{r,\alpha}_n(\cos\gamma)\sin^{2r+1}\gamma d\gamma.$$

From this equality, by virtue of lemma 5, Lemma 6 and (3.3) we conclude that

$$\int_0^\delta \left|\frac{d}{d\gamma}K^{r,\alpha}_n(\cos\gamma)\right|\sin^{2r+2}\gamma d\gamma<c.$$

If $m=\pm1$, then by virtue of (2.6) it is sufficient to show that

$$\int_0^\delta \left|\frac{d}{d\gamma}\Phi^{r,\alpha}_n(\cos\gamma)\right|\sin^{2r+2}\gamma d\gamma=O(1). \tag{3.10}$$

Since

$$\frac{d}{d\gamma}\Phi^{r,\alpha}_n(\cos\gamma)=\frac{\sin\gamma}{(1+\cos\gamma)^2}\int_\gamma^\pi K^{r,\alpha}_n(\cos t)\sin t dt-\frac{\sin\gamma}{1+\cos\gamma}K^{r,\alpha}_n(\cos\gamma),$$

we have

$$\begin{aligned}\int_0^\delta \left|\frac{d}{d\gamma}\Phi^{r,\alpha}_n(\cos\gamma)\right|\sin^{2r+2}\gamma d\gamma &\le \left|\int_0^\delta \frac{\sin^{2r+3}\gamma}{(1+\cos\gamma)^2}\left\{\int_\gamma^\pi K^{r,\alpha}_n(\cos t)\sin t dt\right\}d\gamma\right|+\\ &+\int_0^\delta \frac{\sin^{2r+3}\gamma}{(1+\cos\gamma)}|K^{r,\alpha}_n(\cos\gamma)|d\gamma=\left|\int_0^\delta K^{r,\alpha}_n(\cos t)\sin t\left\{\int_0^t \frac{\sin^{2r+3}\gamma}{(1+\cos\gamma)^2}d\gamma\right\}dt+\right.\\ &\left.+\int_\delta^\pi K^{r,\alpha}_n(\cos t)\sin t\left\{\int_0^\delta\frac{\sin^{2r+3}\gamma}{(1+\cos\gamma)^2}d\gamma\right\}\right|+\int_0^\delta\frac{\sin^{2r+3}\gamma}{1+\cos\gamma}|K^{r,\alpha}_n(\cos\gamma)|d\gamma\le\\ &\le c\left\{\int_0^\delta |K^{r,\alpha}_n(\cos t)|\sin^{2r+1}t dt+\int_\delta^\pi |K^{r,\alpha}_n(\cos t)|\sin t dt\right\}.\end{aligned}$$

Hence, by virtue of Lemma 5, Lemma 6 and (3.8), the estimate (3.10) is obtained. □

Theorem 3. *Let us say that $V_m(\mathcal{P})$, with $m=0,\ \pm1$ is an integrable function defined on S^2. If, for nonnegative number r there exists $\tilde{D}^rV_m(\mathcal{P})$, then the series $\Delta^rS(V_m;\mathcal{P})$ is (C,α)-summable to the number $\tilde{D}^r(V_m;\mathcal{P})$ when $\alpha>2r+2$.*

Proof. Let us assume that the point P is the north pole of the sphere S^2, and the origin of the coordinates is located at the center of S^2. If $\tilde{D}^{(i)}V_m(\mathcal{P})=0$ holds for $i=0,1,\dots,r$, then, using the accelerated form of Lemma 4 it is sufficient to show that the series $\Delta^rS(V_m;P)$ is summable to zero by the (C,α) method, when $\alpha>2r+2$.

Since $\tilde{D}^rV_m(\mathcal{P})=0$ holds for $i=0,1,\dots,r$, thus for any number $\varepsilon>0$, there exists $\delta>0$ such that

$$\left|\iint\limits_{\varphi(P,\gamma)} V_m(Q)e^{-im(\varphi_1+\varphi_2)}dS_Q\right|<c\varepsilon\left(\sin\frac{\gamma}{2}\right)^{2r+2},\ \text{when}\ \gamma<\delta.$$

That is,

$$\int_0^\gamma\left\{\int_{(P,Q)=\cos h}V_m(Q)e^{-im(\varphi_1+\varphi_2)}dS_Q\right\}dh<c\varepsilon\left(\sin\frac{\gamma}{2}\right)^{2r+2}, \tag{3.11}$$

when $\gamma < \delta$.

Let us assume that $\eta = \min(\delta; \frac{\pi}{4})$, $\frac{1}{n} \leq \eta$. We have

$$|\sigma_{n,m}^{r,\alpha}(V_m; \mathcal{P})| \leq \frac{1}{2\pi} \left| \int_0^\eta \left[\int_0^{2\pi} V_m(\gamma, \overline{\varphi}) e^{-im(\overline{\phi}_1 + \overline{\phi}_2)} d\overline{\varphi} \right] K_{n,m}^{r,\alpha}(\cos\gamma) \sin\gamma d\gamma \right| +$$
$$+ \frac{1}{\pi} \left| \int_\eta^\pi \int_0^{2\pi} V_m(\gamma, \overline{\varphi}) e^{-im(\overline{\phi}_1 + \overline{\phi}_2)} d\overline{\varphi} K_{n,m}^{r,\alpha}(\cos\gamma) \sin\gamma d\overline{\phi} \right|.$$

By theorem 2, the second summand on the right-hand side of the last inequality tends to zero as $n \to \infty$. For the first summand, let us perform partial integration, and obtain

$$\int_0^\eta \left[\int_0^{2\pi} V_m(\gamma, \overline{\varphi}) e^{-im(\overline{\varphi}_1 + \overline{\varphi}_2)} d\overline{\varphi} \right] K_{n,m}^{r,\alpha}(\cos\gamma) \sin\gamma d\gamma =$$
$$= K_{n,m}^{r,\alpha}(\cos\eta) \int\limits_0^\eta \int\limits_0^{2\pi} V_m(\gamma, \overline{\varphi}) e^{-im(\overline{\varphi}_1 + \overline{\varphi}_2)} \sin\gamma d\gamma d\overline{\varphi} -$$
$$- \int_0^\eta \left\{ \int_{(P,Q) = \cos\gamma} V_m(Q) e^{-im(\overline{\varphi}_1 + \overline{\varphi}_2)} dS_Q \right\} \frac{d}{d\gamma} K_{n,m}^{r,\alpha}(\cos\gamma) d\gamma.$$

By virtue of inequality (3.11) and Lemma 7, we have

$$\left| \int_0^\eta \left\{ \int_{(\mathcal{P},Q) = \cos\gamma} V_m(Q) e^{-im(\overline{\varphi}_1 + \overline{\varphi}_2)} dS_Q \right\} \frac{d}{d\gamma} K_{n,m}^{r,\alpha}(\cos\gamma) d\gamma \right| < c\varepsilon,$$

Moreover, by (3.3) the evaluation $\lim\limits_{n\to\infty} K_{n,m}^{r,\alpha}(\cos\eta) = 0$ holds. Thus, the theorem is proved. □

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