# ON THE EXISTENCE OF BOUNDED SOLUTIONS OF NONLINEAR SYSTEMS OF A CLASS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS AND NONLINEAR IMPULSIVE DIFFERENTIAL SYSTEMS ON INFINITY AXES

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Abstract. Effective sufficient conditions are established for existence of bounded solutions of systems of nonlinear generalized ordinary differential equations with nondecreasing matrix-function on the real axis. Sufficient conditions are established for the existence of unique solution. Results are realized for nonlinear system of impulsive differential equations
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## 1. Statement of the problem. Basic notation and definitions

For the system of the nonlinear generalized ordinary differential equations

$$dx = dA(t) \cdot f(t, x) \quad \text{for} \quad t \in \mathbb{R} \tag{1.1}$$

consider the problem on the existence of solutions satisfying one from the two conditions

$$\sup\{\|x(t)\| : t \in \mathbb{R}\} < +\infty, \tag{1.2}$$

or

$$\sup\{\|x(t)\| : t \in \mathbb{R}_+\} < +\infty, \tag{1.3}$$

where $A = (a_{ik})_{i,k=1}^{n}$; $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a nondecreasing matrix-function, and $f = (f_k)_{k=1}^{n} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector-function restriction of which on the every closed interval $[a, b] \subset \mathbb{R}$ belongs to the Carathéodory class $Car([a, b] \times \mathbb{R}^n, \mathbb{R}^n; A)$.

The generalized ordinary differential equations has been introduced by J. Kurzweil [15]. To a considerable extent, the interest to the theory has also been stimulated by the fact that this theory enabled one to investigate ordinary differential, impulsive differential and difference equations from a unified point of view (see, [1] – [8], [18], [20] and references therein).

Therefore, we can consider the ordinary differential, impulsive differential and difference equations as equations of the same type.

Problem, analogous to problem considered in present paper, for linear and nonlinear cases, is investigated in [13], [14] (see, also references therein) in the case of systems of ordinary differential equations, and as to generalized case, it is considered in [8] only for linear systems.

As we know, the question connected to the nonlinear case was not investigated in earlier papers. So, the considered problem is actually. In the paper, an

Some general questions of impulsive differential theory are investigated in [9], [10],[16],[17],[19] (see, also references therein).

The use will be made of the following notation and definitions

$\mathbb{R} = ]-\infty,+\infty[$, $\mathbb{R}_+ = [0,+\infty[$; $[a,b]$ and $]a,b[$ $(a,b \in \mathbb{R})$ are, respectively, closed and open intervals. $I$ is an arbitrary finite or infinite interval from $\mathbb{R}$.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max\limits_{j=1,\dots,m} \sum\limits_{i=1}^{n} |x_{ij}|$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column $n$-vectors $x = (x_i)_{i=1}^n$;

If $X \in \mathbb{R}^{n\times n}$, then $X^{-1}$ and $\det(X)$ are, respectively, the matrix inverse to $X$ and the determinant of $X$. $I_n$ is the identity $n\times n$-matrix.

$\operatorname{diag}(h_1,,\dots,h_n)$ is a diagonal matrix-functions with diagonal elements $h_1.\dots,h_n$.

The inequalities between the matrices are understood componentwise.

$\overset{b}{\underset{a}{\mathrm{V}}}(X)$ is the sum total variation of the components of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$; $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of $X$ at the point $t \in [a,b]$ (we assume that $X(a-) = X(a)$, $X(b+) = X(b)$, and $X(t)$ is defined by continuity outside of $[a,b]$). $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$

If $X = (x_{ij})_{i,j=1}^{n,m} : [a,b] \to \mathbb{R}^{n\times m}$, then $V(X)(t) = \big(\overset{t}{\underset{a}{\mathrm{V}}}(x_{ij})\big)_{i,j=1}^{n,m}$.

$\mathrm{BV}_{loc}(\mathbb{R};\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : \mathbb{R} \to \mathbb{R}^{n\times m}$ whose restrictions on every closed interval $[a,b]$ from $\mathbb{R}$ belong to $\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

If $g \in \mathrm{BV}([a,b];\mathbb{R})$, $f : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then we assume

$$\int\limits_s^t x(\tau)\,dg(\tau) = (L-S)\int\limits_{]s,t[} x(\tau)\,dg(\tau) + f(t)d_1g(t) + f(s)d_2g(s).$$

where $(L-S)\int\limits_{]s,t[} f(\tau)\,dg(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$. It is known (see, [18, 20]) that if the integral exists, than the right side of the integral equality equals to the Kurzeil–Stieltjes integral $(K-S)\int\limits_s^t f(\tau)\,dg(\tau)$ and, hence, $\int\limits_s^t f(\tau)\,dg(\tau) = (K-S)\int\limits_s^t f(\tau)\,dg(\tau)$. If $a = b$, then we assume $\int\limits_a^b x(t)\,dg(t) = 0$.

We introduce the operator $\mathcal{A}(X,Y)$s in the following way:

if $X \in \mathrm{BV}_{loc}(\mathbb{R};\mathbb{R}^{n\times n})$, $\det(I_n - (-1)^j d_j X(t)) \neq 0$ $(j = 1,2)$ for $t \in \mathbb{R}$, and $Y \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m})$, then

$$\mathcal{A}(X,Y)(0) = O_{n\times m},$$

$$\mathcal{A}(X,Y)(t) - \mathcal{A}(X,Y)(s) = Y(t) - Y(s) + \sum_{s<\tau\le t} d_1X(\tau)\,(I_n - d_1X(\tau))^{-1}\,d_1Y(\tau)$$

$$- \sum_{s\le\tau<t} d_2X(\tau)\,(I_n + d_2X(\tau))^{-1}\,d_2Y(\tau) \ \text{ for } \ s < t.$$

If $G = (g_{ik})_{i,k=1}^{n} \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and $x = (x_k)_k^n : [a,b] \to \mathbb{R}^n$, then

$$\int_a^b dG(\tau)\cdot x(\tau) = \left(\sum_{k=1}^{n}\int_a^b x_k(\tau)\,dg_{ik}(\tau)\right)_i^n.$$

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to \mathbb{R}^{l\times n}$ is a nondecreasing matrix-function, and $D_1 \subset \mathbb{R}^{l\times n}$ and $D_2 \subset \mathbb{R}^{n\times m}$, then $Car([a,b]\times D_1, D_2; G)$ is the Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that for each $i \in \{1,\dots,l\}$, $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$:

(a) the function $f_{kj}(\cdot,x) : [a,b] \to D_2$ is $\mu(g_{ik})$-measurable for every $x \in D_1$;

(b) the function $f_{kj}(t,\cdot) : D_1 \to D_2$ is continuous for $\mu(g_{ik})$-a. a. $t \in [a,b]$, and

$$\sup\big\{|f_{kj}(\cdot,x)| : \ x \in D_0\big\} \in L([a,b],\mathbb{R};g_{ik})$$

for every compact $D_0 \subset D_1$;

If $G(t) \equiv \mathrm{diag}(t,\dots,t)$ then under $Car([a,b]\times D_1, D_2; G)$ we mean the classical Carathéodory class $Car([a,b]\times D_1, D_2)$.

We assume that $A(0) = O_{n\times n}$ without loss of generality for every system of type (1.1). Let, moreover,

$$\det(I_n - (-1)^j d_j A(t)) \neq 0 \ \text{ for } \ t \in \mathbb{R} \ \ (j = 1,2). \tag{1.4}$$

The inequalities (1.4) guarantee the unique solvability of the Cauchy problem for the corresponding linear systems (see [18, 20]).

By a solution of system (1.1) we understand a vector-function $x \in \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^n)$ such that

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot f(\tau, x(\tau))d\tau \ \text{ for } \ s < t, \ s,t \in \mathbb{R}.$$

Let $\alpha \in \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R})$ and $t_0 \in \mathbb{R}$ are such that $1 + (-1)^j d_j\alpha(t) \neq 0$ for $(-1)^j(t - t_0) < 0$, $t \neq t_0$ $(j = 1,2)$. Then it is known that (see [11, 12] the initial problem

$$d\xi = \xi d\alpha(t), \ \ \xi(t_0) = 1$$

has the unique solution $\xi_\alpha$ and it is defined by

$$\xi_\alpha(t) = \exp(\alpha(t) - \alpha(t_0))\cdot\mu_\alpha(t),$$

where

$$\xi_\alpha(t)=\begin{cases}\exp(s_c(\alpha)(t)-s_c(\alpha)(t_0))\prod\limits_{t_0<\tau\le t}(1-d_1\alpha(\tau))^{-1}\prod\limits_{t_0\le\tau<t}(1+d_2\alpha(\tau)) & \text{for } t>t_0,\\ \exp(s_c(\alpha)(t)-s_c(\alpha)(t_0))\prod\limits_{t<\tau\le t_0}(1-d_1\alpha(\tau))\prod\limits_{t\le\tau<t_0}(1+d_2\alpha(\tau))^{-1} & \text{for } t<t_0.\end{cases}$$

Let $\gamma_\alpha(t,s)\equiv\xi_\alpha(t)\xi_\alpha^{-1}(s)$ be the Cauchy function of the problem. Then

$$\gamma_\alpha(t,s)=\exp\big(J(\alpha)(t)-J(\alpha)(s)\big)\prod_{s<\tau\le t}\operatorname{sgn}\big(1-d_1\alpha(\tau)\big)$$
$$\times\prod_{s\le\tau<t}\operatorname{sgn}\big(1+d_2\alpha(\tau)\big)\ \text{ for }\ t>s,$$
$$\gamma_\alpha(t,s)=\gamma_\alpha^{-1}(s,t)\ \text{ for }\ t<s.$$

Note that the following equality holds (see, [6, 7])

$$d\xi_\alpha^{-1}(t)\equiv-\xi_\alpha^{-1}(t)d\mathcal{A}(\alpha,\alpha)(t)$$

If $\sigma=(\sigma_i)_{i=1}^n$, where $\sigma_i\in\{-1;1\}$, then by $\mathcal{N}_+(\sigma)$ (by $\mathcal{N}_-(\sigma)$) we denote the set of $i\in\{1,...,n\}$ such that $\sigma_i=1$ ($\sigma_i=-1$). The sets has been introduced by I. Kiguradze for the ordinary differential equations (see, [13, 14]).

Everywhere, under $t_1,\dots,t_n$ we assume that

$$t_i=a\ \text{ for }\ i\in\mathcal{N}_+(\sigma)\quad\text{and}\quad t_i=b\ \text{ for }\ i\in\mathcal{N}_-(\sigma).$$

## 2. Formulation of the results

The rezults given in the section immediatelly follows from similarly rezults for the matrix-function $A$ with bouded variation.

**Theorem 2.1.** *Let the matrix-function $A$ and $\sigma_i\in\{-1;1\}$ $(i=1,\dots,n)$, vector-functions $\alpha_l=(\alpha_{li})_{i=1}^n\in\mathrm{BV}_{loc}(\mathbb{R};\mathbb{R}^n)$ $(l=1,2)$ and matrix-functions $(\beta_{lik})_{i,k=1}^n$, $\beta_{lik}\in L_{loc}(\mathbb{R},\mathbb{R};a_{ik})$ $(l=1,2)$ be such that*

$$\alpha_{li}(t)-\alpha_{li}(\tau)=\sum_{k=1}^n\int_\tau^t\beta_{lik}(\tau)da_{ik}(\tau)\ \textit{ for }\ t,\tau\in\mathbb{R}\ \ (l=1.2;\ i=1,\dots,n),\tag{2.1}$$

$$\sup\{|\alpha_{li}(t)|:t\in\mathbb{R}\}<+\infty\ \ (l=1.2;\ i=1,\dots,n),\tag{2.2}$$

$$\alpha_1(t)\le\alpha_2(t)\ \textit{ for }\ t\in\mathbb{R};\tag{2.3}$$

*and inequalities*

$$(-1)^l\sigma_i\big(f_k(t,x_1,\dots,x_{i-1},\alpha_{ji}(t),x_{i+1},\dots,x_n)-\beta_{lik}(t)\big)\le0$$
$$(l,j=1,2;\ i,k=1,\dots,n),\tag{2.4}$$

$$(-1)^l\left(x_i + (-1)^j \sum_{k=1}^{n} f_k(t, x_1, \dots, x_n) d_j\alpha_{ik}(t) - \alpha_{li}(t) - (-1)^j d_j\alpha_{li}(t)\right) \leq 0$$
$$\text{for } (-1)^j\sigma_i \geq 0 \ \ (l, j = 1, 2;\ i = 1, \dots, n) \tag{2.5}$$

*are fulfilled on the set*

$$\{(t, x_1, \dots, x_n) : t \in \mathbb{R},\ \ \alpha_{1i}(t) \leq x_i \leq \alpha_{2i}(t)\ \ (i = 1, \dots, n)\}.$$

*Then problem* (1.1), (1.2) *is solvable.*

**Theorem 2.1$_1$.** *Let the matrix-function* $A$ *and* $\sigma_i \in \{-1; 1\}$ $(i = 1, \dots, n)$, *vector-functions* $\alpha_l = (\alpha_{li})_{i=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}; \mathbb{R}^n)$ $(l = 1, 2)$ *and matrix-functions* $(\beta_{lik})_{i,k=1}^n$, $\beta_{lik} \in L_{loc}(\mathbb{R}, \mathbb{R}; a_{ik})$ $(l = 1, 2)$ *be such that* (2.1) *holds for* $t, \tau \in \mathbb{R}_+$,

$$\sup\{|\alpha_{li}(t)| : t \in \mathbb{R}_+\} < +\infty \ \ (l = 1, 2;\ i = 1, \dots, n),$$
$$\alpha_1(t) \leq \alpha_2(t) \ \ \text{for } \ t \in \mathbb{R}_+;$$

*and inequalities* (2.4) *and* (2.5) *are fulfilled on the set*

$$\{(t, x_1, \dots, x_n) : t \in \mathbb{R}_+,\ \ \alpha_{1i}(t) \leq x_i \leq \alpha_{2i}(t)\ \ (i = 1, \dots, n)\}.$$

*Then system* (1.1), *for every* $c_i \in [\alpha_{1i}(0), \alpha_{2i}0)]$ $(i \in \mathcal{N}_+(\sigma))$, *where* $\sigma = (\sigma_i)_{i=1}^n$, *has a solution satisfying satisfying conditions* (1.3) *and*

$$x_i(0) = c_i \ \ \text{if} \ \ i \in \mathcal{N}_+(\sigma). \tag{2.6}$$

(If $\mathcal{N}_+(\sigma) = \emptyset$, then condition (2.6) is eliminated).

In Theorems 2.2 and 2.3, we will assume that

$$b_{ii}(t) \equiv \sum_{k=1}^{n} a_{ik}(t) \ \ (i = 1, \dots, n)$$

end

$$\zeta_i(t) \equiv \sigma_i\eta_{ii}b_{ii}(t) \ \ (i = 1, \dots, n).$$

It is evident that $b_{ii}$ $(i = 1, \dots, n)$ are nondecreasing functions.

**Theorem 2.2.** *Let the matrix-function* $A$ *and* $\sigma_i \in \{-1, 1\}$ $(i = 1. \dots, n)$ *be such that*

$$\sigma_i f_i(t, x_1, \dots, x_n) \operatorname{sgn} x_i \leq \sum_{l=1}^{n} \eta_{il}|x_l| + q_i(t) \ \ (i = 1, \dots, n), \tag{2.7}$$

*and*

$$\left(|f_i(t, x_1, \dots, x_n)| - \sum_{l=1}^{n} \eta_{il}|x_l| - q_i(t)\right) d_j a_{ii}(t) \leq 0 \ \ (j = 1, 2;\ \ i = 1, \dots, n) \tag{2.8}$$

*be fulfilled on* $\mathbb{R} \times \mathbb{R}^n$ *and let, in addition,*

$$\rho_i = \sup\left\{\left|\ \bigvee_{t}^{\zeta_i(t)+1} \big(\mathcal{A}(\zeta_i, \sigma_i g_i)\big)\right|\ t \in \mathbb{R}\right\} << \infty \ \ (i = 1, \dots, n), \tag{2.9}$$

$$1+(-1)^j d_j\zeta_i(t)>0 \ \ for \ \ t\in\mathbb{R} \ \ (j=1,2;\ i=1,\dots,n). \tag{2.10}$$

*and*

$$\sum_{k=1}^{n}|\eta_{ki}|d_jb_{kk}(t)<1 \ \ for \ \ t\in\mathbb{R}\ (j=1,2;\ \ i=1,\dots,n), \tag{2.11}$$

*where* $\eta_{il}\in\mathbb{R}_+$ $(i\neq l)$, $\eta_{ii}<0$, $q_i\in \mathrm{BV}_{loc}(\mathbb{R};\mathbb{R})$ $(i,l=1,\dots,n)$, $g_i(t)\equiv\int\limits_{t_i}^{t}q_i(\tau)db_{ii}(\tau)$ $(i,l=1,\dots,n)$ *and the real part of every characteristic value of the matrix* $(\eta_{il})_{i,l=1}^n$ *be negative. Then problem* (1.1), (1.2) *is solvable.*

Note that if $d_ja_{ii}(t)=0$, then the corresponding inequality is eliminated in (2.8);

**Theorem 2.3.** *Let the matrix-function* $A$ *and* $\sigma_i\in\{-1,1\}$ $(i=1.\dots,n)$ *be such that*

$$\sigma_i\big(f_i(t,x_1,\dots,x_n)-f_i(t,y_1,\dots,y_n)\big)\operatorname{sgn}(x_i-y_i)\le\sum_{l=1}^{n}\eta_{il}|x_l-y_i| \ \ (i=1,\dots,n) \tag{2.12}$$

*and*

$$\left(|f_i(t,x_1,\dots,x_n)-f_i(t,y_1,\dots,y_n)|-\sum_{l=1}^{n}\eta_{il}|x_l-y_l|\right)d_ja_{ii}(t)\le 0 \quad (j=1,2;\ \ i=1,\dots,n)$$

*be fulfilled on* $\mathbb{R}\times\mathbb{R}^n$ *and let, in addition, conditions* (2.7), (2.9) *and* (2.10) *hold, where* $\eta_{il}\in\mathbb{R}_+$ $(i\neq l)$, $\eta_{ii}<0$, $g_i(t)\equiv\int\limits_{t_i}^{t}f_i(\tau,0,\dots,0)db_{ii}(\tau)$ $(i,l=1,\dots,n)$ *and the real part of every characteristic value of the matrix* $(\eta_{il})_{i,l=1}^n$ *is negative. Then problem* (1.1), (1.2) *has the unique solution.*

## 3. Formulation of the results for impulsive problems

Obtained results we use for nonlinear impulsive differential system

$$\frac{dx}{dt}=f(t,x) \ \ \text{for a.a.} \ \ t\in\mathbb{R}\setminus T, \tag{3.1}$$

$$x(\tau_l+)-x(\tau_l-)=h(\tau_l,x(\tau_l)) \ \ (l=1,2,\dots), \tag{3.2}$$

where $f=(f_k)_{k=1}^n:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is a vector-function restriction of which on the every closed interval $[a,b]\subset\mathbb{R}$ belongs to the Carathéodory class $Car([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$, the vector-function $h(\tau_l,.)=(h_k(\tau_l,.))_{k=1}^n:\mathbb{R}^n\to\mathbb{R}^n$ $(l=1,2,\dots)$ are continuous, $T=\{\tau_1,\tau_2,\dots\}$, $\tau_l\in\mathbb{R}$, $\tau_l<\tau_{l+1}$ $(l=1,2,\dots)$.

In the section, in addition, we use the designations:

$\mathrm{AC}([a,b];D)$ is the set of all absolutely continuous matrix-functions $X:[a,b]\to D$;

$\mathrm{AC}_{loc}(I;D)$ is the set of all matrix-functions $X:I\to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from $I$ belong to $AC([a,b];D)$;

$\mathrm{AC}_{loc}(I \setminus T; D)$, where $T = \{\tau_1, \tau_2, \dots\}$, $\tau_l \in I$ $(l = 1, 2, \dots)$, $\tau_l \neq \tau_k$ $(i \neq k)$, is the set of all matrix-functions $X : I \to D$ whose restrictions to an arbitrary closed interval $[a, b]$ from $I \setminus T$ belong to $AC([a, b], D)$;

By a solution of the impulsive differential system (3.1), (3.2) we understand a continuous from the left vector-function $x \in \mathrm{BV}(\mathbb{R}; \mathbb{R}^n) \cap \mathrm{AC}_{loc}(\mathbb{R} \setminus T; \mathbb{R}^n)$ satisfying both the system

$$x'(t) = f(t, x(t)) \text{ for a.a. } t \in \mathbb{R} \setminus T$$

and relation (3.2) for every $l \in \{1, 2, \dots\}$.

We assume that the impulsive points $\tau_l$ $(l = 1, 2, \dots)$ are such that $0 \le \tau_1 < \tau_2 < \dots < \tau_l < \tau_{l+1} < \dots$ and $\lim\limits_{l \to +\infty} \tau_l = +\infty$.

Let $\alpha(t) \equiv t + \iota(t)$, where $\iota : \mathbb{R} \to \mathbb{N}$ is function defined by $\iota(t) = \max\{l : \tau_l < t\}$. It is evident that functions $\iota$ and $\alpha$ are nondecreasing, $\iota(t) = l$ for $t \in ]\tau_l, \tau_{l+1}]$ and $d_2\iota(t_l) = l$ $(l = 1, 2, \dots)$. Then the matrix-function $\widetilde{A}(t) \equiv \mathrm{diag}(t + \iota(t), \dots, t + \iota(t))$ is nondecreasing, as well, and $d_1\widetilde{A}(t) \equiv O_{n\times n}$, $d_2\widetilde{A}(t) = O_{n\times n}$ for $t \in \mathbb{R} \setminus T$ and $d_2\widetilde{A}(t) = I_n$ for $t \in T$. Therefore, we can rewrite system (3.1), (3.2) in the form

$$dx = d\widetilde{A}(t) \cdot \widetilde{f}(t, x) \quad \text{for} \quad t \in \mathbb{R}, \tag{3.3}$$

where

$$\widetilde{f}(t, x) = \begin{cases} f(t, x) & \text{for } t \in \mathbb{R} \setminus T, \\ h(\tau_l, x) & \text{for } t = \tau_l \ (l = 1, 2, \dots). \end{cases}$$

Note that, by (3.3) we can assume that $h(\tau_l, x) \equiv f(\tau_l, x)$ for $t = \tau_l$ $(l = 1, 2, \dots)$ with out loss of generallity.

Using for system (3.3) results given above, we obtain following ones.

**Theorem 3.1.** *Let numbers* $\sigma_i \in \{-1; 1\}$ $(i = 1, \dots, n)$, *vector-functions* $\alpha_l = (\alpha_{li})_{i=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}; \mathbb{R}^n)$ *and* $(\beta_{li})_{i=1}^n \in L_{loc}(\mathbb{R}, \mathbb{R}^n)$ $(l = 1, 2)$ *be such that conditions* (2.2), (2.3),

$$\alpha_{li}(t) - \alpha_{li}(\tau) = \int\limits_{\tau}^{t} \beta_{li}(\tau)d\tau + \sum_{\tau \le \tau_l < t} \beta_{li}(\tau_l) \ \textit{for } t, \tau \in \mathbb{R} \ \ (l = 1.2;\ i = 1, \dots, n), \tag{3.4}$$

*and inequalities*

$$(-1)^l \sigma_i \big(f_i(t, x_1, \dots, x_{i-1}, \alpha_{ji}(t), x_{i+1}, \dots, x_n) - \beta_{li}(t)\big) \le 0$$
$$\textit{for } t \in \mathbb{R} \setminus T, \ \ \alpha_{1i}(t) \le x_i \le \alpha_{2i}(t) \ \ (l, j = 1, 2;\ i, k = 1, \dots, n), \tag{3.5}$$

$$(-1)^l \sigma_i \big(g_i(\tau_m, x_1, \dots, x_{i-1}, \alpha_{ji}(\tau_m), x_{i+1}, \dots, x_n) - \beta_{li}(\tau_m)\big) \le 0$$
$$\textit{for } \alpha_{1i}(\tau_m) \le x_i \le \alpha_{2i}(\tau_m) \ \ (l, j = 1, 2;\ i, k = 1, \dots, n;\ m = 1, 2, \dots), \tag{3.6}$$

$$(-1)^l \big(x_i + h_i(\tau_m, x_1, \dots, x_n) - \alpha_{li}(\tau_m+)\big) \le 0 \ \textit{for } (-1)^j \sigma_i \ge 0,$$
$$\alpha_{1i}(\tau_m) \le x_i \le \alpha_{2i}(\tau_m), \ \ (l = 1, 2;\ i = 1, \dots, n\ m = 1, 2, \dots) \tag{3.7}$$

*are fulfilled. Then problem* (3.1), (3.2); (1.2) *is solvable.*

**Theorem 3.1$_1$.** *Let numbers $\sigma_i \in \{-1;1\}$ $(i=1,\dots,n)$, vector-functions $\alpha_l = (\alpha_{li})_{i=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^n)$ $(l=1,2)$ and $(\beta_{li})_{i=1}^n \in L_{loc}(\mathbb{R},\mathbb{R}^n)$ $(l=1,2)$ be such that*

$$\sup\{|\alpha_{li}(t)| : t\in\mathbb{R}_+\} < +\infty \ \ (l=1,2;\ i=1,\dots,n),$$
$$\alpha_1(t)\le\alpha_2(t) \ \ \textit{for}\ \ t\in\mathbb{R}_+;$$

*and conditions* (3.4), (3.5), (3.6) *and* (3.7) *are fulfilled on the set*

$$\{(t,x_1,\dots,x_n) : t\in\mathbb{R}_+,\ \ \alpha_{1i}(t)\le x_i\le\alpha_{1i}(t)\ \ (i=1,\dots,n)\}.$$

*Then system* (3.1), (3.2), *for every $c_i\in[\alpha_{1i}(0),\alpha_{2i}0)]$ $(i\in\mathcal{N}_+(\sigma))$, where $\sigma=(\sigma_i)_{i=1}^n$, has a solution satisfying conditions* (1.3) *and* (2.6).

In Theorems 3.2 and 3.3, for the case, we use the equalities.

$$b_{ii}(t)\equiv\alpha(t),\ \ d_1b_{ii}(t)\equiv 0,\ \ d_2b_{ii}(t)=0\ \ \text{for}\ \ t\in\mathbb{R},\ \ d_2b_{ii}(t)=1\ \ \text{for}\ \ t\in T$$

and

$$\zeta_i(t)\equiv\sigma_i\eta_{ii}\alpha(t),\ \ d_1\zeta_i(t)\equiv 0,\ \ d_2\zeta_i(t)=0\ \ \text{for}\ \ t\in\mathbb{R},\ \ d_2\zeta_i(t)=\sigma_i\eta_{ii}\ \ \text{for}\ \ t\in T \quad (i=1,\dots,n);$$

$$g_i(t)\equiv\int_0^t q_i(\tau)d\tau+\sum_{0\le\tau_l<t} q_i(\tau_l),\ \ d_1g_i(t)\equiv 0,$$
$$d_2g_i(t)\equiv 0\ \ \text{for}\ \ t\in\mathbb{R}\setminus T,\ \ d_2g_i(t)=q_i(t)\ \ \text{for}\ \ t\in T\ \ (i=1,\dots,n);$$
$$\mathcal{A}(\zeta_i,\sigma_ig_i)(t)\equiv\sigma_i\int_0^t q_i(\tau)d\tau+\sum_{0\le\tau_l<t}\eta_{ii}(1+\sigma_i\eta_{ii})^{-1}q_i(\tau_l)\ \ (i=1,\dots,n).$$

In the case, we have that conditions (2.9), (2.10), (2.11) are equivalent to the conditions, respectively,

$$\sup\left\{\left|\int_t^{\zeta_i(t)+1}|q_i(\tau)|d\tau+\sum_{t\le\tau_l<\zeta_i(t)+1}\eta_{ii}(1+\sigma_i\eta_{ii})^{-1}q_i(\tau_l)\right| : t\in\mathbb{R}\right\}<+\infty \quad (i=1,\dots,n); \tag{3.8}$$

$$\eta_{ii}>-1\ \ \text{if}\ \ \sigma_i=1\ \ (i=1,\dots,n), \tag{3.9}$$

$$\sum_{k=1}^n|\eta_{ki}|<1\ \ (i=1,\dots,n), \tag{3.10}$$

Note that if $h(i,x)\equiv 0_n$ for some $i\in\{1,\dots,n\}$, then corresponding inequality is eliminated. in (2.8).

**Theorem 3.2.** *Let the matrix-function $A$ and $\sigma_i \in \{-1,1\}$ $(i = 1.\dots,n)$ be such that inequalities* (2.7) *are fulfilled for $t \in \mathbb{R} \setminus T$, $(x_i)_{i=1}^n \mathbb{R}^n$, and*

$$|h_i(\tau_m, x_1, \dots, x_n)| \leq \sum_{l=1}^{n} \eta_{il}|x_l| + q_i(\tau_m) \ \text{ for } \ (x_l)_{l=1}^n \in \mathbb{R}^n$$
$$(i = 1, \dots, n, \ m = 1, 2, \dots).$$

*Let, in addition, conditions* (3.8), (3.9) *and* (3.10) *be fulfilled, where $\eta_{il} \in \mathbb{R}_+$ $(i \neq l)$, $\eta_{ii} < 0$, $\zeta_i(t) \equiv \sigma_i \eta_{ii} \alpha(t)$, $q_i \in \mathrm{BV}_{loc}(\mathbb{R};\mathbb{R})$ $(i,l = 1,\dots,n)$. and the real part of every characteristic value of the matrix $(\eta_{il})_{i,l=1}^n$ be negative. Then problem* (3.1), (3.2); (1.2) *is solvable.*

**Theorem 3.3.** *Let the matrix-function $A$ and $\sigma_i \in \{-1,1\}$ $(i = 1.\dots,n)$ be such that inequalities* (2.12) *are fulfilled for $t \in \mathbb{R} \setminus T$, $(x_i)_{i=1}^n \mathbb{R}^n$, and*

$$\left(|h_i(\tau_m, x_1, \dots, x_n) - h_i(\tau_m, y_1, \dots, y_n)| - \sum_{l=1}^{n} \eta_{il}|x_l - y_l|\right) \leq 0$$
$$\text{for } \ (x_l)_{l=1}^n, \ (y_l)_{l=1}^n \in \mathbb{R}^n \ \ (i = 1, \dots, n, \ m = 1, 2, \dots).$$

*Let, in addition, conditions* (3.8), (3.9) *and* (3.10) *be fulfilled, where $\eta_{il} \in \mathbb{R}_+$ $(i \neq l)$, $\eta_{ii} < 0$, $\zeta_i(t) \equiv \sigma_i \eta_{ii} \alpha(t)$, $q_i(t) \equiv f_i(t, 0, \dots, 0)$ $(i,l = 1,\dots,n)$ and the real part of every characteristic value of the matrix $(\eta_{il})_{i,l=1}^n$ be negative. Then problem* (3.1), (3.2); (1.2) *has the unique solvable.*

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