

ON FACTORIZATION OF PREORDERED GROUPS

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Abstract. We provide a complete classification of factorizations of preordered groups, expressed in terms of certain quasi-projections on the underlying groups.

1. INTRODUCTION

The problem of describing factorizations of algebraic structures, particularly groups and monoids, has received substantial attention due to its deep connections with matched pairs, bicrossed products, and non-abelian extensions (see, e.g., [1, 2, 3, 4, 5]). In this note, we extend this study to the context of preordered groups.

The main purpose of this note is to provide a complete classification of all factorizations of a preordered group $(A, \leq_A)$ through preordered subgroups. We establish that a factorization (X, B) of the underlying group A lifts to a factorization of preordered groups if and only if the corresponding positive cone monoid A^+ factorizes through the submonoids X^+ and B^+ . Furthermore, we demonstrate that such factorizations are parameterized precisely by left quasi-projections on A that preserve the positive cone.

2. PRELIMINARIES

We briefly review the necessary background on preordered sets, preordered groups, and factorizations of monoids and groups.

2.1. The Category of Preordered Sets. A *preorder* on a set X is a reflexive and transitive binary relation $\leq_X$. A set equipped with a preorder is called a *preordered set*, denoted $(X, \leq_X)$. A map $f : (X, \leq_X) \rightarrow (Y, \leq_Y)$ between preordered sets is *monotone* (or *order-preserving*) if $x \leq_X x'$ implies $f(x) \leq_Y f(x')$. We denote by **PreOrd** the category of preordered sets and monotone maps. **PreOrd** admits finite products. For preordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$, the product $(X \times Y, \leq_{\text{Prod}})$ is defined by:

$$(x_1, y_1) \leq_{\text{Prod}} (x_2, y_2) \iff x_1 \leq_X x_2 \text{ and } y_1 \leq_Y y_2.$$

An isomorphism in **PreOrd** is an order-isomorphism, i.e., a bijective map that preserves and reflects the preorder.

2.2. The Category of Preordered Groups. A *preordered group* is a pair $(A, \leq_A)$ consisting of a group A and a preorder $\leq_A$ compatible with multiplication:

$$a_1 \leq_A a'_1 \text{ and } a_2 \leq_A a'_2 \implies a_1 a_2 \leq_A a'_1 a'_2, \quad \forall a_1, a'_1, a_2, a'_2 \in A.$$

The *positive cone* of $(A, \leq_A)$ is the set of all positive elements:

$$A^+ := \{a \in A \mid a \geq_A 1\}.$$

The set A^+ forms a submonoid of A containing the identity 1. The preorder $\leq_A$ is completely determined by A^+ via $a_1 \leq_A a_2 \iff a_1 a_2^{-1} \in A^+$.

A *preordered subgroup* of $(A, \leq_A)$ is a subgroup $X \subseteq A$ equipped with the induced preorder from A , so that $X^+ = A^+ \cap X$. The inclusion $X \hookrightarrow A$ reflects and preserves the order.

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We denote by **PreOrdGrp** the category whose objects are preordered groups and whose morphisms are monotone group homomorphisms (or equivalently, group homomorphisms $f : A \rightarrow B$ such that $f(A^+) \subseteq B^+$). An isomorphism in **PreOrdGrp** is a group isomorphism that is also an isomorphism in **PreOrd**.

3. FACTORIZATIONS OF MONOIDS AND GROUPS

Recall (for example, from [1, 3]) that a **factorization** of a monoid A is a pair (X, B) of submonoids of A such that the set-theoretic multiplication map

$$\gamma = \gamma_{X,B} : X \times B \longrightarrow A, \quad (x, b) \longmapsto xb$$

is a bijection. One sometimes says that A **factorizes** through its submonoids X and B . When A is a group, any factorization (X, B) necessarily has both X and B as subgroups of A ; in this case one speaks of a **factorization of a group**.

4. FACTORIZATIONS OF PREORDERED GROUPS

A natural way to define factorizations of preordered groups $(A, \leq_A)$ is to start with the usual notion of a factorization (X, B) of groups and require the multiplication map to reflect the product preorder on $X \times B$.

Definition 1. A **factorization** of a preordered group $(A, \leq_A)$ is a pair of preordered subgroups $(X, \leq_X)$ and $(B, \leq_B)$ such that the set-theoretic multiplication map

$$\gamma_{X,B} : X \times B \longrightarrow A, \quad (x, b) \longmapsto xb$$

is an isomorphism in the category **PreOrd** from $(X \times B, \leq_{\text{Prod}})$ to $(A, \leq_A)$.

In explicit terms, (X, B) is a factorization of $(A, \leq_A)$ if and only if:

- (1) The map $\gamma_{X,B}$ is a bijection (i.e., (X, B) is a factorization of the ordinary group A).
- (2) The map $\gamma_{X,B}$ preserves and reflects the order:

$$xb \leq_A x'b' \iff x \leq_X x' \quad \text{and} \quad b \leq_B b'.$$

Given a factorization (X, B) of the underlying group A , transporting the preorder $\leq_A$ back through $\gamma_{X,B}$ does not automatically yield a factorization of preordered groups. The following theorem characterizes precisely when this holds in terms of the positive cone monoid A^+ .

Theorem 1. Let $(A, \leq_A)$ be a preordered group, and let $X, B \subseteq A$ be subgroups such that (X, B) is a factorization of the underlying group A . Then the multiplication map

$$\gamma_{X,B} : (X \times B, \leq_{\text{Prod}}) \rightarrow (A, \leq_A)$$

is an isomorphism in **PreOrd** if and only if the positive cone monoid A^+ factorizes through the submonoids X^+ and B^+ , i.e., if and only if $\gamma_{X,B}$ restricts and corestricts to a map

$$\gamma^+ : X^+ \times B^+ \longrightarrow A^+.$$

5. CLASSIFICATION VIA LEFT QUASI-PROJECTIONS

Recall from [1] that a **left quasi-projection** on a group A is a set-theoretic endomap $\ell : A \rightarrow A$ satisfying the following identities for all $a, a' \in A$:

- (1) $\ell(a) \cdot \ell(a') = \ell(\ell(a) \cdot a')$,
- (2) $\ell(a \cdot a') = \ell(a \cdot \ell(a'))$.

With every left quasi-projection $\ell : A \rightarrow A$, one associates two canonical subsets:

$$A^\ell := \{a \in A \mid \ell(a) = a\} \quad \text{and} \quad A_\ell := \{a \in A \mid \ell(a) = 1\}.$$

Both A^ℓ and A_ℓ are subgroups of A ([1]). When A is a preordered group, we denote by $\leq^\ell$ and $\leq_\ell$ the preorders on A^ℓ and A_ℓ induced by the preorder on A .

The main theorem of this note provides a complete classification of all preordered group factorizations via left quasi-projections.

Theorem 2. *Let $(A, \leq_A)$ be a preordered group. The assignment*

$$\ell \longmapsto ((A^\ell, \leq^\ell), (A_\ell, \leq_\ell))$$

yields a one-to-one correspondence between:

- (1) *Left quasi-projections $\ell : A \rightarrow A$ satisfying $1 \leq_A \ell(a) \leq_A a$ for all $a \in A^+$, and*
- (2) *The set of all factorizations of the preordered group $(A, \leq_A)$.*

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