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TRACE AND OLSEN'S SHARP INEQUALITIES

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Abstract. We give a necessary and sufficient condition on an absolutely continuous Borel measure μ governing the trace inequality:

$$\|I_\alpha f\|_{L^p_q(\mu)} \lesssim \|f\|_{L^{p,1}_q},$$

where $L^p_q(\mu)$ is the Morrey space with measure μ, and $L^{p,1}_q$ is the Lorentz-Morrey space.

Consequently, the following sharp Olsen's inequality in the diagonal case holds:

$$\|g(I_\alpha f)\|_{L^p_q} \lesssim \|g\|_{L^{n/\alpha}_q} \|f\|_{L^q_{p,1}}.$$

The problem was studied for ball Banach function spaces setting as well.

1. Introduction

It is well-known that (see [1]) if $1 < p < q < \infty$ and $0 < \alpha < \frac{n}{p}$, then the necessary and sufficient condition on a Borel measure ν governing the trace inequality

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^q d\nu(x)\right)^{1/q} \lesssim \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p}. \tag{1.1}$$

is that

$$\nu(B) \le C|B|^{(\alpha/n - 1/p)q}. \tag{1.2}$$

Here I_α is the fractional integral operator:

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n, \ x \in \mathbb{R}^n.$$

Although several characterizations of (1.1) are available (see [4, 17, 18]), the Frostman condition (1.2) alone is no longer sufficient. Its necessity is immediate by testing (1.1) on characteristic functions of balls, whereas the converse implication is false. Counterexamples were constructed in [3, 13, 16]; see also [21, Book II, Proposition 257].

A breakthrough in this direction was obtained independently in [12] and [15]. The key observation is that replacing the Lebesgue space on the right-hand side with the Lorentz space $L^{p,1}$ restores the exact analogue of Adams' theorem. More precisely, for the weighted estimate

$$\|I_\alpha f\|_{L^p(\mu)} \lesssim \|f\|_{L^{p,1}}, \tag{1.3}$$

where μ is absolutely comtinuous measure $d\mu(x) = v(x)dx$, Frostman's condition (1.2) is not only necessary but also sufficient.

Throughout the note, for a measurable function v and a measurable set E, we write

$$v(E) = \int_E v(x), dx.$$

With this notation, the endpoint characterization of (1.1) can be stated as follows.

Theorem A. *Let $1 < p < \infty$ and $0 < \alpha < n/p$. Suppose that v is a non-negative locally integrable function on $\mathbb{R}^n$. Then the following statements are equivalent:*

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(i) *There exists a positive constant C such that*

$$\|I_\alpha f\|_{L^p(v)} \le C\,\|f\|_{L^{p,1}}, \quad f \in L^{p,1}(\mathbb{R}^n).$$

(ii)

$$[v]_{p,\alpha} := \sup_{B\subset\mathbb{R}^n} \big(v(B)\big)^{\frac{1}{p}} |B|^{\frac{\alpha}{n}-\frac{1}{p}} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

When the measure is absolutely continuous, Theorem A naturally suggests a formulation in terms of Morrey spaces. Let $\mathscr{Q}$ denote the family of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

For $1 \le q < r < \infty$, and Borel measure μ on $\mathbb{R}^n$, the weighted Morrey norm is defined by

$$\|f\|_{L^q_r(\mu)} := \sup_{Q\in\mathscr{Q}} |Q|^{\frac{1}{r}-\frac{1}{q}} \left(\int_Q |f(x)|^q \, d\mu(x)\right)^{1/q}.$$

The exponent q describes local integrability, whereas r controls the global behavior.

If μ is absolutely continuous, i.e., $d\mu(x) = V(x)dx$, where V is a non-negative function on $\mathbb{R}^n$, then we demote $L^q_r(\mu)$ by $L^q_r(V)$. For constant V, the space $L^q_r(V)$ is the classical Morrey space denoted by L^q_r.

A sharp Olsen-type inequality in Morrey spaces of Spanne type was established in [7, Theorem 3.1]; see also [6] for the corresponding trace inequality.

Theorem B. *Let ℓ, p, q, r, s, α satisfy $1 < q \le r < \infty$, $1 < p < q < \infty$, $1 < p \le s < \infty$, and $0 < \alpha < \frac{n}{s}$, together with*

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{s} - \frac{1}{r} = \frac{\alpha}{n} - \frac{1}{\ell}.$$

Then there exists a constant $C > 0$ such that

$$\|g \cdot I_\alpha f\|_{L^q_r} \le C\,\|g\|_{L^q_\ell}\,\|f\|_{L^p_s}$$

for all $f \in L^p_s(\mathbb{R}^n)$ and $g \in L^q_\ell(\mathbb{R}^n)$.

Moreover, the exponent q in the norm $\|g\|_{L^q_\ell}$ is optimal and cannot be replaced by any smaller value of q.

Theorem B belongs to the family of inequalities introduced by Olsen [19]. Such estimates play an important role in the study of perturbed Schrödinger equations; see [23–29] for further developments to various directions and applications.

Our aim in this note is to give a complete characterization of a trace inequality and a sharp Olsen's inequality in the limiting (diagonal) case, in Morrey spaces. Moreover, we formulate the main statement in the setting of ball Banach function spaces (BBFS briefly).

2. Preliminaries

2.1. **Ball Banach function spaces.** To give the definition of *ball Banach function space* BBFS briefly) we use a notion proposed in [22] (see Section 2 for the precise definition).

Write

$$\mathbb{M}^+(\mathbb{R}^n) := \{f : \mathbb{R}^n \to [0,\infty] : f \text{ is Lebesgue measurable}\}.$$

Definition 1 (Ball Banach function norm)**.** Let

$$\rho : \mathbb{M}^+(\mathbb{R}^n) \to [0,\infty]$$

be a mapping. We say that ρ is a *ball Banach function norm* if, for all $f, g, \{f_j\}_{j=1}^\infty \subset \mathbb{M}^+(\mathbb{R}^n)$, all constants $a \ge 0$, and all balls $B \subset \mathbb{R}^n$, the following properties hold:

(P1) **Norm properties.** $\rho(f) = 0 \iff f = 0$ a.e., $\rho(af) = a\,\rho(f)$, and $\rho(f+g) \le \rho(f) + \rho(g)$.
(P2) **Lattice property.** If $0 \le g \le f$ a.e., then $\rho(g) \le \rho(f)$.
(P3) **Fatou property.** If $0 \le f_j \uparrow f$ a.e., then $\rho(f_j) \uparrow \rho(f)$.

(P4) **Finite norm of characteristic functions of balls.** For every ball $B \subset \mathbb{R}^n$, $\rho(\chi_B) < \infty$.

(P5) **Local integrability.** For every ball $B \subset \mathbb{R}^n$, $\|\chi_B f\|_{L^1(\mathbb{R}^n)} \lesssim_B \rho(f)$, where the implicit constant depends on B and ρ, but is independent of f.

The collection

$$X(\mathbb{R}^n) := \{f \in \mathbb{M}(\mathbb{R}^n) : \rho(|f|) < \infty\},$$

equipped with the norm

$$\|f\|_X := \rho(|f|),$$

is called the *BBFS* generated by ρ.

Definition 2 (Köthe dual)**.** Let $X = X(\mathbb{R}^n)$ be a BBFS. The *Köthe dual* of X is defined by

$$X' := \{g \in \mathbb{M}(\mathbb{R}^n) : \|g\|_{X'} < \infty\},$$

where

$$\|g\|_{X'} := \sup \left\{ \int_{\mathbb{R}^n} |f(x)g(x)|\, dx : f \in X,\ \|f\|_X \leq 1 \right\}.$$

The space X', equipped with the norm $\|\cdot\|_{X'}$, is called the *Köthe dual* of X.

Let f be a measurable function on $\mathbb{R}^n$ and let $1 \leq p < \infty$, $1 \leq s \leq \infty$.

We say that f belongs to the Lorentz space $L^{p,s}(\mathbb{R}^n)$ if

$$\|f\|_{L^{p,s}} = \begin{cases} \left(s \displaystyle\int_0^\infty \left(|\{x \in \mathbb{R}^n : |f(x)| > \tau\}|\right)^{\frac{s}{p}} \tau^{s-1}\, d\tau \right)^{\frac{1}{s}}, & 1 \leq s < \infty, \\ \displaystyle\sup_{\tau>0} \tau\, |\{x \in \mathbb{R}^n : |f(x)| > \tau\}|^{\frac{1}{p}}, & s = \infty, \end{cases} \tag{2.1}$$

and this quantity is finite.

If $p = s$, then $L^{p,s}(\mathbb{R}^n)$ coincides with the Lebesgue space $L^p(\mathbb{R}^n)$.

Based on [20], we define the Lorentz–Morrey space $L^{r,s}_p(\mathbb{R}^n)$ by the norm

$$\|f\|_{L^{r,s}_p} := \sup_{Q \in \mathcal{Q}} |Q|^{\frac{1}{r}-\frac{1}{p}} \|f\|_{L^{p,s}(Q)}, \qquad p \leq r.$$

See [8] for the diversity of this class of function spaces. Recently, a more general framework is established; see [10]. Let $1 < p \leq q < \infty$, $0 < \alpha < \frac{n}{q}$. If we define s and t by $\frac{1}{t} = \frac{1}{q} - \frac{\alpha}{n}$ and $\frac{s}{t} = \frac{p}{q}$, then I_α is a bounded linear operator from $L^p_q(\mathbb{R}^n)$ to $L^s_t(\mathbb{R}^n)$ due to a fundamental result by Adams [2].

3. Main Results

Theorem 3.1. *Let $0 < \alpha < n$ and*

$$1 < p < \frac{n}{\alpha}.$$

Let X and Y be BBFSs with the following properties:

- *The Hardy–Littlewood maximal operator M is bounded on X.*
- *There exists a constant $C > 0$ such that*

$$\left\| \left(M(|g|^{p'})\right)^{\frac{1}{p'}} \right\|_{X'} \leq C\, \|g\|_{Y'} \tag{3.1}$$

for every $g \in Y'$.

- *There exist $1 < r < \infty$ and a constant $C > 0$ such that*

$$\left\| \sum_{j=1}^\infty (Mf_j)^r \right\|_X \leq C \left\| \sum_{j=1}^\infty |f_j|^r \right\|_X \tag{3.2}$$

for all sequences $\{f_j\}_{j=1}^\infty$ of measurable functions.

Then there exists a constant $C > 0$ such that

$$\|g \cdot I_\alpha f\|_Y \le C\, \|g\|_{L^p_{n/\alpha}} \|f\|_X$$

for all $f \in X$ and $g \in L^p_{n/\alpha}(\mathbb{R}^n)$.

It is noteworthy that if $X = L^u_v$ with $1 \le u \le v < \infty$ and $p > u$, then this result extends the ones in [11, 23].

Adams-type trace inequalities in Lorentz–Morrey spaces were studied in [5] in the off-diagonal case.

For $s = p$, the space $L^{r,s}_p(\mathbb{R}^n)$ coincides with the classical Morrey space $L^r_p(\mathbb{R}^n)$.

Let V be a non-negative, locally integrable function on $\mathbb{R}^n$. Let $1 < p \le q < \infty$. The weighted Morrey space $L^r_p(V)$ cane be given by

$$\|f\|_{L^p_q(V)} := \|V^{\frac{1}{p}} f\|_{L^p_q}.$$

For Morrey spaces we have

Theorem 3.2 (Trace inequality characterization in Morrey spaces: Diagonal case)**.** *Let $1 < p \le q < \infty$ and $0 < \alpha < \frac{n}{q}$. Let V be a non-negative measurable function. Then the inequality*

$$\|I_\alpha f\|_{L^p_q(V)} \lesssim \|f\|_{L^{p,1}_q}$$

holds for any non-negative measurable function f with the implicit constant independent of f if and only if

$$[V]_{p,\alpha} := \|V^{\frac{1}{p}}\|_{L^p_{n/\alpha}} < \infty.$$

Moreover, $\|I_\alpha f\|_{L^{p,1}_q \to L^p_q(V)} \sim [V]_{p,\alpha}$.

In particular letting $f = \chi_E$ in the above, we have the following norm estimates.

Corollary. *Let $1 < p \le q < \infty$, $0 < \alpha < \frac{n}{q}$. Then there is a positive constant C depending only on α, p, q and n such that for all compact sets E,*

$$\|I_\alpha \chi_E\|_{L^p_q(V)} \lesssim [V]_{p,\alpha} |E|^{\frac{1}{q}}.$$

Here the implicit constant is independent of E.

Let $k \in \mathbb{N}$. The Sobolev–Lorentz space $W^k_{p,1}(\mathbb{R}^n)$ is the set of all $f \in L^{p,1}(\mathbb{R}^n)$ for which the weak partial derivatives up to order k belongs to $L^{p,1}(\mathbb{R}^n)$. Furthermore, the norm is given by

$$\|f\|_{W^k_{p,1}} = \sum_{|\alpha| \le k} \|\partial^\alpha f\|_{L^{p,1}}.$$

Let $k, d \in \mathbb{N}$. Denote by $W^k_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$ the higher order Sobolev–Lorentz space of all d-dimensional vector field F for which each component F_j, $j = 1, \dots, d$, belongs to $W^k_{p,1}(\mathbb{R}^n)$. Furthermore, the norm is given by

$$\|F\|_{W^k_{p,1}(\mathbb{R}^n,\mathbb{R}^d)} = \sum_{j=1}^d \|F_j\|_{W^k_{p,1}}.$$

Trace inequalities for fractional integrals have a wide range of applications. Suppose $0 < k - 1 < n$. Using the definition of the appropriate Sobolev–Lorentz space $W^k_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$ of Sobolev–Lorentz mappings, together with the classical estimate $|\nabla v| \le C\, I_{k-1}[|\nabla^k v|]$, Theorem 3.2 yields the following consequence:

Corollary. *Let $1 < p \le q < \infty$, and let $k \in \{1, \dots, n\}$ satisfy $k < n/q + 1$. Suppose that V is a non-negative locally integrable function on $\mathbb{R}^n$. Then, for every function $v \in W^k_{p,1}(\mathbb{R}^n, \mathbb{R}^n)$, the following estimate holds*:

$$\|\nabla v\|_{L^p_q(V)} \lesssim [V]_{p,k-1}\, \|\nabla^k v\|_{L^{p,1}_q}.$$

We next transfer our results to the setting of Morrey spaces.

Theorem 3.3 (Sharp Olsen's Inequality: Diagonal case)**.** *Let $1 < p \le q < \infty$ and $0 < \alpha < \frac{n}{q}$. Then the following inequality holds*:

$$\|g\, I_\alpha f\|_{L^p_q} \lesssim \|g\|_{L^p_{n/\alpha}} \|f\|_{L^{p,1}_q}.$$

Moreover, this inequality is sharp in the sense that p in the Morrey norm $\|g\|_{L^p_{n/\alpha}}$ cannot be replaced by any strictly smaller value.

By the next statement we see that the optimality of the second parameter Lorentz-Lebesgue (Lorentz-Morrey) space on the right-hand side of the trace (Olsen's) inequality. In particular, the following statement holds:

Theorem 3.4. *Let $1 < r \le r_0 < \frac{n}{\alpha}$ and $\theta > 1$. Then for any $c > 0$ there exist functions $f, g \in \mathbb{M}^+(\mathbb{R}^n)$ such that*

$$\|g \cdot I_\alpha f\|_{L^r_{r_0}} > c\, \|g\|_{L^r_{\frac{n}{\alpha}}} \|f\|_{L^{r,\theta}_{r_0}} > 0.$$

Tanaka showed that the analogue of Proposition 3.4 fails for $r = 1$; see [26, Theorem 1.2].

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