FAITHFUL R-COMPLETION OF EXPONENTIAL MR-GROUPS

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Abstract. This paper is devoted to partial exponential MR-groups that are isomorphically embeddable in their tensor R-completions. As a consequence, the free MR-groups and free MR-products are described in terms of usual group-theoretical free constructions.

1. PRELIMINARIES

The concept of an exponential R-group, where R is an arbitrary associative ring with identity, was introduced by Lyndon in [1]. In [2] Myasnikov and Remeslennikov refined the concept of an R-group by adding an extra axiom. Specifically, the new concept of an exponential R-group is a direct generalization of the concept of an R-module to noncommutative groups. In the honor of Myasnikov, R-groups with an extra axiom were called in [3] MR-groups (R is a ring). The role of the tensor extension of a ring of scalars for modules is well known. An exact analogue of this construction for an arbitrary MR-group–tensor completion–was defined in [2]. A systematic study of MR-groups was begun in [3–10]. Note that the results of these works turned out to be useful in solving well-known problems of Tarski.

Following [1,2], we recall the basic definitions and facts. Let $L_{\mathrm{gr}} = \{\bullet, ^{-1}, e\}$ be the group language (signature), where $\bullet$ is the binary operation of multiplication, $^{-1}$ is the unary operation of inversion, and e is a constant symbol for the identity element of a group. The group language L_{gr} is enriched to the language $L_{\mathrm{gr}}^R = L_{\mathrm{gr}} \cup \{f_\alpha(g) \mid \alpha \in R\}$, where $f_\alpha(g)$ is a unary algebraic operation (below, by convention, the expression $f_\alpha(g)$ is written for brevity as g^α).

Definition 1.1 [1]. *A Lyndon R-group is a set G with operations $\bullet, ^{-1}, \{f_\alpha(g) \mid \alpha \in R\}$ defined for which the following axioms hold:*

(i) the group axioms;

(ii) for all $g, h \in G$ and all $\alpha, \beta \in R$,

$$g^1 = g, \quad g^0 = e, \quad e^\alpha = e; \tag{1}$$

$$g^{\alpha+\beta} = g^\alpha g^\beta, \quad g^{\alpha\beta} = (g^\alpha)^\beta; \tag{2}$$

$$(h^{-1}gh)^\alpha = h^{-1}g^\alpha h. \tag{3}$$

There exist Abelian Lyndon R-groups that are not R-modules (see [11], where the structure of a free Abelian R-group was studied in detail). The authors of [2] augmented Lyndon's axioms by the extra one (quasi-identity)

$$(MR\text{-axiom}) \quad \forall g, h \in G, \alpha \in R \quad [g, h] = e \to (gh)^\alpha = g^\alpha h^\alpha \tag{4}$$

where $[g, h] = g^{-1}h^{-1}gh$.

Definition 1.2 [2]. *An MR-group is a group G on which the operation g^α is defined for all $g \in G$ and $\alpha \in R$ and axioms (1)–(4) hold.*

Let $\mathcal{L}_R$ and $\mathcal{M}_R$ denote the classes of all exponential Lyndon R-groups and all MR-groups, respectively. Obviously, $\mathcal{L}_R \supset \mathcal{M}_R$. The concepts of an MR-subgroup, MR-generatedness, MR-normal subgroup, etc., are introduced in a standard manner. Clearly, this class is a quasi-variety in the language L_{gr}^R and free MR-groups, R-homomorphisms, etc., are defined in it. Additionally, every Abelian MR-group is an R-module, and vice versa.

Definition 1.3 [2]. *A homomorphism of R-groups $\varphi\colon G \to G^*$ is called an R-homomorphism if $\varphi(g^\alpha) = \varphi(g)^\alpha$ for any $g \in G$ and $\alpha \in R$.*

Most natural examples of exponential R-groups belong to the class $\mathcal{M}_R$. For example, a free exponential Lyndon R-group is an MR-group, a unipotent group over a field K of zero characteristic is an MK-group, an arbitrary pro-p-group is an $M\mathbb{Z}_p$-group over the ring $\mathbb{Z}_p$ of p-adic integers, and so on (for other examples, see [2]).

It was shown in [2] that a key role in the study of exponential MR-groups is played by the operation of tensor completion. It naturally generalizes the extension of a ring of scalars for modules to the noncommutative case.

Definition 1.4 [2]. *Let G be an MR-group and $\mu\colon R \to S$ be a ring homomorphism. Then an MS-group $G^{S,\mu}$ is called the* tensor S-completion *of the MR-group G if $G^{S,\mu}$ satisfies the following universal properties:*

(1) *There exists an homomorphism $\lambda\colon G \to G^{S,\mu}$ such that $\lambda(G)$ S-generates $G^{S,\mu}$, i.e., $\langle\lambda(G)\rangle_S = G^{S,\mu}$.*

(2) *For any MS-group H and any homomorphism $\varphi\colon G \to H$ compatible with μ (i.e., $\varphi(g^\alpha) = \varphi(g)^{\mu(\alpha)}$) there exists an S-homomorphism $\psi\colon G^{S,\mu} \to H$ such that the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "\lambda"] \arrow[d, "\varphi"'] & G^{S,\mu} \arrow[dl, dashed, "\exists\psi"] \\ H & \end{tikzcd} \qquad (\varphi = \lambda\psi).$$

Note that, if G is an Abelian MR-group, then $G^{S,\mu}$ is Abelian as well, i.e., it is an S-module and $G^{S,\mu} \simeq G \bigotimes_R S$ is the tensor product of the R-module G and the ring S. It was proved in [2] that, for any group $G \in \mathcal{M}_R$ and any homomorphism $\mu\colon R \to S$, there always exists a tensor completion of $G^{S,\mu}$ and it is unique up to an R-homomorphism. A method based on combinatorial group theory was proposed for constructing a tensor completion in [9]. In what follows, the ring homomorphism $\mu\colon R \to S$ is fixed, so, instead of $G^{S,\mu}$, we write G^S. In applications, μ is most frequently an embedding of rings, but, even in this case, the R-homomorphism $\lambda\colon G \to G^S$ is not always an embedding. A sufficient condition for such an embedding was given in [2, Proposition 11]. Obviously, the class $\mathcal{M}_R(\mathcal{L}_R)$ of all exponential MR-groups (Lyndon R-groups) is a category in which morphisms are R-homomorphisms of groups. In the language of category theory, the above-described completion operation is a tensor completion functor.

In the category $\mathcal{M}_R$ or $\mathcal{L}_R$, it is convenient to perform many constructions step by step, gradually defining powers. This leads to the concept of a partial R-group.

Definition 1.5 [2]. *A group G is called a* partial MR-group *if the operation of raising to power is defined for some pairs (g, α), but not necessarily for all pairs;* moreover, if one part of an equality in axioms (1)–(4) is defined, then the other part is defined as well and they satisfy axioms (1)–(4) in the definition of an exponential MR-group.

The class of partial MR-groups is denoted by $\mathcal{P}_R$. For example, if R is a subring of a ring S, then any R-group is a partial S-group. A group homomorphism $\varphi\colon G \to H$ is called a *partial R-homomorphism* if $\varphi(g^\alpha) = \varphi(g)^\alpha$ for all pairs (g, α) for which the element g^α is defined. Straightforward verification shows that $\mathcal{P}_R$ is a supercategory of $\mathcal{M}_R$.

2. FAITHFUL R-COMPLETION

In the rest of this paper, we assume that R is a ring containing the ring of integers $\mathbb{Z}$ as a subring.

Definition 2.1. *A group $G \in \mathcal{P}_R$ is said to be* faithful *with respect to a ring R if the canonical mapping $\lambda\colon G \to G^R$ is an embedding. A group G is faithful if it is faithful with respect to any ring containing $\mathbb{Z}$.*

The following useful facts about exact groups hold:

Proposition 2.1. *Let* $G = \prod_{i \in I} G_i$ *be the direct product of groups* G_i*. Then the group* G *is exact under the ring* R *if and only if all* G_i *are exact groups under the ring* R*.*

Proposition 2.2. *A partial* R*-group* G *is exact under a ring* R *if and only if all of its subgroups that are finitely generated in the category of partial* R*-groups are exact under the ring* R*.*

Proposition 2.3. *Let*

$$e \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} G/N \longrightarrow e$$

be a short exact sequence of partial R*-groups. The group* G *is exact if and only if the canonical mapping* $\lambda_{N,G}\colon N \to G^R$ *is an embedding and* G/N *is an exact group.*

Below, we formulate theorems on the faithfulness of rather broad classes of groups. Additionally, examples of nonfaithful groups are given.

Example 2.1. Let G be a simple group containing a nonidentity element of finite order. Then G is not faithful over the field $\mathbb{Q}$ of rational numbers. Moreover, $G^{\mathbb{Q}} = e$. Indeed, since any $M\mathbb{Q}$-group does not contain elements of finite order, G is not a subgroup of $G^{\mathbb{Q}}$. However, in this case, the homomorphism $\lambda\colon G \to G^{\mathbb{Q}}$ has a nontrivial kernel and the simplicity of G implies $\lambda(G) = e$, while $\lambda(G)$ generates $G^{\mathbb{Q}}$. Therefore, $G^{\mathbb{Q}} = e$.

Example 2.2. Let G be a torsion-free group with a nonunique extraction of roots. Then $G^{\mathbb{Q}}$ is a group with a unique extraction of roots. This easily follows from Proposition 3 in [2]. Clearly, G cannot be isomorphically embedded in $G^{\mathbb{Q}}$ and, hence, is not faithful.

In the category $\mathcal{P}_R$, we consider a class of groups $\mathcal{P}_R^0$. By definition, a group G from $\mathcal{P}_R$ is said to belong to $\mathcal{P}_R^0$ if it satisfies the following conditions:

(1) For any maximal Abelian subgroup M of G and for any $x \notin M$, it is true that $M \cap M^x = e$.

(2) The canonical homomorphism $j\colon M \to M \bigotimes_R R$ is an embedding.

Let us indicate some properties of groups of class $\mathcal{P}_R^0$ and give examples.

Proposition 2.4. *Let* $G \in \mathcal{P}_R^0$*. Then the following assertions hold:*

(a) *If* M_1 *and* M_2 *are different maximal Abelian subgroups of* G*, then* $M_1 \cap M_2 = e$*.*

(b) *If G is a torsion-free group, then the operation of root extraction is uniquely defined in G.*

(c) *The commutation relation on nonidentity elements is an equivalence relation.*

(d) *The centralizer of any nonidentity element is a maximal Abelian normal subgroup.*

(e) *If G is a torsion-free group and $x^r = y^s$, then $[x, y] = e$.*

(f) *If x and y are elements of a maximal Abelian subgroup M and x and y are conjugate in G, then $x = y$.*

Proposition 2.5. *The free product $G = \underset{i \in I}{*} G_i$ of groups G_i, $i \in I$, of class $\mathcal{P}_R^0$ also belongs to the class $\mathcal{P}_R^0$.*

Below are examples of groups of class $\mathcal{P}_R^0$.

I. For any ring $R \supseteq \mathbb{Z}$, free groups are partial MR-groups. Clearly, these groups belong to the class $\mathcal{P}_R^0$.

II. Generalizing the preceding example, we consider a class of groups with the following properties:

(1) G is a torsion-free group;

(2) the centralizer of any nonidentity element is an infinite cyclic subgroup.

Then G belongs to $\mathcal{P}_R^0$. Indeed, the maximal Abelian subgroup M of G is infinite and cyclic. Let $x \notin M$. Then, if $M \cap M^x \neq e$ and $M = \langle y \rangle$, for suitable integers r and s, it is true that $x^{-1} y^s x = y^r$. Introducing the notation $x^{-1} y x = z$, we have $z^s = y^r$, which implies, by Proposition 1(e), that z commutes with y and, hence, $x^{-1} y x = y^p$. If $|p| > 1$, then the subgroup $\langle x, y \rangle$ contains the subgroup p-adic rationals. The last group is not a cyclic subgroup, which contradicts condition (2). If $p = 1$, then x commutes with y, a contradiction. Finally, if $x^{-1} y x = y^{-1}$, then $x^{-2} y x^2 = y$, which is again a contradiction to item (e) in Proposition 2.4.

Theorem 2.1[3]. *Suppose that $\mathbb{Z}$ is a subring of a ring R, G is a group of class $\mathcal{P}_R^0$, and there are no elements of order 2 in G and R^+ (the additive group of the ring). Then G is faithful, i.e., the canonical mapping $\lambda \colon G \to G^R$ is an embedding.*

Remark 2.1. The main theorem provides a sufficient condition for the faithfulness of a tensor R-completion. Note that condition (1) in the definition of the class $\mathcal{P}_R^0$ is a necessary condition. The class $\mathcal{P}_R^0$ contains free groups. It is closed under direct limits, free products, and extensions of special form.

Remark 2.2. Let us define a class of groups $\mathcal{P}_R^*$ that is larger than $\mathcal{P}_R^0$. We say that a group G belongs to $\mathcal{P}_R^*$ if any of its maximal subgroups M satisfies the following condition: M is either an R-module or satisfies conditions (1) and (2) in the definition of the class $\mathcal{P}_R^0$. Then the main theorem also holds for groups of class $\mathcal{P}_R^*$.

3.APPLICATIONS TO FREE CONSTRUCTIONS

In the category $\mathcal{M}_R$, the concept of a free MR-group $F_R(X)$ with a free MR-generating set X and the concept of free MR-product $\underset{R}{*} G_i$ of groups $G_i, i \in I$, are introduced in a standard manner. In these definitions, the homomorphisms have to be replaced by R-homomorphisms.

Theorem 3.1. *For any X and R, where R contains the ring of integers $\mathbb{Z}$, a free MR-group $F_R(X)$ exists and is unique up to an R-isomorphism, and $F_R(X) = F(X)^R$, where $F(X)$ is an absolutely free group with basis X.*

Theorem 3.2. *Let R be a ring containing the ring of integers $\mathbb{Z}$, and let G_i be a set of MR-groups, $i \in I$. Then*

(1) $\underset{R}{*} G_i \cong (*G_i)^R$;

(2) *the canonical mapping $\lambda\colon G_i \to (*G_i)^R$ is an embedding.*

Theorem 3.3. *The class $\mathcal{P}_R^0$ is closed under free MR-products.*

Acknowledgement

This work was supported by the Shota Rustaveli National Science Foundation of Georgia (SRNSFG) (Grant No. FR-25-7138).

REFERENCES

1. Lyndon, Roger C. Groups with parametric exponents. Trans. Amer. Math. Soc. 96 (1960), 518–533.

2. Myasnikov, A. G.; Remeslennikov, V. N. Degree groups. I. Foundations of the theory and tensor completions. Siberian Math. J. 35 (1994), no. 5, 986–996.

3. Amaglobeli, M. G.; Remeslennikov, V. N. Free nilpotent R-groups of class 2. Dokl. Math. 85 (2012), no. 2, 236–239.

4. Myasnikov, A. G.; Remeslennikov, V. N. Exponential groups. II. Extensions of centralizers and tensor completion of CSA-groups. Internat. J. Algebra Comput. 6 (1996), no. 6, 687–711.

5. Baumslag, Gilbert; Myasnikov, Alexei; Remeslennikov, Vladimir. Discriminating completions of hyperbolic groups. Dedicated to John Stallings on the occasion of his 65th birthday. Geom. Dedicata 92 (2002), 115–143.

6. Amaglobeli, M. G.; Remeslennikov, V. N. Extension of a centralizer in nilpotent groups. (Russian) Sibirsk. Mat. Zh. 54 (2013), no. 1, 8–19.

7. Amaglobeli, Mikheil; Remeslennikov, Vladimir. Algorithmic problems for class-2 nilpotents MR-groups. Georgian Math. J. 22 (2015), no. 4, 441–449.

8. Amaglobeli, M. G.; Remeslennikov, V. N. Foundations of the theory of varieties of nilpotent MR-groups. Sib. Math. J. 57 (2016), no. 6, 935–942.

9. Amaglobeli, M. G. The tensor completion functor in categories of exponential MR-groups. Algebra Logic 57 (2018), no. 2, 89–97.

10. Amaglobeli, M. G.; Myasnikov, A. G.; Nadiradze, T. T. Varieties of exponential R-groups. Algebra Logic 62 (2023), no. 2, 119–136.

11. Baumslag, Gilbert. Free abelian X-groups. Illinois J. Math. 30 (1986), no. 2, 235–245.

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