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SOLUTION OF A MIXED PROBLEM FOR A VARIABLE-COEFFICIENT PARABOLIC PARTIAL DIFFERENTIAL EQUATION DESCRIBING POPULATION DYNAMICS

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Abstract. This paper presents a new mathematical partial differential equation model designed, for example, to describe the population dynamics of Kartvelian-speaking populations. The variable-coefficient parabolic partial differential equation incorporates diffusion and demographic terms, as well as an additional term representing the increasing degree of population protection associated with increasing elevation above sea level. By means of a transformation that homogenizes the boundary conditions of the mixed problem, the original problem is reduced to a boundary-value problem for a second-order ordinary differential equation with variable coefficients, whose solution can be expressed in terms of Airy functions of the first and second kinds. The Fourier method is employed to solve the linear homogeneous parabolic partial differential equation. The spatially dependent factor is determined from a Sturm–Liouville boundary-value problem, which is solved analytically in a particular case.

Introduction

Previously, we developed new approaches to the mathematical and computational modeling of social processes, thereby contributing to the development of modern approaches to the analysis of complex systems in the social sciences. Mathematical models not only describe demographic, economic, political, and social transformations but also provide new opportunities for prediction and practical application [1–9].

We also proposed and developed continuous mathematical models describing the transformation of the Proto-Kartvelian population into Georgian-, Laz-, Mingrelian-, and Svan-speaking populations. These processes were represented by systems consisting of two, three, and four nonlinear ordinary differential equations, respectively [10–12].

Fisher studied a constant-coefficient parabolic partial differential equation that constitutes one of the simplest diffusion mathematical models of the logistic growth of a given population [13]. The heuristic and fundamental genetic conclusions associated with this equation were also obtained independently by Kolmogorov, Petrovsky, and Piskunov, whose work laid the foundation for a more rigorous analytical investigation of the Fisher equation [14].

Of particular historical interest is the question of how Kartvelian-speaking populations managed, over the course of millennia, to withstand the threats of complete extinction or complete assimilation. Proto-Kartvelian civilization, dating approximately to the 6th-3rd millennia BCE, was predominantly located in lowland and plain regions. Subsequently, however, the numerous surrounding populations - including Indo-European- and Semitic-speaking peoples - and later imperial powers increasingly constrained their territorial and demographic space. From the perspective of population survival, one possible response was to resist assimilation by establishing settlements in mountainous regions and by taking advantage of favorable natural geographical and ecological conditions.

This consideration motivated us to develop a partial differential equation model capable of describing the population dynamics of Kartvelian-speaking populations and of representing the possibility of reducing or avoiding complete or partial assimilation through settlement in regions situated at higher elevations above sea level.

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1. Formulation of the Mathematical Model. The Mixed Problem

The partial differential equation should take into account changes in population size resulting from several factors, including population diffusion, demographic effects, and an additional term representing the increasing degree of population protection associated with increasing elevation above sea level. The latter effect can be incorporated by introducing a variable coefficient of the unknown function, for example, in the form of a linear function.

Let us consider the diffusion process of a given population (for example, the Kartvelian-speaking population, the Basques, the Swiss, the Albanians, and others), which is described by the following linear mathematical model, formulated as a first boundary-value problem for a second-order parabolic partial differential equation:

$$u_t = Du_{xx} + ru - \mu(1-x)u, \tag{1.1}$$

$$\left\{\begin{array}{l} u(x,0) = \varphi(x), \quad x \in [0,1] \\ u(0,t) = \delta_1 = \text{const}, \quad t \in [0,1] \\ u(1,t) = \delta_2 = \text{const}, \quad t \in [0,1] \end{array}\right., \tag{1.2}$$

where $x,\ t \in [0,1]$, $D, \mu = const > 0$, $r = const$, $u \in C_x^2(\Omega)$, $\Omega \equiv \{(x,t)|\ 0 < x < 1;\ 0 < t < 1\}$, $\varphi(x) \in C^2[0,1]$.

For the regularity of the solution, the following compatibility conditions must be satisfied:

$$\left\{\begin{array}{l} \varphi(0) = \delta_1 \\ \varphi(1) = \delta_2 \end{array}\right. . \tag{1.3}$$

The mixed problem (1.1)–(1.2) is written in dimensionless form, where

$$x = \frac{x_{real}}{H}, \qquad t = \frac{t_{real}}{T}, \qquad u = \frac{u_{real}}{u_{initial}}, \qquad u_{initial} = \int_0^H \varphi(y)dy, \tag{1.4}$$

H is the maximum altitude above sea level at which the population lives, and T is the time interval under consideration.

2. Analytical solution of the problem

We introduce a transformation that annihilates the boundary conditions:

$$u(x,t) = v(x,t) + x\delta_2 + (1-x)\delta_1. \tag{2.1}$$

Then, according to (2.1), the problem (1.1)–(1.2) is rewritten as

$$v_t = Dv_{xx} + (r - \mu(1-x))\,v + [x\delta_2 + (1-x)\delta_1][r - \mu(1-x)], \tag{2.2}$$

$$\left\{\begin{array}{l} v(x,0) = \varphi(x) + x(\delta_1 - \delta_2) - \delta_1, \quad x \in [0,1] \\ v(0,t) = 0, \quad t \in [0,1] \\ v(1,t) = 0, \quad t \in [0,1] \end{array}\right. . \tag{2.3}$$

In addition, the compatibility conditions (1.3) remain in force.

We rewrite (2.2) in the form

$$\begin{aligned} v_t = Dv_{xx} + (r - \mu + \mu x)v + \delta_1(r-\mu) + \\ + [(\delta_2 - \delta_1)(r-\mu) + \mu\delta_1]x + \mu(\delta_2 - \delta_1)x^2. \end{aligned} \tag{2.4}$$

Thus, introducing the notation

$$\begin{aligned} &\alpha(x) \equiv r - \mu + \mu x, \\ &\beta(x) \equiv \delta_1(r-\mu) + [(\delta_2 - \delta_1)(r-\mu) + \mu\delta_1]x + \mu(\delta_2 - \delta_1)x^2, \end{aligned} \tag{2.5}$$

we obtain the initial-boundary-value problem

$$\begin{aligned} &v_t = Dv_{xx} + \alpha(x)v + \beta(x), \\ &\left\{\begin{array}{l} v(x,0) = \varphi(x) + x(\delta_1 - \delta_2) - \delta_1, \quad x \in [0,1], \\ v(0,t) = 0, \quad t \in [0,1], \\ v(1,t) = 0, \quad t \in [0,1]. \end{array}\right. \end{aligned} \tag{2.6}$$

By means of the next transformation we pass to a new unknown function

$$v(x,t) = w(x,t) + h(x), \tag{2.7}$$

where the function $h(x)$ is the solution of the following boundary-value problem for an ordinary differential equation:

$$\begin{aligned} &Dh''(x) + \alpha(x)h + \beta(x) = 0, \\ &h(0) = h(1) = 0. \end{aligned} \tag{2.8}$$

We introduce the notation

$$\begin{aligned} &a = r - \mu, \qquad \mu = b, \qquad a_0 = \delta_1(r-\mu), \\ &a_1 = (\delta_2 - \delta_1)(r-\mu) + \mu\delta_1, \qquad a_2 = \mu(\delta_2 - \delta_1). \end{aligned} \tag{2.9}$$

Then (2.8) is rewritten as

$$\begin{aligned} &Dh''(x) + (a+bx)h + a_0 + a_1 x + a_2 x^2 = 0, \\ &h(0) = h(1) = 0. \end{aligned} \tag{2.10}$$

Consider the homogeneous equation corresponding to (2.10):

$$h''(x) + \frac{(a+bx)}{D}h = 0. \tag{2.11}$$

In (2.11) we perform the change of the independent variable

$$x \to \xi : \xi = \left(\frac{b}{D}\right)^{\frac{1}{3}} \left(x + \frac{a}{b}\right). \tag{2.12}$$

This yields the equation

$$\frac{d^2 h(\xi)}{d\xi^2} + \xi h(\xi) = 0. \tag{2.13}$$

The general solution of (2.13) has the form

$$h(\xi) = C_1 Ai(\xi) + C_2 Bi(\xi), \tag{2.14}$$

where

$$Ai(\xi) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + \xi t\right) dt$$

and

$$Bi(\xi) = \frac{1}{\pi}\int_0^\infty \left[\exp\left(-\frac{t^3}{3} + \xi t\right) + \sin\left(\frac{t^3}{3} + \xi t\right)\right] dt$$

are the Airy functions of the first and second kinds, respectively.

We rewrite the boundary-value problem (2.10) as

$$Dh''(x) + (a+bx)h = f(x), \qquad h(0) = h(1) = 0, \tag{2.15}$$

where $f(x) \equiv -(a_0 + a_1 x + a_2 x^2)$.

A particular solution of equation (2.15) is written in the form

$$h_{sp}(x) = -h_1(x)\int \frac{h_2(x)f(x)}{DW(x)}dx + h_2(x)\int \frac{h_1(x)f(x)}{DW(x)}dx, \tag{2.16}$$

where $h_1(x) = Ai(\xi(x))$, $h_2(x) = Bi(\xi(x))$, and

$$W(x) = \begin{vmatrix} Ai & Bi \\ Ai' & Bi' \end{vmatrix} = \frac{1}{\pi}$$

(the Wronskian of the linearly independent solutions). Thus

$$h(x) = C_1 Ai(\xi(x)) + C_2 Bi(\xi(x)) + h_{sp}(x), \tag{2.17}$$

In (2.17) the constants C_1, C_2 are determined from the boundary conditions:

$$h(0) = h(1) = 0,$$

$$\begin{cases} C_1 Ai(\xi(0)) + C_2 Bi(\xi(0)) + h_{sp}(0) = 0 \\ C_1 Ai(\xi(1)) + C_2 Bi(\xi(1)) + h_{sp}(1) = 0 \end{cases}, \tag{2.18}$$

$$C_1 = \frac{\begin{vmatrix} -h_{sp}(0) & Bi(\xi(0)) \\ -h_{sp}(1) & Bi(\xi(1)) \end{vmatrix}}{\begin{vmatrix} Ai(\xi(0)) & Bi(\xi(0)) \\ Ai(\xi(1)) & Bi(\xi(1)) \end{vmatrix}},$$

$$C_2 = \frac{\begin{vmatrix} Ai(\xi(0)) & -h_{sp}(0) \\ Ai(\xi(1)) & -h_{sp}(1) \end{vmatrix}}{\begin{vmatrix} Ai(\xi(0)) & Bi(\xi(0)) \\ Ai(\xi(1)) & Bi(\xi(1)) \end{vmatrix}}, \tag{2.19}$$

$$\begin{vmatrix} Ai(\xi(0)) & Bi(\xi(0)) \\ Ai(\xi(1)) & Bi(\xi(1)) \end{vmatrix} \neq 0.$$

For $w(x,t)$, we obtain a mixed problem for a homogeneous parabolic-type equation with homogeneous boundary conditions and the corresponding initial condition. The problem is solved by means of the Fourier method. For the factor depending on the spatial variable, we obtain a Sturm–Liouville boundary-value problem in which the coefficient of the unknown function is a linear function. According to (2.7), (2.16), (2.17), and (2.19), we obtain the function, while from (2.1) we obtain the desired function.

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