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# ON THE ASYMPTOTIC SOLUTION ON THE EXPLOSION PROBLEM OF A TRIAXIAL ROTATING HOMOGENEOUS SELF-GRAVITATING GASEOUS ELLIPSOID OF JACOBI

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**Abstract.** The paper considers a non-self-similar problem of a central explosion in a homogeneous triaxial gaseous ellipsoid that exists within its own gravitational field and rotates about the $z$-axis with a constant $\omega$ angular velocity as a rigid body. The initial conditions are based on the exact solution of the stationary rigid-body rotation problem for a homogeneous triaxial Jacobi gaseous ellipsoid. It is assumed that, at the initial moment of time, a central explosion occurs, releasing a finite amount of energy. As a result, a propagating detonation shock wave (a first-kind discontinuity surface of the unknown functions) is generated at the center of the gaseous ellipsoid.

To solve the problem, the asymptotic thin shock-layer method previously proposed by one of the authors is employed. Analytical solutions in quadratures are obtained for the zeroth- and first-order approximations of the singular asymptotic expansions describing the law of motion and velocity behind the detonation shock wave, as well as the thermodynamic characteristics of the self-gravitating gas.

## Introduction

To solve many problems of astrophysics, it is necessary to study the dynamics of the interaction of gas bodies with the gravitational field. The concept of studying celestial phenomena should be based on posing and solving a number of dynamic problems about the motion of gravitational gas, which are considered to be important mathematical models of stellar motion and evolution [1]. Then we solved several one-dimensional and two-dimensional non-self-similar problems about explosive processes in gas bodies and the spread of a detonation or shock wave to the surface of the body, followed by a separation into the vacuum [2–8]. In these mathematical models, we considered mainly homogeneous gas bodies.

## 1. Statement of the mixed problem for the system of equations in partial derivatives

When considering explosive phenomena in gravitational bodies, it is convenient to use the equations of three-dimensional motion in Lagrangian coordinates, which are

$$\frac{\partial^2 r}{\partial t^2} - r\left[\left(\frac{\partial\theta}{\partial t}\right)^2 + \sin^2\theta\left(\frac{\partial\varphi}{\partial t}\right)^2\right] = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\partial\Phi}{\partial r}, \tag{1.1}$$

$$\frac{\partial}{\partial t}\left[r^2\frac{\partial\theta}{\partial t}\right] - \frac{r^2\sin 2\theta}{2}\left(\frac{\partial\varphi}{\partial t}\right)^2 = -\frac{1}{\rho}\frac{\partial p}{\partial\theta} + \frac{\partial\Phi}{\partial\theta}, \tag{1.2}$$

$$\frac{\partial}{\partial t}\left[r^2\sin^2\theta\frac{\partial\varphi}{\partial t}\right] = \sin\theta\left[-\frac{1}{\rho}\frac{\partial p}{\partial\varphi} + \frac{\partial\Phi}{\partial\varphi}\right], \tag{1.3}$$

$$\rho r^2\sin\theta\det\left|\frac{\partial x^i}{\partial\xi^j}\right| = \frac{\sin\widehat{\theta}}{4\pi}, \tag{1.4}$$

$$p(m,\widehat{\theta},\widehat{\varphi},t) = (\gamma_2 - 1)f(m,\widehat{\theta},\widehat{\varphi})\rho^{\gamma_2}, \tag{1.5}$$

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$r = r(m, \widehat{\theta}, \widehat{\varphi}, t)$, $\theta = \theta(m, \widehat{\theta}, \widehat{\varphi}, t)$, $\varphi = \varphi(m, \widehat{\theta}, \widehat{\varphi}, t)$, $p(m, \widehat{\theta}, \widehat{\varphi}, t)$, $\rho(m, \widehat{\theta}, \widehat{\varphi}, \tau)$ - are unknown functions, $m = \frac{4\pi}{3}\rho_1 \widehat{r}^3$, $\rho_1 = \text{const}$, $m, \widehat{\theta}, \widehat{\varphi}$ - are Lagrangian coordinates, $f(m, \widehat{\theta}, \widehat{\varphi})$ - is the function related to the distribution of entropy by Lagrangian coordinates, $\Phi$ - is the potential of the gravitational field.

$$\Phi(r,\theta,\varphi) = \Phi_0 - Pr^2 \sin^2\theta \cos^2\varphi - Qr^2 \sin^2\theta \sin^2\varphi - Rr^2 \cos^2\theta, \tag{1.6}$$

$$Q = \pi k \rho_1 abc \int_0^\infty \frac{1}{b^2+s}\frac{ds}{\Gamma}, \qquad R = \pi k \rho_1 abc \int_0^\infty \frac{1}{c^2+s}\frac{ds}{\Gamma}, \qquad \Phi_0 = \pi k \rho_1 abc \int_0^\infty \frac{ds}{\Gamma},$$

$$P = \pi k \rho_1 abc \int_0^\infty \frac{1}{a^2+s}\frac{ds}{\Gamma}, \qquad \Gamma(s) = \sqrt{(c^2+s)(b^2+s)(a^2+s)}, \qquad a \neq b \neq c, \quad a > b, \quad a > c,$$

$$R > P, \qquad Q > P.$$

## 2. Problem statement

Let a homogeneous gas tree-axis ellipsoid in its own gravitational field rotate solid about the axis $z$ with $\omega$ angular velocity. Suppose that at a moment in time an explosion occurs with the release of $E$ energy. In this case, a divergent detonation shock wave forms in the center. It is necessary to determine the law and the velocity of the medium behind the detonation shock wave, as well as the thermodynamic characteristics of the medium.

The conditions on the discontinuity surface of the first kind (detonation wave), permitted with respect to the parameters of the gas behind the detonation shock wave, noted with index 2, have the form:

$$\rho_2 = \frac{\gamma_2+1}{\gamma_2-1}\rho_1 \Lambda^{-1}, \qquad p_2 = \frac{p_1\left[1+(\gamma_2-1)(1-B^2)\right] + \rho_1 V^2(1+g)}{(\gamma_2+1)B^2}, \tag{2.1}$$

$$\Lambda \equiv \left\{1 + \frac{B^2}{\gamma_2-1}\left(\frac{\gamma_2}{\gamma_1}\frac{a_1^2}{V^2} + 1 - g\right)\right\},$$

$$B \equiv \sqrt{1 + \frac{\left(\frac{\partial R}{\partial \theta}\right)^2}{R^2} + \frac{\left(\frac{\partial R}{\partial \varphi}\right)^2}{R^2\sin^2\theta}}, \qquad V^2 = \left[\frac{\partial R}{\partial t} - \left(\frac{\partial r}{\partial t} - \frac{\partial R}{\partial \theta}\frac{\partial \theta}{\partial t} - \frac{\partial R}{\partial \varphi}\frac{\partial \varphi}{\partial t}\right)_1\right]^2,$$

$$g = \left[\left(1 - \frac{\gamma_2}{\gamma_1}\frac{a_1^2}{V^2}\right)^2 + \frac{2(\gamma_2+1)(\gamma_1-\gamma_2)a_1^2}{\gamma_1(\gamma_1-1)V^2} - \frac{2(\gamma_2^2-1)Q}{V^2}\right]^{1/2},$$

$$\left(\frac{\partial r}{\partial t} - \frac{\partial R}{\partial \theta}\frac{\partial \theta}{\partial t} - \frac{\partial R}{\partial \varphi}\frac{\partial \varphi}{\partial t}\right)_2 - \frac{\partial R}{\partial t} = \left[\left(\frac{\partial r}{\partial t} - \frac{\partial R}{\partial \theta}\frac{\partial \theta}{\partial t} - \frac{\partial R}{\partial \varphi}\frac{\partial \varphi}{\partial t}\right)_1 - \frac{\partial R}{\partial t}\right]\frac{\gamma_2-1}{\gamma_2+1}\Lambda,$$

$$R^2\left[1 + \frac{1}{R^2\sin^2\theta}\left(\frac{\partial R}{\partial \varphi}\right)^2\right]\left(\frac{\partial \theta}{\partial t}\right)_2 + \frac{\partial R}{\partial \theta}\left[\left(\frac{\partial r}{\partial t}\right)_2 - \frac{\partial R}{\partial \varphi}\left(\frac{\partial \varphi}{\partial t}\right)_2\right]$$

$$= R^2\left[1 + \frac{1}{R^2\sin^2\theta}\left(\frac{\partial R}{\partial \varphi}\right)^2\right]\left(\frac{\partial \theta}{\partial t}\right)_1 + \frac{\partial R}{\partial \theta}\left[\left(\frac{\partial r}{\partial t}\right)_1 - \frac{\partial R}{\partial \varphi}\left(\frac{\partial \varphi}{\partial t}\right)_1\right],$$

$$\sin^2\theta\frac{\partial R}{\partial \theta}\left[R^2\left(\frac{\partial \theta}{\partial t}\right)_2 + \frac{\partial R}{\partial \theta}\left(\frac{\partial r}{\partial t}\right)_2\right] + \frac{\partial R}{\partial \varphi}\left[R^2\sin^2\theta\left(\frac{\partial \varphi}{\partial t}\right)_2 + \frac{\partial R}{\partial \varphi}\left(\frac{\partial r}{\partial t}\right)_2\right]$$

$$= \sin^2\theta\frac{\partial R}{\partial \theta}\left[R^2\left(\frac{\partial \theta}{\partial t}\right)_1 + \frac{\partial R}{\partial \theta}\left(\frac{\partial r}{\partial t}\right)_1\right] + \frac{\partial R}{\partial \varphi}\left[R^2\sin^2\theta\left(\frac{\partial \varphi}{\partial t}\right)_1 + \frac{\partial R}{\partial \varphi}\left(\frac{\partial r}{\partial t}\right)_1\right],$$

$$R^2\sin^2\theta\left[1+\frac{1}{R^2}\left(\frac{\partial R}{\partial\theta}\right)^2\right]\left(\frac{\partial\varphi}{\partial t}\right)_2+\frac{\partial R}{\partial\varphi}\left[\left(\frac{\partial r}{\partial t}\right)_2-\frac{\partial R}{\partial\theta}\left(\frac{\partial\theta}{\partial t}\right)_2\right]$$

$$=R^2\sin^2\theta\left[1+\frac{1}{R^2}\left(\frac{\partial R}{\partial\theta}\right)^2\right]\left(\frac{\partial\varphi}{\partial t}\right)_1+\frac{\partial R}{\partial\varphi}\left[\left(\frac{\partial r}{\partial t}\right)_1-\frac{\partial R}{\partial\theta}\left(\frac{\partial\theta}{\partial t}\right)_1\right],$$

$$[r]_1^2=0,\quad [\theta]_1^2=0,\quad [\varphi]_1^2=0,\quad [m]_1^2=0,\quad [\widehat{\theta}]_1^2=0,\quad [\widehat{\varphi}]_1^2=0.$$

## 3. Exact solution of a problem on rotation of a gravitating homogeneous gas ellipsoid of Jacobi (pre-detonation shock wave solution)

As initial data, consider the exact solution to the problem of stationary solid rotation of a homogeneous three-axis Jacobi gas ellipsoid.

In spherical coordinates, the exact solution gives the distribution of velocity, pressure and gravitational potential (1.6) within a three-axis homogeneous ellipsoid of Jacobi [9]

$$u_r=u_\theta=0,\qquad u_\varphi=r\sin\theta\frac{d\varphi}{dt},\qquad \frac{d\varphi}{dt}=\omega, \tag{3.1}$$

$$p(r,\theta,\varphi)=\frac{\rho_1 a^2}{2}\left\{\omega^2\sin^2\theta-2\left[P\sin^2\theta\cos^2\varphi+Q\sin^2\theta\sin^2\varphi+R\cos^2\theta\right]\right\}S, \tag{3.2}$$

$$S\equiv\left(\frac{r}{a}\right)^2-\frac{1}{\sin^2\theta\left[1+I_1^2\sin^2\varphi\right]+(I_2^2+1)\cos^2\theta}, \tag{3.3}$$

$$I_1^2=\frac{a^2-b^2}{b^2},\qquad I_2^2=\frac{a^2-c^2}{c^2},\qquad (-\omega^2+2P)I_1^2=2(Q-P), \tag{3.4}$$

$$(1+I_2^2)(Q-P)=I_1^2R. \tag{3.5}$$

Thus, for the stability of the rotating **Jacobi ellipsoid**, it is necessary that conditions (3.4) and (3.5) be satisfied. This is possible when $a>b$, $a>c$, that is, $Q>P$, $R>P$, and the **angular velocity of rotation** satisfies the following inequality:

$$\omega^2=2P-\frac{2(Q-P)}{I_1^2}<2P. \tag{3.6}$$

## 4. Asymptotic solution behind the detonation shock wave

Behind the detonation shock wave, we seek the solution in the form of the following singular asymptotic expansion:

$$\begin{gathered}
r(m,\widehat{\theta},\widehat{\varphi},\tau)=R_0(\tau)+\varepsilon H(m,\widehat{\theta},\widehat{\varphi},\tau)+\ldots,\\
R(\theta,\varphi,\tau)=R_0(\tau)+\varepsilon R_1(\theta,\varphi,\tau)+\ldots,\\
M(\theta,\varphi,\tau)=M_0(\tau)+\varepsilon M_1(\theta,\varphi,\tau)+\ldots,\\
\theta(m,\widehat{\theta},\widehat{\varphi},\tau)=\widehat{\theta}+\varepsilon\theta_1(m,\widehat{\theta},\widehat{\varphi},\tau)+\ldots,\\
\varphi(m,\widehat{\theta},\widehat{\varphi},\tau)=\widehat{\varphi}+\sqrt{\varepsilon}\,\varphi_1(m,\widehat{\theta},\widehat{\varphi},\tau)+\ldots,\\
p(m,\widehat{\theta},\widehat{\varphi},\tau)=\frac{p_0(m,\tau)}{\varepsilon}+p_1(m,\widehat{\theta},\widehat{\varphi},\tau)+\ldots,\\
\rho(m,\widehat{\theta},\widehat{\varphi},\tau)=\frac{\rho_0(m,\widehat{\theta},\widehat{\varphi},\tau)}{\varepsilon}+\rho_1(m,\widehat{\theta},\widehat{\varphi},\tau)+\ldots,\\
\tau=\frac{t}{\sqrt{\varepsilon}},\qquad E=\frac{E_0}{\varepsilon^2},\qquad E_0=\underline{\underline{O(1)}},\qquad \varepsilon=\frac{\gamma_2-1}{\gamma_2+1},\qquad Q=\frac{Q_0}{\varepsilon},\qquad Q_0=\underline{O}(1).
\end{gathered} \tag{4.1}$$

Substituting the asymptotic decomposition (4.1) into the system of equations (1.1)- (1.5) and boundary conditions (2.1), given also the integral energy equation, for zero (main approximation) approximations of the unknown functions, we obtain

$$
\begin{gathered}
p_0(m,\tau) = R_0'^2(\tau) + \frac{R_0''(\tau)(M_0(\tau)-m)}{4\pi R_0^2(\tau)}, \\
M_0(\tau) = \frac{4}{3}\pi R_0^3(\tau), \qquad R_0(\tau) = \left(\frac{75}{4\pi}E_0\right)^{1/5}\tau^{2/5}, \\
\rho_0(m,\widehat{\theta},\widehat{\varphi},\tau) = p_0^{1/\gamma_2}(m,\tau)\left[R_0'^2(T_0(m))\right]^{-1/\gamma_2}\left[1+\frac{a_1^2(m,\widehat{\theta},\widehat{\varphi})}{\gamma_2 R_0'^2(T_0(m))}\right]^{-1}, \\
T_0(m) = \left(\frac{3}{4\pi}\right)^{5/6}\left(\frac{4\pi}{75E_0}\right)^{1/2} m^{5/6}, \\
a_1^2(m,\widehat{\theta},\widehat{\varphi}) = \frac{\gamma_1 p}{\rho_1} = \frac{\gamma_1 A(m,\widehat{\theta},\widehat{\varphi})}{2}\left[\omega^2\sin^2\widehat{\theta} - 2\left(P\sin^2\widehat{\theta}\cos^2\widehat{\varphi} + Q\sin^2\widehat{\theta}\sin^2\widehat{\varphi} + R\cos^2\widehat{\theta}\right)\right], \\
A(m,\widehat{\theta},\widehat{\varphi}) = \left(\frac{3m}{4\pi}\right)^{2/3} - \frac{a^2}{\sin^2\widehat{\theta}\left(1+I_1^2\sin^2\widehat{\varphi}\right)+(I_2^2+1)\cos^2\widehat{\theta}}.
\end{gathered}
\tag{4.2}
$$

From the continuity equation (1.4) in the zeroth-order approximation, we obtain the first-order approximation of the radial component of the equation of motion

$$
\frac{\partial H(m,\widehat{\theta},\widehat{\varphi},\tau)}{\partial m} = \frac{1}{4\pi R_0^2\rho_0(m,\widehat{\theta},\widehat{\varphi},\widehat{\tau})}.
$$

The remaining four first-order terms of the asymptotic expansion $\theta_1$, $\varphi_1$ $p_1$ $\rho_1$ are obtained from the three equations of motion and the adiabatic equation, thus, the zeroth-order and first-order approximations of the singular asymptotic expansion (4.1) are completely determined.

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