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ON THE EXISTENCE OF BOUNDED SOLUTIONS FOR SYSTEMS OF LINEAR DIFFERENCE EQUATIONS

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Abstract. Effective sufficient conditions for the existence of bounded solutions satisfying the Nicoletti conditions for systems of linear difference equations are established. A constructive method for obtaining such solutions is presented. Sufficient conditions guaranteeing the existence, uniqueness and positivity of bounded solutions are derived.

1. Statement of the problem. Basic notation and definitions

Consider the system of linear difference equations

$$y(k)-y(k-1)=G_1(k)y(k)+G_2(k-1)y(k-1)+\varphi_1(k)+\varphi_2(k-1), \qquad k\in\mathbb{N} \tag{1.1}$$

and the problem of existence of bounded solutions on $\widetilde{\mathbb{N}}$, i.e.

$$\sup\{\|y(k)\| : k\in\widetilde{\mathbb{N}}\}<+\infty, \tag{1.2}$$

where $G_j(k)\in\mathbb{R}^{n\times n}$ and $\varphi_j(k)\in\mathbb{R}^n$ $(j=1,2;\ k=0,1,\dots)$.

In this paper, sufficient conditions for the existence of solutions to problem (1.1), (1.2) are established. These systems represent discrete analogues of generalized linear differential equations and are employed to study the dynamics of processes in discretized time. Analogous results have been obtained for systems of ordinary differential, impulsive differential and generalized differential equations (see [1]–[8] and the references therein).

The following notations and definitions will be used.

$\mathbb{N}$ and $\mathbb{R}$ are, respectively, the sets of all natural and real numbers, $\widetilde{\mathbb{N}}=\{0\}\cup\mathbb{N}$, $\mathbb{R}_+=[0,+\infty[$ and $[a,b]$ is a closed interval.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X=(x_{ij})$ with the norm

$$\|X\|=\max_{j=1,\dots,m}\sum_{i=1}^{n}|x_{ij}|.$$

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors.

$O_{n\times m}$ (or O) is the zero $n\times m$ matrix, I_n is the identity $n\times n$ matrix and 0_n is the zero n-vector.

If $X\in\mathbb{R}^{n\times n}$, then X^{-1}, $\det(X)$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X. δ_{ij} is the Kronecker symbol.

$E(J,\mathbb{R}^{n\times m})$, where $J\subseteq\widetilde{\mathbb{N}}$, is the space of all matrix-functions $Y:J\to\mathbb{R}^{n\times m}$ with the norm

$$\|Y\|_J=\max\{\|Y(k)\| : k\in J\}.$$

The first-order difference operator is $\Delta Y(k-1)=Y(k)-Y(k-1)$.

We assume

$$\det(I_n+(-1)^jG_j(k))\neq 0,\ j=1,2,\ k\in\widetilde{\mathbb{N}}.$$

If $\beta_j\in E(\widetilde{\mathbb{N}},\mathbb{R})$ is such that $1+(-1)^j\beta_j(k)\neq 0$ $(j=1,2,\ k\in\widetilde{\mathbb{N}})$. Then the initial problem

$$\Delta\xi(k-1)=\beta_1(k)\xi(k)+\beta_2(k-1)\xi(k-1),\ \xi(0)=1,$$

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has the unique solution ξ and it is defined by

$$\xi_{(\beta_1,\beta_2)}(k)=\prod_{i=1}^{k}(1-\beta_1(i))^{-1}(1+\beta_2(i-1)),\ k\in\widetilde{\mathbb{N}}.$$

Let $\gamma_{(\beta_1,\beta_2)}(k,s)\equiv\xi_{(\beta_1,\beta_2)}(k)\xi^{-1}_{(\beta_1,\beta_2)}(s)$ be the Cauchy function of the problem. Then

$$\gamma_{(\beta_1,\beta_2)}(k,s)=\begin{cases}\prod\limits_{j=s+1}^{k}(1-\beta_1(j))^{-1}(1+\beta_2(j-1)), & k>s,\\ 1, & k=s,\\ \gamma^{-1}_{(\beta_1,\beta_2)}(s,k), & k<s.\end{cases}$$

Note that the following equality holds

$$\Delta\xi^{-1}_{(\beta_1,\beta_2)}(k-1)=\xi^{-1}_{(\beta_1,\beta_2)}(k-1)[(1-\beta_1(k))(1+\beta_2(k-1))^{-1}-1].$$

Remark 1.1. Let $\beta_j(k)\in\mathbb{R}$ $(j=1,2)$ be such that

$$1+(-1)^j\beta_j(k)>0 \text{ for all } k\in\widetilde{\mathbb{N}},$$

and one of the functions $\sum_{i=0}^{k-1}\beta_1(i)+\sum_{i=1}^{k}\beta_2(i-1)$, $\sum_{i=0}^{k-1}\ln(1+\beta_2(i-1))-\sum_{i=0}^{k-1}\ln(1-\beta_1(i))$, and $\sum_{i=0}^{k-1}\beta_1(i)(1-\beta_1(i))^{-1}+\sum_{i=1}^{k}\beta_2(i-1)(1+\beta_2(i-1))^{-1}$ be nondecreasing (nonincreasing). Then the other two functions will be nondecreasing (nonincreasing), as well.

We introduce the operator

$$V(\zeta)(k)=\sup\{s\geq k:\zeta(s)\leq\zeta(k+1)+1\}$$

for nondecreasing $\zeta:\widetilde{\mathbb{N}}\to\mathbb{R}$, and

$$V(\zeta)(k)=\inf\{s\leq k:\zeta(s)\leq\zeta(k-1)+1\}$$

for nonincreasing $\zeta:\widetilde{\mathbb{N}}\to\mathbb{R},\ k\geq 1$.

2. Formulation of the results

For every $k_i\in\widetilde{\mathbb{N}}\cup\{+\infty\}$ $(i=1,\dots,n)$ we put $N_0(k_1,\dots,k_n)=\{i:k_i\in\widetilde{\mathbb{N}}\}$. It is evident that $N_0(k_1,\dots,k_n)=\{1,...,n\}$ if $k_i\in\widetilde{\mathbb{N}}$ $(i=1,\dots,n)$ and $N_0(k_1,\dots,k_n)=\varnothing$ if $k_i=+\infty$ $(i=1,...,n)$.

Theorem 2.1. Let

$$1+(-1)^j g_{jii}(k)\neq 0,\ (j=1,2;\ i=1,\dots,n) \text{ for } k\in\widetilde{\mathbb{N}} \tag{2.1}$$

and let there exist $k_i\in\widetilde{\mathbb{N}}\cup\{+\infty\}$ $(i=1,\dots,n)$ such that

$$s_{ij}=\sup\Bigg\{\sum_{\ell=0}^{k-1}|\gamma_i(k,\ell)||(1-g_{1ii}(\ell))|^{-1}|g_{1ij}(\ell)|+$$
$$+\sum_{\ell=1}^{k}|\gamma_i(k,\ell-1)||(1+g_{2ii}(\ell-1))|^{-1}|g_{2ij}(\ell-1)|:k\in\widetilde{\mathbb{N}}\Bigg\}<+\infty \tag{2.2}$$

$(i\neq j;\ i,j=1,\dots,n)$,

$$\sup\Bigg\{\sum_{\ell=0}^{k-1}|\gamma_i(k,\ell)||(1-g_{1ii}(\ell))|^{-1}|\varphi_{1i}(\ell)|+$$
$$+\sum_{\ell=1}^{k}|\gamma_i(k,\ell-1)||(1+g_{2ii}(\ell-1))|^{-1}|\varphi_{2i}(\ell-1)|:k\in\widetilde{\mathbb{N}}\Bigg\}<+\infty \tag{2.3}$$

$(i=1,\dots,n)$, and

$$\sup\{|\gamma_i(k,k_i)|:k\in\widetilde{\mathbb{N}}\}<+\infty \text{ for } i\in N_0(k_1,\dots,k_n), \tag{2.4}$$

where $\gamma_i(s,k) \equiv \gamma_{(g_{1ii},g_{2ii})}(s,k)$ $(i=1,\dots,n)$. Let, moreover, the matrix $S=(s_{ij})_{i,j=1}^n$, where $s_{ii}=0$ $(i=1,\dots,n)$, be such that

$$r(S) < 1. \tag{2.5}$$

Then for every $c_i \in \mathbb{R}$, $i \in N_0(k_1,\dots,k_n)$, system (1.1) has at least one bounded solution on $\widetilde{\mathbb{N}}$ satisfying

$$y_i(k_i) = c_i,\ i \in N_0(k_1,\dots,k_n). \tag{2.6}$$

Theorem 2.1′. Suppose that conditions (2.1)–(2.3) hold for some $k_i = +\infty$ $(i=1,\dots,n)$, where $\gamma_i(s,k) \equiv \gamma_{(g_{1ii},g_{2ii})}(s,k)$ $(i=1,\dots,n)$ and the matrix $S=(s_{ij})_{i,j=1}^n$, where $s_{ii}=0$ $(i=1,\dots,n)$, satisfy the condition (2.5). Then system (1.1) has at least one solution bounded on $\widetilde{\mathbb{N}}$.

Corollary 2.1. Let

$$1+(-1)^j g_{jii}(k) > 0,\ k \in \widetilde{\mathbb{N}},\ (j=1,2;\ i=1,\dots,n) \tag{2.7}$$

and let there exist $k_i \in \widetilde{\mathbb{N}} \cup \{+\infty\}$ $(i=1,\dots,n)$ such that conditions (2.2)–(2.5) hold, where $S=(s_{ij})_{i,j=1}^n$, $s_{ii}=0$ $(i=1,\dots,n)$ and $\gamma_i(s,k) \equiv \gamma_{(g_{1ii},g_{2ii})}(s,k)$ $(i=1,\dots,n)$. Let, moreover, the functions

$$\begin{gathered}\operatorname{sgn}(k-k_i)\left[\sum_{\ell=0}^{k-1} g_{1ij}(\ell)(1-g_{1ii}(\ell))^{-1} + \sum_{\ell=1}^{k} g_{2ij}(\ell-1)(1+g_{2ii}(\ell-1))^{-1}\right],\\ \operatorname{sgn}(k-k_i)\left[\sum_{\ell=0}^{k-1} \varphi_{1i}(\ell)(1-g_{1ii}(\ell))^{-1} + \sum_{\ell=1}^{k} \varphi_{2i}(\ell-1)(1+g_{2ii}(\ell-1))^{-1}\right],\\ (i\neq j;\ i,j=1,\dots,n) \quad \text{are nondecreasing on} \quad \widetilde{\mathbb{N}}.\end{gathered} \tag{2.8}$$

Then for every $c_i \in \mathbb{R}_+$, $(i \in N_0(k_1,\dots,k_n))$, system (1.1) has at least one nonnegative bounded solution satisfying (2.6).

If $N_0(k_1,\dots,k_n)=\emptyset$ then Corollary 2.1 has the following form.

Corollary 2.1′. Let conditions (2.7) and (2.8) hold and let $k_i=+\infty$ $(i=1,\dots,n)$ such that conditions (2.2), (2.3) and (2.5) hold, where $S=(s_{ij})_{i,j=1}^n$, $s_{ii}=0$ $(i=1,\dots,n)$, and $\gamma_i(s,k) \equiv \gamma_{(g_{1ii},g_{2ii})}(s,k)$ $(i=1,\dots,n)$. Then system (1.1) has at least one nonnegative bounded solution on $\widetilde{\mathbb{N}}$.

Theorem 2.2. Let (2.1) hold and there exist $k_i \in \widetilde{\mathbb{N}} \cup \{+\infty\}$ $(i=1,\dots,n)$ such that conditions (2.2)–(2.5) hold, where $S=(s_{ij})_{i,j=1}^n$, $s_{ii}=0$ $(i=1,\dots,n)$, and $\gamma_i(s,k) \equiv \gamma_{(g_{1ii},g_{2ii})}(s,k)$ $(i=1,\dots,n)$.

Let, moreover,

$$\liminf_{k\to k_i} \gamma_i(0,k) = 0 \text{ for } i \in \{1,\dots,n\} \setminus N_0(k_1,\dots,k_n).$$

Then for every $c_i \in \mathbb{R}$ $(i \in N_0(k_1,\dots,k_n))$ system (1.1) has a unique bounded solution $(y_1)_{i=1}^n$ on $\widetilde{\mathbb{N}}$ satisfying condition (2.6), and

$$\sum_{i=1}^n |y_i(k) - y_{im}(k)| \le \rho_0 \alpha^m,\ k \in \widetilde{\mathbb{N}}\ (m=1,2,\dots),$$

where ρ_0 and $0<\alpha<1$ are positive numbers independent of m, and $(y_{im})_{i=1}^n$ $(m=0,1,\dots)$ is the sequence of vector-functions whose components are defined by

$$y_{i0}(k) \equiv 0,$$

$$\begin{aligned} y_{im}(k) \equiv u_i(k) + \sum_{\substack{j=1\\ j\neq i}}^n \Bigg[& \sum_{\ell=k_i}^{k-1} \gamma_i(k,\ell)(1-g_{1ii}(\ell))^{-1} g_{1ij}(\ell) y_{jm-1}(\ell) \\ & + \sum_{\ell=k_i+1}^{k} \gamma_i(k,\ell-1)(1+g_{2ii}(\ell-1))^{-1} g_{2ij}(\ell-1) y_{jm-1}(\ell-1)\Bigg] \end{aligned}$$

$(i=1,\dots,n;\ m=1,2,\dots)$, and the functions u_i $(i=1,\dots,n)$ are defined by

$$u_i(k)=c_i\gamma_i(k,k_i)+\sum_{\ell=k_i}^{k-1}\gamma_i(k,\ell)(1-g_{1ii}(\ell))^{-1}\varphi_{1i}(\ell)\\
+\sum_{\ell=k_i+1}^{k}\gamma_i(k,\ell-1)(1+g_{2ii}(\ell-1))^{-1}\varphi_{2i}(\ell-1)$$

for $i\in N_0(k_1,\dots,k_n)$, and

$$u_i(k)=\sum_{\ell=k_i}^{k-1}\gamma_i(k,\ell)(1-g_{1ii}(\ell))^{-1}\varphi_{1i}(\ell)+\sum_{\ell=k_i+1}^{k}\gamma_i(k,\ell-1)(1+g_{2ii}(\ell-1))^{-1}\varphi_{2i}(\ell-1)$$

for $i\in\{1,\dots,n\}\setminus N_0(k_1,\dots,k_n)$.

Corollary 2.2. Let (2.7) hold and let there exist $k_i\in\widetilde{\mathbb{N}}\cup\{+\infty\}$ $(i=1,\dots,n)$ such that the functions

$$\operatorname{sgn}(k-k_i)\left[\sum_{l=0}^{k-1}g_{1ii}(l)+\sum_{l=1}^{k}g_{2ii}(l-1)\right]\ (i=1,\dots,n)$$

are nonincreasing on $\widetilde{\mathbb{N}}$,

$$\liminf_{k\to k_i}\left[\sum_{l=0}^{k-1}g_{1ii}(l)+\sum_{l=1}^{k}g_{2ii}(l-1)\right]=+\infty$$

for $i\in\{1,\dots,n\}\setminus N_0(k_1,\dots,k_n)$,

$$\sum_{l=0}^{k-1}|g_{1ij}(l)|\big(1-g_{1ii}(l)\big)^{-1}+\sum_{l=1}^{k}|g_{2ij}(l-1)|\big(1+g_{2ii}(l-1)\big)^{-1}\le\\
h_{ij}\operatorname{sgn}(k-k_i)\left[\sum_{l=0}^{k-1}g_{1ii}(l)\big(1-g_{1ii}(l)\big)^{-1}+\right.\\
\left.+\sum_{l=1}^{k}g_{2ii}(l-1)\big(1+g_{2ii}(l-1)\big)^{-1}\right], \tag{2.9}$$

for $k\in\widetilde{\mathbb{N}}$ $(i\ne j;\ i,j=1,\dots,n)$, and

$$r(H)<1, \tag{2.10}$$

where h_{ij} $(i,j=1,\dots,n)$ are such that $H=((1-\delta_{ij})h_{ij})_{i,j=1}^n$.

Let, moreover,

$$\rho_i=\sup\left|\sum_{l=k}^{V(\zeta_i)(k)-1}|\varphi_{1i}(l)|\big(1-g_{1ii}(l)\big)^{-1}+\sum_{l=k+1}^{V(\zeta_i)(k)}|\varphi_{2i}(l-1)|\big(1+g_{2ii}(l-1)\big)^{-1}\right|<+\infty \tag{2.11}$$

$$(i=1,\dots,n),$$

where $\zeta_i(k)\equiv\xi_{(g_{1ii},g_{2ii})}\operatorname{sgn}(k-k_i)$ $(i=1,\dots,n)$. Then the conclusion of Theorem 2.2 is true.

Corollary 2.3. Let there exist the points $k_i\in\widetilde{\mathbb{N}}\cup\{+\infty\}$ $(i=1,\dots,n)$, and the functions $\beta_{ji}:\widetilde{\mathbb{N}}\to\mathbb{R}$ $(j=1,2;\ i=1,\dots,n)$, such that

$$\operatorname{sgn}(k-k_i)\left[\sum_{l=0}^{k-1}\beta_{1i}(l)+\sum_{l=1}^{k}\beta_{2i}(l-1)\right]\ (i=1,\dots,n),$$

are nondecreasing on $\widetilde{\mathbb{N}}$ and conditions

$$g_{1ii}(k)\le\eta_{1ii}\beta_{1i}(k)<1,\ -1<g_{2ii}(k)\le\eta_{2ii}\beta_{2i}(k)\ (i=1,\dots,n), \tag{2.12}$$

$$|g_{lij}(k)|\le\eta_{lij}|\beta_{li}(k)|\ (l=1,2;\ i,j=1,\dots,n), \tag{2.13}$$

hold on $\widetilde{\mathbb{N}}$.

Define $\eta_{ij} = \max\{\eta_{1ij}, \eta_{2ij}\}$ $(i,j = 1,\dots,n)$, $H = \big((1-\delta_{ij})\eta_{ij}|\eta_{ii}|^{-1}\big)_{i,j=1}^{n}$ and condition (2.10) holds. Let, moreover,

$$\rho_i = \sup\left|\sum_{l=k}^{V(\zeta_i)(k)-1} |\varphi_{1i}(l)|\left|(1-\eta_{1ii}\beta_{1i}(l))^{-1}\right| + \sum_{l=k+1}^{V(\zeta_i)(k)} |\varphi_{2i}(l-1)|\left|(1+\eta_{2ii}\beta_{2i}(l-1))^{-1}\right|\right| < +\infty \tag{2.14}$$

$$(i = 1,\dots,n),$$

where $\zeta_i(k) \equiv \xi_{(\eta_{1ii}\beta_{1i},\,\eta_{2ii}\beta_{2i})}\ \mathrm{sgn}(k-k_i)$ $(i=1,\dots,n)$. Then conclusion of Theorem 2.2 is true.

Theorem 2.2′. Let (2.1) hold and let there exist $k_i = +\infty$ $(i=1,\dots,n)$ such that conditions (2.2), (2.3) and (2.5) hold, where $S = (s_{ij})_{i,j=1}^{n}$, $s_{ii} = 0$ $(i=1,\dots,n)$, and $\gamma_i(s,k) \equiv \gamma_{(g_{1ii},g_{2ii})}(s,k)$ $(i = 1,\dots,n)$. Let, moreover,

$$\liminf_{k\to+\infty} \gamma_i(0,k) = 0 \text{ for } i \in \{1,\dots,n\}.$$

Then system (1.1) has a unique bounded solution $(y_i)_{i=1}^{n}$ on $\widetilde{\mathbb{N}}$ and

$$\sum_{i=1}^{n} |y_i(k) - y_{im}(k)| \le \rho_0\alpha^m \text{ for } k \in \widetilde{\mathbb{N}}\ (m = 1,2,\dots),$$

where ρ_0 and $0 < \alpha < 1$ are positive numbers independent of m, and $(y_{im})_{i=1}^{n}$ $(m = 0,1,\dots)$ is the sequence of vector-functions whose components are defined by

$$y_{i0}(k) \equiv 0,$$

and

$$y_{im}(k) = u_i(k) + \sum_{\substack{j=1\\ j\ne i}}^{n}\Bigg[\sum_{\ell=k_i}^{k-1}\gamma_i(k,\ell)(1-g_{1ii}(\ell))^{-1}g_{1ij}(\ell)y_{jm-1}(\ell)$$
$$+ \sum_{\ell=k_i+1}^{k}\gamma_i(k,\ell-1)(1+g_{2ii}(\ell-1))^{-1}g_{2ij}(\ell-1)y_{jm-1}(\ell-1)\Bigg],$$

$(i=1,\dots,n;\ m=1,2,\dots)$, where

$$u_i(k) = \sum_{\ell=k_i}^{k-1}\gamma_i(k,\ell)(1-g_{1ii}(\ell))^{-1}\varphi_{1i}(\ell) + \sum_{\ell=k_i+1}^{k}\gamma_i(k,\ell-1)(1+g_{2ii}(\ell-1))^{-1}\varphi_{2i}(\ell-1)$$

$(i=1,\dots,n)$.

Corollary 2.2′. Let condition (2.7) hold and let $k_i = +\infty$ $(i=1,\dots,n)$ such that the functions

$$\sum_{\ell=0}^{k-1} g_{1ii}(\ell) + \sum_{\ell=1}^{k} g_{2ii}(\ell-1) \qquad (i=1,\dots,n)$$

are nondecreasing on $\widetilde{\mathbb{N}}$ and

$$\liminf_{k\to+\infty}\left[\sum_{\ell=0}^{k-1} g_{1ii}(\ell) + \sum_{\ell=1}^{k} g_{2ii}(\ell-1)\right] = +\infty \text{ for } i \in \{1,\dots,n\}$$

hold, where $\zeta_i(k) \equiv \xi_{(g_{1ii},g_{2ii})}\ \mathrm{sgn}(k-k_i)$ $(i=1,\dots,n)$, and the numbers h_{ij} $(i,j=1,\dots,n)$ such that $H = ((1-\delta_{ij})\eta_{ij})_{i,j=1}^{n}$ and condition (2.10) holds. Then the conclusion of Theorem 2.2′ is true.

Corollary 2.3′. Let $k_i = +\infty$ $(i=1,\dots,n)$ and let there exist the functions $\beta_{ji} : \widetilde{\mathbb{N}} \to \mathbb{R}$ $(j = 1,2;\ i=1,\dots,n)$ such that

$$\sum_{\ell=0}^{k-1} \beta_{1i}(\ell) + \sum_{\ell=1}^{k} \beta_{2i}(\ell-1) \qquad (i=1,\dots,n)$$

are nondecreasing on $\widetilde{\mathbb{N}}$. Let conditions (2.10), (2.12)–(2.14) hold on $\widetilde{\mathbb{N}}$, where

$$\zeta_i(k) \equiv -\xi_{(\eta_{1ii}\beta_{1i},\, \eta_{2ii}\beta_{2i})} \qquad (i = 1, \dots, n),$$

and the numbers $\eta_{\ell ij} < 0$ ($\ell = 1, 2;\ i, j = 1, \dots, n$) are such that

$$H = \big((1 - \delta_{ij})\eta_{ij}\eta_{ii}^{-1}\big)_{i,j=1}^{n},$$

where $\eta_{ij} = \max\{\eta_{1ij}, \eta_{2ij}\}$ $(i, j = 1, \dots, n)$. Then the conclusion of Theorem 2.2′ is true.

Corollary 2.4. Assume that the conditions of Theorem 2.2, Corollary 2.2, or Corollary 2.3 are satisfied. In addition, suppose that condition (2.8) holds. Then for every $c_i \in \mathbb{R}_+$ $(i \in N_0(k_1, \dots, k_n))$ system (1.1) has a unique bounded solution on $\widetilde{\mathbb{N}}$ satisfying (2.6), and this solution is nonnegative.

Corollary 2.4′. Let the conditions of Theorem 2.2′, Corollary 2.2′ or Corollary 2.3′ be satisfied. Let, in addition, the functions

$$\sum_{\ell=0}^{k-1} g_{1ij}(\ell)(1 - g_{1ii}(\ell))^{-1} + \sum_{\ell=1}^{k} g_{2ij}(\ell - 1)(1 + g_{2ii}(\ell - 1))^{-1},$$

$$\sum_{\ell=0}^{k-1} \varphi_{1i}(\ell)(1 - g_{1ii}(\ell))^{-1} + \sum_{\ell=1}^{k} \varphi_{2i}(\ell - 1)(1 + g_{2ii}(\ell - 1))^{-1},$$

$(i \neq j;\ i, j = 1, \dots, n)$ are nonincreasing on $\widetilde{\mathbb{N}}$.

Then system (1.1) has a unique bounded solution on $\widetilde{\mathbb{N}}$, and it is nonnegative.

References

1. R. P. Agarwal, Difference equations and inequalities. *Theory, Methods, and Applications. Monographs and Textbooks in Pure and Applied Mathematics,* **155** (1992), Marcel Dekker, Inc., New York.
2. Sh. Akhalaia, M. Ashordia, N. Kekelia, On the necessary and sufficient conditions for the stability of linear generalized ordinary differential, linear impulsive and linear difference systems. *Georgia Math. J.,* **16** (2009), no. 4, pp. 597–616.
3. M. Ashordia, The Initial Problem for Linear Systems of Generalized Ordinary Differential Equations, Linear Impulsive and Ordinary Differential Systems. Numerical Solubility. *Mem. Differential Equations Math. Phys.,* **78** (2019), pp. 1–662.
4. M. Ashordia, The General Boundary Value Problems for Linear Systems of Generalized Ordinary Differential Equations, Linear Impulsive Differential and Ordinary Differential Systems. Numerical Solubility. *Mem. Differential Equations Math. Phys.,* **81** (2020), pp. 1–184.
5. N. Kekelia, On the existence of bounded solutions on the real axis R for systems of linear impulsive differential equations. *Transactions of A. Razmadze Mathematical Institute,* **179** (2025), issue 1, pp. 133–146.
6. I. T. Kiguradze, The singular Nicoletti problem (English. Russian original) Dokt. *Akad. Nauk SSSR,* **186** (1969), no. 4, pp. 769–772.
7. I. Kiguradze, The initial value problem and boundary value problems for systems of ordinary differential equations. *Linear theory, Tbilisi, Metsniereba,* **1** (1997).
8. A. Samarskii, Theory of Difference Schemes (Russian), Second edition. *"Nauka", Moscow,* (1980).

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