Transactions of A. Razmadze
Mathematical Institute
Vol. **1??** (202?), ?–??

A SIMPLE APPROACH TO FATOU'S LEMMA FOR RIEMANN INTEGRALS

YOSHIHIRO SAWANO[1]

Abstract. We establish a variant of Fatou's lemma for the Riemann integral. We give a short and self-contained proof including some auxiliary known results.

1. Introduction

We aim to establish a variant of Fatou's lemma in the setting of the Riemann integral. In formulating such a result, several intrinsic difficulties arise:

(1) Lebesgue measurability of a function does not guarantee the existence of its Riemann integral.
(2) Even if two functions agree almost everywhere, one may be Riemann integrable while the other is not, as illustrated by the pair consisting of the constant function 1 and the indicator function $\chi_{\mathbb{R}\setminus\mathbb{Q}}$. The former is Riemann integrable in $[0,1]$ while the latter is not.
(3) In general, the pointwise limit of a sequence of Riemann integrable functions need not be Riemann integrable. Such examples can be constructed using an enumeration $\{r_j\}_{j=1}^{\infty}$ of the rational numbers in $[0,1]$.

A recent article [7] addresses related questions in the setting of continuous functions and considers differences on null sets. However, when working with lower and upper Riemann integrals, functions that agree almost everywhere are not identified. Let $-\infty < a < b < \infty$, and let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. We briefly recall the basic terminology and definitions related to the Riemann integral. Throughout this paper, we restrict our attention to Riemann integrals over compact intervals in $\mathbb{R}$; extensions to higher-dimensional settings can be carried out without essential difficulty, but will not be pursued here.

(1) A *partition* of $[a,b]$ is a finite increasing sequence
$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$
of arbitrary length. We denote the length of each subinterval by $\Delta x_i := x_i - x_{i-1}$.
(2) Let P be a partition of $[a,b]$ as above. For each subinterval $[x_{i-1}, x_i]$, define
$$M_i := \sup_{x\in[x_{i-1},x_i]} f(x), \qquad m_i := \inf_{x\in[x_{i-1},x_i]} f(x),$$
and the corresponding *upper and lower sums* by
$$U(f,P) := \sum_{i=1}^{n} M_i\,\Delta x_i, \qquad L(f,P) := \sum_{i=1}^{n} m_i\,\Delta x_i.$$
(3) The *upper Riemann integral* and the *lower Riemann integral* of f over $[a,b]$ are defined by
$$\overline{\int_a^b} f(x)\,\mathrm{d}x := \inf_P U(f,P), \qquad \underline{\int_a^b} f(x)\,\mathrm{d}x := \sup_P L(f,P),$$
where the infimum and supremum are taken over all partitions P of $[a,b]$.

2020 *Mathematics Subject Classification.* 26A42, 26A39.
Key words and phrases. Riemann integral; Fatou's lemma; Arzelà's classical convergence theorem .

(4) The function f is said to be *Riemann integrable* on $[a,b]$ if

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x.$$

In this case, the common value is called the *Riemann integral* of f over $[a,b]$.

By definition, the upper and lower Riemann integrals are finite only for bounded functions; thus the boundedness of functions is essential when considering Riemann integrability. Moreover, the values of the upper and lower Riemann integrals may change if f is replaced by another function that agrees with it almost everywhere as we mentioned.

The main result of this note is the following theorem:

Theorem 1.1. *Let $\{f_j\}_{j=1}^\infty$ be a sequence of real-valued functions defined on a bounded closed interval $[a,b]$, and assume that the sequence is uniformly bounded, that is, there exists a constant $M>0$ such that*

$$\sup_{j\in\mathbb{N}, x\in[a,b]} |f_j(x)| \le M \tag{1.1}$$

Then

$$\liminf_{j\to\infty} \overline{\int_a^b} f_j(x)\,\mathrm{d}x \ \ge \ \underline{\int_a^b} \Big(\liminf_{j\to\infty} f_j(x)\Big)\,\mathrm{d}x.$$

It is worth noting that Arzelà's classical convergence theorem can be deduced from Theorem 1.1 [1, 2].

2. Proof of Theorem 1.1

To prove Theorem 1.1, we need two lemmas, both of which are known. Here for the sake of self-containedness, we supply the proof without using the Lebesgue integral.

The proof of Theorem 1.1 is based on a classical result due to W. A. J. Luxemburg [5], originally developed in the context of the dominated convergence for Riemann integrals using Dini's theorem [3]. We recall this lemma as Lemma 2.

Suppose that O is an open subset of $\mathbb{R}$. Since every open subset of $\mathbb{R}$ can be written uniquely as a countable disjoint union of open intervals, we define $|O|$ to be the sum of the lengths of these intervals. Likewise if F is the set of a finite disjoint union of intervals, we define $|F|$ to be the sum of the lengths of these intervals. We use the observation due to [4, Theorem 4].

Lemma 1. *Let $\{Q_j\}_{j=1}^\infty$ be a decreasing sequence of bounded open subsets of $\mathbb{R}$: $Q_1 \supset Q_2 \supset Q_3 \supset \cdots$. Assume that, for every $j\in\mathbb{N}$, the set Q_j contains a finite union of open intervals and satisfies $|Q_j| \ge 2$. Then $\bigcap_{j=1}^{\infty} Q_j \neq \emptyset$.*

Proof. For each $j\in\mathbb{N}$, choose a compact set $K_j \subset Q_j$ that is realized as a finite disjoint union of closed intervals such that

$$|K_j| \ge |Q_j| - 2^{-j} \ge 1. \tag{2.1}$$

This is possible because Q_j is a countable union of open intervals.

Define $F_j := \bigcap_{k=1}^{j} K_k$. Then each F_j is compact and $F_1 \supset F_2 \supset F_3 \supset \cdots$.

We claim that $F_j \neq \emptyset$ for every j. Indeed, if $F_j = \emptyset$ with some $j \ge 2$, then we would have $K_j \subset \bigcup_{k=1}^{j-1} (\mathbb{R}\setminus K_k)$. Since $K_j \subset Q_j \subset Q_k$ for $1 \le k \le j-1$, we have $K_j \subset \bigcup_{k=1}^{j-1}(Q_k \setminus K_k)$. Therefore,

$|K_j| \le \sum_{k=1}^{j-1} |Q_k \setminus K_k|$. From (2.1), $|Q_k \setminus K_k| \le 2^{-k}$. Hence $2 - 2^{-j} \le |Q_j| - 2^{-j} \le |K_j| \le \sum_{k=1}^{j-1} 2^{-k} = 1 - 2^{-j}$. This contradiction proves that $F_j \neq \emptyset$.

Since $\{F_j\}_{j=1}^\infty$ is a decreasing sequence of nonempty compact sets, the Cantor intersection theorem yields $\bigcap_{j=1}^\infty F_j \neq \emptyset$. Finally, because $F_j \subset Q_j$ for every j, we conclude that $\bigcap_{j=1}^\infty Q_j \supset \bigcap_{j=1}^\infty F_j \neq \emptyset$. This completes the proof. $\square$

We now recall another observation, which we refer to as Luxemburg's lemma in this paper.

Lemma 2 ([5, (2.2) Lemma], see also [6, Lemma 2])**.** *Let $\{f_j\}_{j=1}^\infty$ be a bounded sequence of functions on a bounded closed interval $[a,b]$ such that*

$$0 \le f_{j+1}(x) \le f_j(x), \qquad \lim_{j\to\infty} f_j(x) = 0$$

for all $x \in (a,b)$. Then we have

$$\lim_{j\to\infty} \underline{\int_a^b} f_j(x)\,\mathrm{d}x = 0. \tag{2.2}$$

Proof. Suppose (2.2) fails. Then, due to the monotonicity, there exists $\delta > 0$ such that

$$\underline{\int_a^b} f_j(x)\,\mathrm{d}x > 2\delta.$$

for all $j \in \mathbb{N}$.

From the definition of the lower integral, we can find a finite partition $P_j = \{a = x_{0,j} < x_{1,j} < \cdots < x_{N_j,j} = b\}$ such that $L(f_j, P_j) > 2\delta$.

Define

$$E_j = \left\{k = 1, 2, \ldots, N_j : \inf_{x_{k-1,j} \le x \le x_{k,j}} f_j(x) < \frac{\delta}{b-a}\right\}, \quad F_j = \{1, 2, \ldots, N_j\} \setminus E_j.$$

Then F_j is nonempty.

Furthermore,

$$L(f_j, P_j) = \sum_{k\in E_j} (x_{k,j} - x_{k-1,j}) \inf_{x_{k-1,j} \le x \le x_{k,j}} f_j(x) + \sum_{k\in F_j} (x_{k,j} - x_{k-1,j}) \inf_{x_{k-1,j} \le x \le x_{k,j}} f_j(x).$$

Since

$$\sum_{k\in E_j} (x_{k,j} - x_{k-1,j}) \inf_{x_{k-1,j} \le x \le x_{k,j}} f_j(x) \le \frac{\delta}{b-a} \sum_{k\in E_j} (x_{k,j} - x_{k-1,j}) \le \delta,$$

it follows that

$$\sum_{k\in F_j} (x_{k,j} - x_{k-1,j}) \inf_{x_{k-1,j} \le x \le x_{k,j}} f_j(x) > \delta.$$

Apply the scaled version of Lemma 1 to the set

$$Q_k = \bigcup_{j=k}^\infty \bigcup_{k\in F_j} (x_{k-1,j}, x_{k,j}),$$

which yields $x \in Q_1$ such that

$$x \in \bigcup_{k\in F_j} (x_{k-1,j}, x_{k,j})$$

for infinitely many j. We have $f_j(x) \ge \inf_{x_{k-1,j} \le \bar{x} \le x_{k,j}} f_j(\bar{x}) \ge \dfrac{\delta}{b-a}$ for such j. This contradicts the assumption that $\lim_{j\to\infty} f_j(x) = 0$. $\square$

We now prove Theorem 1.1. We may assume that $f_j \geq 0$ by considering $f_j + M$ if necessary. For each $j \in \mathbb{N}$, define

$$g_j(x) := \inf\{f_j(x), f_{j+1}(x), \dots\}, \qquad x \in [a,b].$$

By construction, we have

$$f_j(x) \geq g_j(x) \geq 0, \quad g_j(x) \leq g_{j+1}(x)$$

for all $x \in (a,b)$ and all $j \in \mathbb{N}$, and

$$\liminf_{j\to\infty} f_j(x) = \lim_{j\to\infty} g_j(x), \quad x \in [a,b].$$

Hence, it suffices to show

$$\lim_{j\to\infty} \overline{\int_a^b} g_j(x)\,\mathrm{d}x \;\geq\; \underline{\int_a^b} \lim_{j\to\infty} g_j(x)\,\mathrm{d}x. \tag{2.3}$$

Set

$$g(x) := \lim_{j\to\infty} g_j(x), \qquad x \in [a,b].$$

For any subinterval $I \subset [a,b]$, observe that

$$\sup_{x\in I} g_j(x) \geq \inf_{x\in I} g(x) + \sup_{x\in I}\big(g_j(x) - g(x)\big).$$

It follows from the definitions of the upper and lower Riemann integrals that

$$\overline{\int_a^b} \big(g_j(x) - g(x)\big)\,\mathrm{d}x = -\underline{\int_a^b} \big(g(x) - g_j(x)\big)\,\mathrm{d}x,$$

and therefore

$$\overline{\int_a^b} g_j(x)\,\mathrm{d}x \geq \underline{\int_a^b} g(x)\,\mathrm{d}x + \overline{\int_a^b} \big(g_j(x) - g(x)\big)\,\mathrm{d}x = \underline{\int_a^b} g(x)\,\mathrm{d}x - \underline{\int_a^b} \big(g(x) - g_j(x)\big)\,\mathrm{d}x.$$

Passing to the limit $j \to \infty$, we obtain

$$\lim_{j\to\infty} \overline{\int_a^b} g_j(x)\,\mathrm{d}x \geq \underline{\int_a^b} g(x)\,\mathrm{d}x - \lim_{j\to\infty} \underline{\int_a^b} \big(g(x) - g_j(x)\big)\,\mathrm{d}x.$$

Since $g(x) - g_j(x) \geq 0$ decreases pointwise to 0 and is bounded, Lemma 2 implies

$$\lim_{j\to\infty} \underline{\int_a^b} \big(g(x) - g_j(x)\big)\,\mathrm{d}x = 0.$$

Hence, we conclude

$$\liminf_{j\to\infty} \overline{\int_a^b} f_j(x)\,\mathrm{d}x \;\geq\; \lim_{j\to\infty} \overline{\int_a^b} g_j(x)\,\mathrm{d}x \;\geq\; \underline{\int_a^b} g(x)\,\mathrm{d}x = \underline{\int_a^b} \liminf_{j\to\infty} f_j(x)\,\mathrm{d}x,$$

which completes the proof of Theorem 1.1.

Acknowledgements

The author is grateful to Professor Jun Kawabe at Shinshu University for kindly introducing the paper [5].
Disclosure statement. : The author declares that there are no conflicts of interest regarding the publication of this paper.

References

1. C. Arzelà, Sulla integrazione per serie. (Italian) *Atti Accad. Lincei Rend.* (4) **1** (1885), 532–537, 596–599.
2. C. Arzelà, Sulle serie di funzioni. (Italian) *Mem. Ist. Bologna* (5) **8** (1900), 131–186, 701–744.
3. U. Dini, *Fondamenti per la teorica delle funzioni di variabili reali.* Pisa: Tipografia T. Nistri e Co., 1878.
4. R. A. Gordon, A convergence theorem for the Riemann integral. *Math. Mag.* **73** (2000), no. 2, 141–147.
5. W. A. J. Luxemburg, Arzelà's dominated convergence theorem for the Riemann integral. *Amer. Math. Monthly* **78** (1971), no. 9, 970–979.
6. A. Sarić, Arzelà's bounded convergence theorem. *Analysis* **44** (2024), no. 2, 115–119.
7. B. S. Thomson, The bounded convergence theorem. *Amer. Math. Monthly* **127** (2020), no. 6, 483–503.

[1]Department of Mathematics, Graduate School of Science and Engineering, Chuo University, 13-27 Kasuga 1-chome, Bunkyo-ku, Tokyo 112-8551, Japan

Email address: yoshihiro-sawano@celery.ocn.ne.jp