Transactions of A. Razmadze
Mathematical Institute
Vol. **180** (2026), issue 3, 1–5

NONLINEAR NONLOCAL BOUNDARY VALUE PROBLEMS FOR FIRST ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

NINO PARTSVANIA

Abstract. For first order differential equations with deviating argument, sufficient conditions for the solvability and unique solvability of nonlocal nonlinear boundary value problems are established.

In the present paper, on the finite interval $[a, b]$ we consider the differential equation

$$u'(t) = f(t, u(\tau(t))) \tag{1}$$

with the nonlocal boundary condition

$$\varphi(u) = 0, \tag{2}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a function from the Carathéodory space, $\tau : [a, b] \to [a, b]$ is a measurable function, while $\varphi : C([a, b]) \to \mathbb{R}$ is a continuous functional.

Everywhere below we use the following notation and definitions.

$\mathbb{R}$ is the set of real numbers; $\mathbb{R}_+ = [0, +\infty[$;

$C([a, b])$ is the space of continuous functions defined on $[a, b]$;

$C([a, b]; \mathbb{R}_+)$ is the set of nonnegative continuous functions defined on $[a, b]$;

$L([a, b])$ is the space of Lebesgue integrable functions defined on $[a, b]$.

If $u_i \in C([a, b])$ $(i = 1, 2)$, then

$$u_1 < u_2 \quad \Leftrightarrow \quad u_1(t) < u_2(t) \text{ for } a \le t \le b.$$

A functional φ is said to be increasing on the set $C([a, b]; \mathbb{R}_+)$ if

$$\varphi(u_1) < \varphi(u_2) \text{ for } u_i \in C([a, b]; \mathbb{R}_+) \ (i = 1, 2), \ \ u_1 < u_2.$$

We say that a function $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ belongs to the Carathéodory space if $f(\cdot, x) : [a, b] \to \mathbb{R}$ is measurable for every $x \in \mathbb{R}$, $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for almost all $t \in [a, b]$, and

$$f^*(\cdot, r) \in L([a, b]) \text{ for } r \in \mathbb{R}_+,$$

where

$$f^*(t, r) = \max\left\{|f(t, x)| : \ |x| \le r\right\}.$$

$f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is said to be locally Lipschitz in the second argument if for any positive number r there exists a nonnegative function $\ell_r \in L([a, b])$ such that

$$|f(t, x) - f(t, y)| \le \ell_r(t)|x - y| \text{ for } t \in [a, b], \ \ |x| \le r, \ \ |y| \le r.$$

In the monograph by R. Hakl, A. Lomtatidze, and J. Šremr [4], for first order functional differential equations, and in particular for equations of type (1), boundary value problems of periodic and antiperiodic types have been studied in detail. As for the nonlocal problem (1), (2), it still remains unstudied in the case where the function f is rapidly increasing in the phase variable and φ is a nonlinear functional (see, for example, [1–3, 5] and the references therein). Theorems 1 and 2 below, containing sufficient conditions for the existence and uniqueness of a nonnegative solution to problem (1), (2), fill this gap to some extent. It should be noted that these theorems and their corollaries are new also for the case $\tau(t) \equiv t$.

2020 *Mathematics Subject Classification.* 34B10, 34B15, 34K10.

Key words and phrases. Boundary value problem; Nonlinear; Nonlocal; Differential equation; First order; Deviating argument; Solvability; Unique solvability.

Along with problem $(1),(2)$, we consider its particular case

$$u'(t)=\sum_{i=1}^{k} p_i(t) f_i(u(\tau(t))), \tag{1_1}$$

$$\sum_{i=1}^{m} \int_a^b |u(s)|^{\lambda_i} d\sigma_i(s)=c_0. \tag{2_1}$$

Here k and m are natural numbers, c_0 is a positive constant, $p_i \in L([a,b])$, $f_i \in C(\mathbb{R})$ $(i=1,\dots,k)$, $\lambda_i>0$ $(i=1,\dots,m)$, and $\sigma_i : [a,b] \to \mathbb{R}$ are nondecreasing functions $(i=1,\dots,m)$.

Theorem 1. *Let*

$$f(t,0)=0, \quad f(t,x)\ge 0 \ \ \textit{for} \ \ t\in[a,b], \quad x\in\mathbb{R}_+, \tag{3}$$

$$\tau(t)\le t \ \ \textit{for} \ \ t\in[a,b], \tag{4}$$

$$\varphi(0)<0, \tag{5}$$

and let on the set $C([a,b];\mathbb{R}_+)$ the condition

$$\liminf_{u(b)\to+\infty} \varphi(u)>0 \tag{6}$$

be satisfied. Then problem $(1),(2)$ has at least one nonnegative solution.

Corollary 1. *Let along with (4) the following conditions*

$$p_i(t)\ge 0 \ \ \textit{for} \ \ t\in[a,b], \quad f_i(0)=0, \quad f_i(x)\ge 0 \ \ \textit{for} \ \ x\in\mathbb{R}_+ \ \ (i=1,\dots,k), \tag{7}$$

$$\sum_{i=1}^{m} (\sigma_i(b)-\sigma_i(b-))>0 \tag{8}$$

hold, where $\sigma_i(b-)$ is a left limit of the function σ_i at the point b. Then problem $(1_1),(2_1)$ has at least one nonnegative solution.

Remark 1. The above formulated theorem and its corollary do not restrict the behavior of the functions in the right-hand sides of equations (1) and (1_1) with respect to the phase variable at infinity. For example, if conditions (4)–(6) are satisfied and

$$f(t,x)=f_0(t,x)(e_k(x)-e_k(0)),$$

where $f_0 : [a,b]\times\mathbb{R}\to]1,+\infty[$ is a function from the Carathéodory space, k is a natural number, and

$$e_1(x)=\exp(x), \quad e_i(x)=e_{i-1}(e_1(x)) \ \ (i=2,3,\dots),$$

then according to Theorem 1, problem $(1),(2)$ has at least one nonnegative solution.

Also, if $p_i\in L([a,b])$ $(i=1,\dots,k)$ are nonnegative functions,

$$f_i(x)=e_i(x)-e_i(0) \ \ (i=1,\dots,k),$$

and inequalities (4), (8) are satisfied, then according to Corollary 1, problem $(1_1),(2_1)$ has at least one nonnegative solution.

Remark 2. Let the conditions of Theorem 1 be fulfilled and let there exist a nonnegative function $\ell\in L([a,b])$ such that

$$f(t,x)\le \ell(t)x \ \ \text{for} \ \ a\le t\le b, \quad 0\le x\le 1. \tag{9}$$

Then every nonnegative solution to problem $(1),(2)$ is positive on $[a,b]$.

Also, if the conditions of Corollary 1 are satisfied and

$$\limsup_{x>0,\, x\to 0} \frac{f_i(x)}{x}<+\infty \ \ (i=1,\dots,k), \tag{10}$$

then every nonnegative solution to problem $(1_1),(2_1)$ is positive on $[a,b]$.

Example 1. Consider the problem

$$u'(t) = 2|u(t)|^{\frac{1}{2}}, \tag{11}$$

$$u(b) = (b - t_0)^2, \tag{12}$$

where $t_0 \in]a, b[$.

It is evident that the function

$$u(t) = \begin{cases} 0 & \text{for } a \le t \le t_0, \\ (t - t_0)^2 & \text{for } t_0 < t \le b \end{cases}$$

is a nonnegative solution to problem $(11), (12)$.

On the other hand, problem $(11), (12)$ coincides with problem $(1), (2)$ when $f(t, x) \equiv 2|x|^{\frac{1}{2}}$, $\varphi(u) = u(b) - (b - t_0)^2$, and $\tau(t) \equiv t$ (coincides with problem $(1_1), (2_1)$ when $k = m = \lambda_1 = 1$, $p_1(t) \equiv 2$, $f_1(x) \equiv |x|^{\frac{1}{2}}$, $\tau(t) \equiv t$, $c_0 = (b - t_0)^2$, $\sigma_1(s) = 0$ for $a \le s < b$, and $\sigma_1(b) = 1$). At that, in this case all the conditions of Theorem 1 (all the conditions of Corollary 1) are satisfied, but condition (9) (condition (10)) is violated.

The given example shows that if all the conditions of Theorem 1 (all the conditions of Corollary 1) hold but condition (9) (condition (10)) is violated, then problem $(1), (2)$ (problem $(1_1), (2_1)$) may have such a nonnegative solution which is zero in certain subintervals of $[a, b]$.

Remark 3. Condition (6) in Theorem 1 cannot be replaced by the condition

$$\liminf_{u(b_0) \to +\infty} \varphi(u) > 0, \tag{13}$$

no matter how $b_0 \in [a, b[$ is.

Indeed, if $f(t, x) \equiv x^2$, $\tau(t) \equiv t$, $b_0 \in [a, b[$, $\varphi(u) = u(b_0) - (b_0 - a)^{-1}$, then problem $(1), (2)$ takes the form

$$u'(t) = u^2(t), \tag{14}$$

$$u(b_0) = (b_0 - a)^{-1}. \tag{15}$$

At that, in this case all the conditions of Theorem 1 are satisfied except condition (6) instead of which condition (13) holds. On the other hand, problem $(14), (15)$ does not have a solution defined on $[a, b]$. This problem has only a blow-up solution

$$u(t) = (b - t)^{-1}.$$

This example also shows that if conditions (4), (7) hold and inequality (8) is violated, then problem $(1_1), (2_1)$ may not have a nonnegative solution.

Theorem 2. *Let conditons* (3)–(6) *hold. If, moreover, f is nondecreasing and locally Lipschitz in the second argument on the set $[a, b] \times \mathbb{R}_+$, and φ is an increasing functional on the set $C([a, b]; \mathbb{R}_+)$, then problem* $(1), (2)$ *has one and only one positive solution.*

Corollary 2. *If conditions* (4), (7), (8) *are satisfied, and f_i $(i = 1, \dots, k)$ are nondecreasing and locally Lipschitz functions on $\mathbb{R}_+$, then problem* $(1_1), (2_1)$ *has one and only one positive solution.*

In the case, where condition (3) is violated, for equation (1) we consider the nonlocal problem with a linear boundary condition

$$\int_a^b u(s)\, d\sigma(s) = c_0, \tag*{(2_2),}$$

where $\sigma : [a, b] \to \mathbb{R}$ is a function with finite variation such that $\sigma(b) \ne \sigma(a)$, and c_0 is a real constant.

The following lemma is true.

Lemma 1. *If $q \in L([a,b])$, then the differential equation*

$$u'(t) = q(t)$$

has a unique solution, satisfying condition (2_2), and this solution admits the representation

$$u(t) = \frac{c_0}{\sigma(b) - \sigma(a)} + \int_a^b g(t,s)q(s)\,ds,$$

where

$$g(t,s) = \begin{cases} \frac{\sigma(s)-\sigma(a)}{\sigma(b)-\sigma(a)} & \text{for } a \le s \le t, \\ \frac{\sigma(s)-\sigma(b)}{\sigma(b)-\sigma(a)} & \text{for } t < s \le b. \end{cases}$$

Due to Lemma 1, problem $(1),(2)$ is equivalent to the integral equation

$$u(t) = \frac{c_0}{\sigma(b) - \sigma(a)} + \int_a^b g(t,s)f(s,u(\tau(s)))\,ds. \tag{16}$$

We set

$$\rho = \sup\left\{|g(t,s)| : \ a \le t,\, s \le b\right\}.$$

Remark 4. If σ is a nondecreasing function, then $\rho = 1$.

By virtue of (16), the following two theorems are valid.

Theorem 3. *If*

$$\limsup_{r\to+\infty} \int_a^b \frac{f^*(s,r)}{r}\,ds < 1/\rho, \tag{17}$$

then problem $(1),(2_2)$ has at least one solution.

Theorem 4. *Let there exist a nonnegative function $\ell \in L([a,b])$ such that*

$$|f(t,x) - f(t,y)| \le \ell(t)|x-y| \ \text{ for } \ a \le t \le b, \quad x,y \in \mathbb{R}.$$

If, moreover,

$$\int_a^b \ell(s)\,ds < 1/\rho, \tag{18}$$

then problem $(1),(2_2)$ has a unique solution.

Theorem 4 yields the following corollary.

Corollary 3. *Let $p,q \in L([a,b])$, and $\tau : [a,b] \to [a,b]$ be a measurable function. If, moreover,*

$$\int_a^b |p(t)|\,dt < 1/\rho, \tag{19}$$

then the linear differential equation

$$u'(t) = p(t)u(\tau(t)) + q(t)$$

has a unique solution, satisfying condition (2_2).

Example 2. Consider the problem

$$u'(t) = p(t)u(a), \tag{20}$$

$$u(b) = c_0, \tag{21}$$

where p is a nonpositive function such that

$$\int_a^b |p(s)|\,ds = 1. \tag{22}$$

Any solution to equation (20) has the form

$$u(t) = x\left(1 - \int_a^t |p(s)|\,ds\right).$$

Hence, due to equality (22), we have $u(b) = 0$. Therefore, if $c_0 \neq 0$, then problem $(20), (21)$ has no solution.

On the other hand, $(20), (21)$ is a particular case of $(1), (2_2)$, when

$$f(t,x) = p(t)x, \quad \tau(t) \equiv a, \quad \sigma(s) = \begin{cases} 0 & \text{for } a \le s < b, \\ 1 & \text{for } s = b. \end{cases}$$

If we now take into account Remark 4, then it becomes evident that the strict inequalities (17), (18), and (19) in Theorems 3, 4 and Corollary 3 cannot be replaced by the nonstrict ones.

References

1. N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, *Introduction to the Theory of Functional-Differential Equations.* (Russian) "Nauka", Moscow, 1991.
2. N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, *Elements of the Modern Theory of Functional Differential Equations. Methods and Applications.* (Russian) Institute for Computer Studies, Moscow, 2022.
3. N. V. Azbelev and L. F. Rakhmatullina, Theory of linear abstract functional-differential equations and applications. *Mem. Differential Equations Math. Phys.* **8** (1996), 1–102.
4. R. Hakl, A. Lomtatidze, and J. Šremr, *Some Boundary Value Problems for First Order Scalar Functional Differential Equations.* Masaryk University, Brno, 2002.
5. I. Kiguradze and B. Půža, *Boundary Value Problems for Systems of Linear Functional Differential Equations.* Masaryk University, Brno, 2003.

(Received 23.08.2026)

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 Merab Aleksidze II Lane, Tbilisi 0193, Georgia

Faculty of Exact and Natural Sciences, I. Javakhishvili Tbilisi State University, 3 Ilia Chavchavadze Avenue, Tbilisi 0179, Georgia

Email address: nino.partsvania@tsu.ge