On locally finite point sets in the plane $\mathbf{R}^2$ and associated decompositions of $\mathbf{R}^2$

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Abstract. Some Delaunay systems Z (in a certain generalized sense) are considered in the plane $\mathbf{R}^2$ and associated with Z two basic parameters r_Z and R_Z are examined.

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Let Z be a point system in the Euclidean plane $\mathbf{R}^2$, satisfying the following two conditions:

(D1) there exists a real number $R > 0$ such that every disc in $\mathbf{R}^2$ with radius R contains at least one point from Z;

(D2) there exists a real number $r > 0$ such that every disc in $\mathbf{R}^2$ with radius r contains at most one point from Z.

In this case Z is called a Delaunay system in the plane $\mathbf{R}^2$ (see the celebrated work by Delaunay [8]) and the above parameters R and r can be denoted by R_Z and r_Z.

In the same work [8] it was proved that if Z is a Delaunay system in $\mathbf{R}^2$, then there exists a uniquely determined decomposition $\{P_i : i \in I\}$ of $\mathbf{R}^2$ into (non-degenerate) convex polygons, such that:

(a) $Z = \cup\{v(P_i) : i \in I\}$, where for each index $i \in I$ the symbol $v(P_i)$ denotes, as usual, the set of all vertices of P_i;

(b) for every $i \in I$, the set $v(P_i)$ is contained in some circumference $S_i \subset \mathbf{R}^2$ and the open disc whose boundary is S_i does not contain any point of Z;

(c) if $i \in I$, $j \in I$ and $i \neq j$, then $S_i \neq S_j$ and the set $P_i \cap P_j$ is either empty, or a common vertex of S_i and S_j, or a common side of S_i and S_j.

Actually, the relation (c) means that the decomposition $\{P_i : i \in I\}$ of $\mathbf{R}^2$ is cellular. The existence of $\{P_i : i \in I\}$ can be demonstrated following a rather simple argument (see [8]; cf. also [9]). Namely, take a sufficiently small disc B in $\mathbf{R}^2$ not containing any point from Z and then, gradually increasing the radius of B and moving its center in an appropriate direction, catch some points from Z on the boundary of B, so that they would constitute the set of all vertices of a certain convex polygon $P \in \{P_i : i \in I\}$. Then one can continue this process in order to find all other polygons from $\{P_i : i \in I\}$.

As known, the decompositions of $\mathbf{R}^2$ associated with Delaunay systems are closely connected with the Dirichlet-Voronoi decompositions of $\mathbf{R}^2$ (cf. [12]). In fact, there is a deep duality between these two important geometric objects. It may happen that the decomposition $\{P_i : i \in I\}$ associated with a given Delaunay system Z in $\mathbf{R}^m$ is such that all polygons P_i $(i \in I)$ are triangles. In this case, $\{P_i : i \in I\}$ is usually called a Delaunay triangulation of $\mathbf{R}^2$.

Remark 1. Let Z be a Delaunay system in the plane $\mathbf{R}^2$ and let $\{P_i : i \in I\}$ denote the decomposition of $\mathbf{R}^2$ into convex polygons, associated with this Z. It is not hard to show that there exists a real number $d(Z) = d > 0$ such that every polygon P_i $(i \in I)$ contains a disc of diameter d (i.e., the width of P_i is greater than or equal to d). Indeed, denote by R $(= R_Z)$ and r $(= r_Z)$, respectively, the two basic parameters of the given Delaunay system Z. Consider any polygon P_i and assume, without loss of generality, that P_i is a triangle. Let a, b, and c stand for the side lengths of P_i. Obviously, $2r < \min\{a, b, c\}$. If R^* denotes the radius of the circumscribed circle of P_i, then $R^* \leq R$. By virtue of the well-known formula from elementary geometry

$$2R^* h_c = ab,$$

one has $h_c \geq ab/2R$ and $1/h_c \leq R/2r^2$. If r^* denotes the radius of the inscribed circle of P_i, then

$$1/r^* = 1/h_a + 1/h_b + 1/h_c \leq 3R/2r^2$$

or, equivalently, $r^* \geq 2r^2/3R$. So, any polygon P_i $(i \in I)$ contains a disc of diameter $d = 4r^2/3R$.

Let Z be a point system in the Euclidean plane $\mathbf{R}^2$, satisfying the following condition:

(D) Z is locally finite and every half-plane of $\mathbf{R}^2$ contains at least one point from Z.

We shall say that such a Z is a generalized Delaunay system in $\mathbf{R}^2$ (cf. [11], Chapter 5).

Clearly, any Delaunay system in $\mathbf{R}^2$ is simultaneously a generalized Delaunay system in $\mathbf{R}^2$. However, there are many generalized Delaunay systems Z in $\mathbf{R}^2$ for which both conditions (D1) and (D2) fail to be true.

Remark 2. For any subset T of $\mathbf{R}^2$, these two assertions are equivalent:

(1) every half-plane of $\mathbf{R}^2$ contains at least one point from T;

(2) for each straight line l in $\mathbf{R}^2$, the orthogonal projection of T to l is simultaneously coinitial and cofinal in l (with respect to the standard linear ordering of l).

The analogous equivalence holds true for the subsets T of the Euclidean space $\mathbf{R}^m$, where $m \geq 3$.

Remark 3. It is not difficult to construct a locally finite point set T in $\mathbf{R}^2$ such that, for each straight line l in $\mathbf{R}^2$, the orthogonal projection of T to l is an everywhere dense subset of l. Moreover, one may additionally suppose that, for such a T, the inequality

$$\inf\{||z - z'|| \ : \ z \in T, \ z' \in T, \ z \neq z'\} \geq 1$$

holds true. Similar examples take place in $\mathbf{R}^m$, where $m \geq 3$.

The following statement holds true.

Theorem 1. *For any locally finite subset Z of the plane $\mathbf{R}^2$, these two assertions are equivalent:*

(1) $\mathbf{R}^2$ admits a unique decomposition $\{P_i : i \in I\}$ into convex polygons satisfying conditions (a), (b), and (c);

(2) every half-plane of $\mathbf{R}^2$ contains at least one point from Z.

Moreover, if (2) holds true for a locally finite subset Z of $\mathbf{R}^2$, then there is an algorithm for constructing the decomposition $\{P_i : i \in I\}$ of $\mathbf{R}^2$ with properties (a), (b), and (c).

In particular, this theorem shows that, for the existence of a decomposition $\{P_i : i \in I\}$ of $\mathbf{R}^2$ satisfying conditions (a), (b), (c), none of the assumptions (D1) and (D2) is necessary (cf. [11], Chapter 5, Exercise 23). Both of them can be replaced by one condition (D).

It is well known that if a subset Z of the Euclidean space $\mathbf{R}^m$ satisfies the natural analogs of (D1) and (D2), then there exists a unique cellular decomposition of $\mathbf{R}^m$ into convex polyhedrons having the properties analogous to (a), (b), and (c) (see again [8], [9]). Theorem 1 also admits a generalization for $\mathbf{R}^m$, where $m \geq 2$ (in fact, it suffices to consider the case $m = 3$). Notice by the way that the proof of Theorem 1 and of its generalized version involves some limit processes and uses the compactness of the unit sphere $\mathbf{S}_{m-1}$.

Remark 4. In connection with Theorem 1, the following simple facts should be mentioned:

(*) there exists a generalized Delaunay system U in the plane such that $R_U = +\infty$ and $r_U > 0$;

(**) there exists a generalized Delaunay system V in the plane such that $R_V = +\infty$ and $r_V = 0$;

(***) there exists a generalized Delaunay system W in the plane such that $R_W < +\infty$ and $r_W = 0$.

Also, a generalized Delaunay system Y in $\mathbf{R}^2$ can be chosen so that $R_Y < +\infty$, all members of the decomposition $\{P_i : i \in I\}$ of $\mathbf{R}^2$ associated with Y are rectangles, and there is no real $d > 0$ such that any rectangle P_i ($i \in I$) contains a disc of diameter d.

Theorem 2. *There exists a generalized Delaunay system Z in $\mathbf{R}^2$ such that:*

(1) the covering $\{P_i : i \in I\}$ of $\mathbf{R}^2$ associated with Z consists of quadrangles and pairwise congruent regular triangles;

(2) for every real $\varepsilon > 0$, there is a quadrangle from $\{P_i : i \in I\}$ one side of which has length strictly less than ε;

(3) there exists a real $d > 0$ such that the widths of all members of the covering $\{P_i : i \in I\}$ are greater than or equal to d;

(4) the circumradii of all members of $\{P_i : i \in I\}$ are pairwise equal.

Thus, for the generalized Delaunay system Z of Theorem 2, one has $r_Z = 0$ and $R_Z < +\infty$.

Remark 5. According to Blaschke's old theorem, if the width of a compact convex body C in the Euclidean space $\mathbf{R}^m$ is $w(C)$, then there exists a ball B in $\mathbf{R}^m$ entirely contained in C and having the radius greater than or equal to $w(C)/(m+1)$. Keeping in mind this theorem for $m = 2$ and the facts presented above, one can conclude that the following three assertions are equivalent for every generalized Delaunay system Z which produces a triangulation of $\mathbf{R}^2$ and satisfies the inequality $R_Z < +\infty$:

(1) for any real $\varepsilon > 0$, in the triangulation of $\mathbf{R}^2$ associated with Z there is a triangle at least one side of which has length strictly less than ε;

(2) for any $\varepsilon > 0$, in the triangulation of $\mathbf{R}^2$ associated with Z there is a triangle whose width is strictly less than ε;

(3) for any real $\varepsilon > 0$, in the triangulation of $\mathbf{R}^2$ associated with Z there is a triangle whose incircle has radius strictly less than ε.

Remark 6. Blaschke's classical theorem mentioned above is not difficult to prove by using a direct argument. However, it should be noticed that this theorem is a logical consequence of some well-known results of convex geometry, which are discussed in many research works, monographs and text-books (see, for instance, [1], [2], [5], [6], [10]). Actually, Blaschke's theorem follows from the conjunction of these two statements:

(H) Helly's theorem on the nonempty intersection of a family of compact convex sets in $\mathbf{R}^m$ such that any at most $m + 1$ members of the family have nonempty intersection;

(B) Bang's theorem that positively solves Tarski's plank problem.

Concerning (H), see e.g. [2], [6], [7], [10], [11]. For (B), we refer the reader to [3] and [4].

Remark 7. It is easy to see that if a point system Z in $\mathbf{R}^2$ is such that any half-plane of $\mathbf{R}^2$ contains at least one element of Z, then for each point $z \in Z$ one has

$$z \in \mathrm{conv}(Z \setminus \{z\}),$$

where conv$(\cdot)$ denotes the operation of taking the convex hulls of subsets of $\mathbf{R}^2$.

On the other hand, there is a locally finite point system Z_0 in $\mathbf{R}^2$ such that:

(1) Z_0 is a set of points in general position in $\mathbf{R}^2$;

(2) $(\forall z \in Z_0)(z \in \mathrm{conv}(Z_0 \setminus \{z\}))$;

(3) conv(Z_0) coincides with some open half-plane of $\mathbf{R}^m$.

Obviously, in view of (3), Z_0 cannot be a generalized Delaunay system in $\mathbf{R}^2$.

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