ON A FAMILY OF INVARIANT MEASURES IN INFINITE DIMENSIONAL LINEAR POLISH SPACES

ALEKS KIRTADZE

A topological space E is a Polish space if E is homeomorphic to a separable metric space. In other words, E is a Polish space if its topological structure can be enriched with the structure of a complete separable metric space.

The next statement is valid.

Lemma 1. For an infinite dimensional separable Banach space X and an infinite dimensional Polish linear space E there is an injective linear operator

$$T : X \to E$$

with dense range.

About of Lemma 1 see [1] (see also [7]).

According to Lemma 1 and the Luzin-Souslin theorem (see [4]) we obtain that if E is an infinite dimensional linear Polish space then there exists a continuous one-to-one compact linear operator

$$T : l_2 \to E$$

such that $T(l_2)$ is a danse Borel subset of E and T^{-1} is a Borel map.

By using the method of inductive limits of finite dimensional invariant measures, A. Kharazishvili constructed a nonzero σ-finite Borel measure χ in infinite dimensional separable Hilbert space l_2, which is invariant under the everywhere dense vector subspace consisting of all finite sequences in l_2 (see [5]).

In the [3] was constructed a nonzero, σ-finite, $T(l_2)$-invariant measure μ defined by formula

$$\mu(A) = \chi(T^{-1}(T(l_2) \cap A))$$

where A is a Borel subset of E.

According to the [2] the next statement is true.

Theorem 1. In an infinite-dimensional linear Polish space there exists a family of non-zero σ-finite invariant Borel measures with cardinality equal to $2^{2^{\mathbf{c}}}$ and each measure lies on a meager linear subspace of E, where $\mathbf{c}$ is cardinality of the continuum.

In the present paper, we construct a family of σ-finite invariant weakly separated measures and estimate the cardinality of this family.

For our purposes, we need the following notations and statements.

A family of σ-finite measures $\{\mu_i : i \in I\}$ defined on a measurable space E is called weakly separated if there exists a family $\{X_i : i \in I\}$ of measurable subsets of E, such that

$$\mu_i(X_j) = 0 \qquad (i \in I, j \in I, i \neq j)$$

and

$$\mu(E \setminus X_i) = 0 \qquad (i \in I).$$

A subset X of E is called μ-massive (or μ-thick) if $\mu_*(E \setminus X) = 0$, where μ_* denotes the inner measure associated with μ;

let E be an infinite set and let $\{X_i : i \in I\}$ be a family of subsets of E. We say that this family is almost disjoint if the following conditions hold:

(a) $card(X_i) = card(E)$ for arbitrary $i \in I$;

(b) $card(X_i \cap X_j) < card(E))$ $(i \in I, j \in I, i \neq j)$.

The notion of an almost disjoint family of sets was first introduced and investigated by W. Sierpinski (see [8]).

Example 1. If $E = \mathbf{N}$ than in the theory (**ZF**) there exists an almost disjoint family $\{X_i : i \in I\}$ of subsets of E, such that

$$card(I) = \mathbf{c}.$$

The next theorem is due to W. Sierpinski.

Lemma 2. Let E be an infinite set. Then there exists an almost disjoint family $\{X_i : i \in I\}$ of subsets of E such that

$$card(I) > card(E).$$

Remark 1. Assume the Generalized Continuum Hypothesis. Then according to the lemma 2 for each an infinite set E, there exists an almost disjoint family $\{X_i : i \in I\}$ of subsets of E such that

$$card(I) = card(P(E)),$$

where $P(E)$ is the set of all subsets of E.

The next statement is valid.

Lemma 3. let E be an infinite set and let $\{Y_j : j \in J\}$ be a family of subsets of E such that:

(1) $card(K) \leq card(E)$;

(2) $card(Y_j) = card(E) \qquad (j \in J)$.

Then there exists an almost disjoint family $\{X_i : i \in I\}$ of subsets of E, satisfying the following relations:

(3) $card(I) > card(E)$;

(4) $card(X_i \cap Y_j) = card(E)) \qquad (i \in I, j \in J)$;

(5) $\cup\{X_i : i \in I\} = E$.

The proofs of the Lemma 2 and the Lemma 3 see [6].

Example 2. If $E = \mathbf{R}$ and taking the family of all nonempty perfect subsets of $\mathbf{R}$ as $\{Y_j : j \in J\}$, that there exists an almost disjoint family $\{X_i : i \in I\}$ of Bernstein sets in for which

$$card(I) > \mathbf{c}.$$

Theorem 2. In infinite dimensional linear Polish spaces there exist a family of σ-invariant weakly separated measures $\{\nu_i : i \in I\}$ such that:

$$card(I) > \mathbf{c}.$$

Proof. Let $\{Y_j : j \in J\}$ be an injective family of all nonempty perfect subsets of an infinite dimensional linear Polish space E. It is clear that

$$card(J) = \mathbf{c}, \quad card(Y_j) = \mathbf{c}.$$

According to the Lemma 3 there exists a family $\{X_i : i \in I\}$ of subsets of E such that
1. $card(I) > \mathbf{c}$;
2. $card(X_i \cap X_{i'}) < \mathbf{c}, \quad (i \in I, i' \in I, i \neq i')$;
3. $card(X_i \cap Y_j) = \mathbf{c}, \quad (i \in I, j \in J)$;
4. $\cup(\{X_i : i \in I\}) = E$.

Now let $\mathbf{S}$ denote the σ-algebra generated by the union

$$dom(\mu) \cup \{X_i, X_i'\}$$

and define the functional

$$\mu_i((X_i \cap Z_1) \cup (X_i' \cap Z_2)) = \mu(Z_1) \quad (i \in I),$$

where $X_i' = E \setminus X_i$, $Z_1 \in dom(\mu)$, $\quad Z_2 \in dom(\mu)$.

The well-definedness of the functional μ_i follows directly from the fact that the set X_i is μ-massive. It is verified that the functional μ_i is a measure and that the following equality holds:

$$\mu_i(E \setminus X_i) = 0.$$

Let $F(E)$ be a class of all non-continuum subsets of E. That is

$$F(E) = \{X : X \subset E, card(X) < \mathbf{c}\}.$$

From the definition of the measure $\mu_i \ (i \in I)$ we obtain that if $Z \in F(E)$, than

$$\mu_i(Z) = 0.$$

For each index $i \in I$ consider the completion of the measure μ_i and extend it to the all subsets of $F(E)$ and denote the resulting measure by $\overline{\mu_i}$.

Taking this into account, we obtain that

$$\overline{\mu_i}(X_{i'}) = 0 \quad (i \neq i').$$

Finally, let $\mathbf{S_1}$ denote the σ-algebra generated by the union

$$dom(\mu) \cup F(E) \cup (\cup\{X_i : i \in I\}).$$

Suppose that

$$\nu_i = \mu_i|_{\mathbf{S}_1}.$$

It follows from the above that the σ-algebra $\mathbf{S}_1$, the family of measures $\{\nu i : i \in I\}$, and the family of sets $\{X_i : i \in I\}$ satisfy the conditions of the theorem, thus completing the proof of Theorem 2.

Remark 2. Example 2 shows that the family of sets $\{X_i : i \in I\}$ obtained in Theorem 2 may be taken to be a family of Bernstein subsets of an infinite dimensional linear Polish spaces.

References.

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Email address: `kirtadze2@yahoo.com`

Georgian Technical University, Kostava str. 77, Tbilisi 0175, TSU A. Razmadze Mathematical Institute, 2, Merab Aleksidze II Lane, Tbilisi 0193, Georgia