Analyticity of the Djerbashyan Canonical Product on the Boundary of the Unit Circle

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Abstract: The paper establishes necessary and sufficient conditions for the Djerbashyan canonical product to be analytic at points on the boundary of the unit circle.

The Blaschke product plays an important role in the study of boundary values of analytic functions in the unit disk (Blaschke, 1915) and has the form

$$B\big(z,(a_n)\big) = z^{\lambda}\prod_{n=1}^{\infty}\frac{|a_n|}{a_n}\frac{a_n - z}{1-\overline{a_n}z} = z^{\lambda}\prod_{n=1}^{\infty}\frac{1}{|a_n|}\left(1-\frac{1-|a_n|^2}{1-\overline{a_n}z}\right),$$

where $\lambda+1$ is a natural number,

$$0<|a_n|\le|a_{n+1}|<1, \qquad \lim_{n\to\infty}|a_n|=1,$$

and

$$\sum_{n=1}^{\infty}(1-|a_n|)<+\infty.$$

In the case when

$$\sum_{n=1}^{\infty}(1-|a_n|)=+\infty,$$

there are different generalizations of the Blaschke product (Picard, 1926; Djerbashyan, 1945, 1948, 1961; Tetvadze, 1980; Tsuji, 1955).

The Djerbashyan canonical product (Djerbashyan, 1945) has the form

$$B_p\big(z,(a_n)\big) = z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{1-|a_n|^2}{1-\overline{a_n}z}\right)\exp\left(\sum_{k=1}^{p}\frac{1}{k}\left(\frac{1-|a_n|^2}{1-\overline{a_n}z}\right)^k\right),$$

where $\lambda+1$ and p are natural numbers,

$$0<|a_n|\le|a_{n+1}|<1, \qquad \lim_{n\to\infty}|a_n|=1, \qquad |z|<1,$$

$$\sum_{n=1}^{+\infty}(1-|a_n|)^p<+\infty.$$

The infinite product $B_p\big(z,(a_n)\big)$ is uniformly and absolutely convergent inside the open unit disk; therefore, it defines an analytic function with zeros

$$\underbrace{0,0,\ldots,0}_{\lambda\text{ times}},\ a_1,\ a_2,\ \ldots,\ a_n,\ \ldots$$

Note that the Djerbashyan canonical product is a particular case of the Djerbashyan product (Djerbashyan, 1948).

Let

$$A = \{a_1, a_2, \dots, a_n, \dots\},$$

and denote by A' the set of its accumulation points.

Tanaka (1963) proved that

Theorem 1. *If $e^{i\theta} \notin A'$, then there exists a neighborhood of the point $e^{i\theta}$ in which the Blaschke product $B\big(z, (a_n)\big)$ converges uniformly and absolutely.*

Result 1. *The point $e^{i\theta}$ is a singular point of the Blaschke product if and only if $e^{i\theta} \in A'$.*

Result 2. *If the Blaschke product $B\big(z, (a_n)\big)$ is analytic at the point $e^{i\theta}$, then there exists a neighborhood of the point $e^{i\theta}$ in which $B\big(z, (a_n)\big)$ converges absolutely and uniformly.*

In the following we consider analogous questions for the Djerbashyan canonical product.

As is known, an infinite product

$$\prod_{n=1}^{\infty}(1 \pm u_n)$$

is called absolutely convergent if the series

$$\sum_{n=1}^{\infty}\ln(1 \pm u_n)$$

is absolutely convergent.

Theorem 2. *If $A = \{a_1, a_2, \dots, a_n, \dots\}$ and $e^{i\theta} \notin A'$, then there exists a neighborhood of the point $e^{i\theta}$ in which the Djerbashyan canonical product converges absolutely and uniformly, and hence it is an analytic function in this neighborhood.*

Result 3. *The point $z = e^{i\theta}$ is a singular point of $B_{p+1}\big(z, (a_n)\big)$ if and only if $e^{i\theta} \in A'$.*

Result 4. *If $B_p\big(z, (a_n)\big)$ is analytic at the point $z = e^{i\theta}$, then there exists a neighborhood of $e^{i\theta}$ in which $B_{p+1}\big(z, (a_n)\big)$ is absolutely and uniformly convergent.*

Proof. If $B_p\big(z, (a_n)\big)$ is analytic in a neighborhood of $e^{i\theta}$, then according to the Result 1 we have $e^{i\theta} \notin A'$. By Theorem 4, $B_p\big(z, (a_n)\big)$ is absolutely and uniformly convergent in some neighborhood of the point $e^{i\theta}$. □

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