Transactions of A. Razmadze
Mathematical Institute
Vol. **1??** (202?), ?–??

APPLYING THE DEEP NEURAL NETWORK FOR NUMERICAL SOLUTION OF ONE FOURTH-ORDER NONLINEAR INTEGRO-DIFFERENTIAL MODEL

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Abstract. The work is concerned with the numerical solution of the initial-boundary value problem for a fourth-order nonlinear partial integro-differential equation. Considered model is a generalization of the second-order nonlinear partial integro-differential equation based on the Maxwell system. Algorithms for finding approximate solutions applying deep neural networks are used. Results of numerical experiments with graphical illustrations and their analysis are given.

Integro-differential equations occur in many applications. Numerous scientific works, monographs and textbooks are devoted to the research of the integro-differential models. The integro-differential equations, often encountered in physics and mathematics, contain derivative of various variables; therefore, these equations are called as integro-differential equations with partial derivatives or partial integro-differential equations (PIDEs).

The purpose of the present note is to study of one fourth-order nonlinear PIDE. The process of electromagnetic field penetration in the substance is described by well-known system of Maxwell equations, which describes the process of propagation of an electromagnetic field into a medium [1]. The literature on the questions of existence, uniqueness, regularity, asymptotic behavior of the solutions and numerical resolutions of the initial-boundary value problems for these types of models is very rich (see, for example, [2] - [5] and references therein).

Note that some of such integro-differential models in the scalar case have the following form

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a\left(\int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau\right)\frac{\partial U}{\partial x}\right]. \tag{1}$$

Such type models have arisen for the first time in [6]. In [7] [9] the unique solvability of the initial-boundary value problems for equations of (1) type is given for rather general assumptions on the function $a = a(S)$, than in [6]. The existence theorems that are proved in the above-mentioned works are based on a priori estimates, modified version of the Galerkin method and on compactness arguments as in [10], [11] for nonlinear parabolic equations.

Based on the works [6] [9], the models of the following type

$$\frac{\partial U}{\partial t} = a\left(\int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau\right)\frac{\partial^2 U}{\partial x^2}, \tag{2}$$

appeared in [12], and the author named those models as the averaged integro-differential equations (AIDEs). Here, $a = a(S)$, as in (1), is a given function defined for $S \in [0, \infty)$. In [12] the author mentioned that investigation of (2) type models requires somewhat different approach, than that of the Volterra-type models (1). The existence and uniqueness of the solutions of the initial-boundary value problems for the AIDEs of type (2) were first studied in [13].

The literature on the questions of the existence, uniqueness, regularity, asymptotic behavior of the solutions and numerical resolutions of the initial-boundary value problems for (1) and (2) type models are very rich. The first publication of (1) type models quickly attracted attention of scientists, and in a short period, following [6] - [9], many scientific papers were published. The numerous scientific works are devoted to the construction and justification of algorithms of the numerical resolution of initial-boundary value problems

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AMS Subject Classification. 45K05, 35K55, 35Q61, 68T05, 68T07.
Key words and phrases. Nonlinear integro-differential equations, Maxwell systems, deep neural networks.
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for the above-stated (1) and (2) models. The possibility of finite difference approximations, application of Galerkin methods and the method of finite elements, as well as the algorithms for their realization are discussed, for example, in [14] - [22]. For relatively complete citations on this study, up to 2019, we refer the reader to [21] and [22]. Since then, this interest has not been reduced but increased. Many works are devoted to the multidimensional and to the higher order integro-differential equations of (1) and (2) types (see, for example, [8], [9], [12] - [14]).

The purpose of the present work is to continue our study. The work aims to formulate the problem rigorously, discuss the analytical challenges, and provide computational strategies based exclusively on deep learning methods, particularly fully connected feedforward neural networks, for its approximate solution (see, for example, [23] - [26]).

The presented work discusses a natural mathematical generalization of the integro-differential models (1) and (2). In particular, in the domain $[0,1]\times[0,T]$, where T is fixed positive number following initial-boundary value problem for the corresponding fourth-order integro-differential equation is investigated:

$$\frac{\partial U}{\partial t}+\frac{\partial^2}{\partial x^2}\left\{\left[1+\int_0^t\left(\frac{\partial U}{\partial x}\right)^2 d\tau\right]\frac{\partial^2 U}{\partial x^2}\right\}=f(x,t), \tag{3}$$

$$U(0,t)=U(1,t)=0, \tag{4}$$

$$\frac{\partial^2 U}{\partial x^2}(0,t)=\frac{\partial^2 U}{\partial x^2}(1,t)=0, \tag{5}$$

$$U(x,0)=U_0(x), \tag{6}$$

where $U=U(x,t)$ is an unknown function, and in equation (3) and in the initial condition (6) $f(x,t)$ and $u_0(x)$ are the given functions of their arguments, respectively.

The stability and uniqueness of the solution to (3) - (6) type problem for the Volterra-type models (1) is studied in [27]. Analogical results for same type nonlinearity are true for (3) - (6) AIDE problem too.

As we mentioned above, many works are devoted to the classical methods of approximate solution of initial-boundary value problems for (1) and (2) type models.

As we already said our goal is to explore an alternative method for solving problem (3) - (6) using Machine Learning techniques. Machine learning, specifically neural networks, is utilized to create surrogate models that predict the investigated solutions at any point within the domain. Neural networks, which consist of input, hidden, and output layers, offer flexibility in terms of architecture and the number of neurons per layer (see, for example, [23]). In this approach, the solution to the problem is approximated by a neural network output, and the network parameters are optimized during training. A significant advantage of using deep neural networks in solving problem is their ability to incorporate physical laws into the learning process, reducing the volume of required training data (as discussed in [23] - [26]).

The above-mentioned differential and integro-differential models using Machine Learning are studied in some other papers too (see, for example, [27] - [30] and references therein).

In our experiment, we have chosen the following exact solution to the problem (3) - (6) with the corresponding right hand side function $f(x,t)$ and initial condition $U_0(x)$:

$$U(x,t)=x^3(1-x)^3(1+t^2).$$

The numerical simulation was carried out using a neural network model implemented in the TensorFlow framework. Developing and training such a model can be implemented in Python environment, for example, with commonly used libraries such as TensorFlow or PyTorch, alongside scientific computing tools like NumPy, SciPy, and Matplotlib for visualization. While training can be done on a standard computer, the use of a GPU (graphics processing unit) is recommended for handling larger problems or reducing training time.

A convenient development environment is Jupyter Notebook hosted on Google Colab, which provides interactive coding capabilities and access to cloud-based computational resources. Google Colab is especially useful, as it supports all required Python libraries for deep learning and offers free GPU access.

To further illustrate the accuracy of model, Table presents numerical values comparing the exact and predicted solutions for selected values of x and t.

Table. Comparison between exact and predicted solutions

x	t	**Exact** $U(x,t)$	**Predicted** $U(x,t)$	**Absolute Error**
0.2	0.7	0.012713	0.01315	0.000438
0,4	0.7	0.028603	0.029312	0.000708
0.6	0.7	0.028603	0.029312	0.000708
0.8	0.7	0.012713	0.01315	0.000438

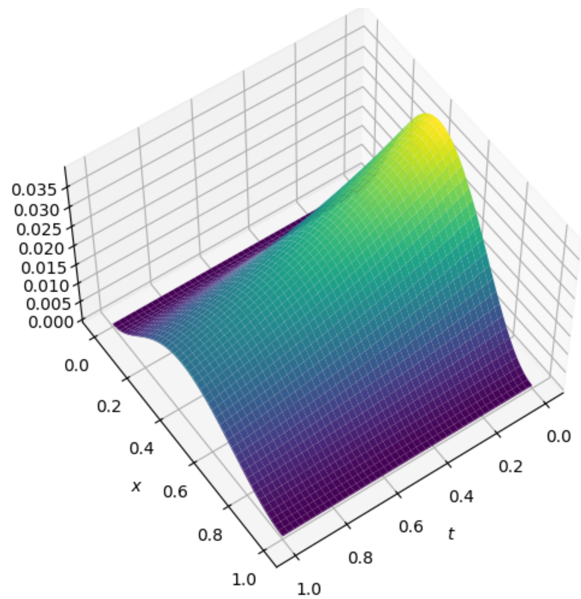

FIGURE 1. Exact solutions

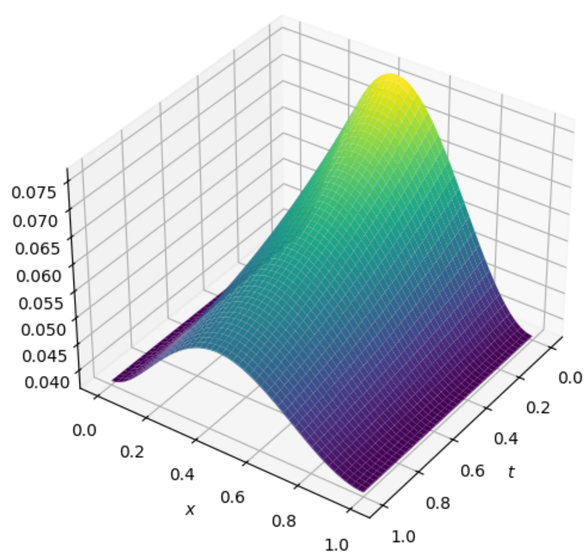

FIGURE 2. Predicted solutions

To visually evaluate the accuracy of the neural networks prediction, Figure 1 presents the approximate solutions, while Figure 2 shows the exact solutions in the form of 3D surface plots. These visualizations clearly indicate that the neural network's output closely matches the true solution across the spatio-temporal domain. Although small discrepancies are noticeable, they remain within acceptable limits, confirming the model's reliability and robustness.

This study proposes a deep learning-based method for solving fourth-order nonlinear parabolic-type integro-differential equations. A fully connected feedforward neural network is used to approximate the solution by embedding the equation, initial, and boundary conditions into a unified loss function.

Numerical results show that the model accurately captures the solutions behavior, with low error and reliable generalization. Both visual and quantitative analysis confirm the methods robustness.

Overall, the findings support deep learning as an effective alternative to traditional mesh-based approaches, especially for complex problems with nonlocal and nonlinear features.

References

1. L. Landau and E. Lifschitz, Electrodynamics of Continuous Media, Course of Theoretical Physics, Vol.8. (Translated from the Russian) Pergamon Press, Oxford-London-New York-Paris; Addision-Wesley Publishing Co., Inc., Reading, Mass., 1960; Russian original: Gosudarstv. Izdat. Tehn-Teor. Lit., Moscow, 1957.
2. I. O. Abuladze, D. G. Gordeziani, T. A. Jangveladze and T. K. Korshiya, Discrete models for a nonlinear magnetic-field scattering problem with thermal conductivity. (Russian) Differ. Uravn. **22** (1986), no. 7, 1119-1129; translation in Differ. Equ. **22** (1986), no. 7, 769-777.
3. T. A. Dzhangveladze, B. Ya. Lyubimov and T. K. Korshiya, On the numerical solution of a class of nonisothermic problems of the diffusion of an electromagnetic field. (Russian) Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy **18** (1986), 5-47.
4. G. I. Laptev, Mathematical singularities of a problem on the penetration of a magnetic field into a substance. (Russian) Zh. Vychisl. Mat. i Mat. Fiz. **28** (1988), no. 9, 1332-1345; translation in U.S.S.R. Comput. Math. and Math. Phys. **28** (1988), no. 5, 35-45 (1990).
5. P. Monk, Finite Element Methods for Maxwells Equations. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003.
6. D. G. Gordeziani, T. A. Dzhangveladze and T. K. Korshiya, Existence and uniqueness of the solution of a class of nonlinear parabolic problems. (Russian) Differ. Uravn. 19 (1983), no. 7, 11971207; translation in Differ. Equ. **19** (1983), 887-895.
7. T. A. Jangveladze, The first boundary value problem for a nonlinear equation of parabolic type. (Russian) Dokl. Akad. Nauk SSSR **269** (1983), no. 4, 839-842; translation in Sov. Phys., Dokl. **28** (1983), 323324.
8. T. Dzhangveladze, An Investigation of the First Boundary Value Problem for Some Nonlinear Parabolic Integro-differential Equations. (Russian) Tbilisi State University, Tbilisi, 1983.
9. T. A. Dzhangveladze, A nonlinear integro-differential equation of parabolic type. (Russian) Differ. Uravn. 21 (1985), no. 1, 41-46; translation in Differ. Equ. 21 (1985), no. 1, 32-36.
10. M. I. Vishik, Solubility of boundary-value problems for quasi-linear parabolic equations of higher orders. (Russian) Mat. Sb. (N.S.) **59** (101) 1962 suppl. 289325.
11. J.-L. Lions, Quelques Mthodes de Rsolution des Problmes aux Limites Non Linaires. (French) Dunod; Gauthier-Villars, Paris, 1969.
12. G. Laptev, Quasilinear Evolution Partial Differential Equations with Operator Coefficients (Russian), Doct. diss., Moscow, 1990.
13. T. Jangveladze, On one class of nonlinear integro-differential parabolic equations. Semin. I. Vekua Inst. Appl. Math. Rep. **23** (1997), 51-87.
14. G. I. Laptev, Quasilinear parabolic equations that have a Volterra operator in the coefficients. (Russian) Mat. Sb. (N.S.) **136** (178) (1988), no. 4, 530-545; translation in Math. USSR-Sb. **64** (1989), no. 2, 527-542.
15. Y. Lin and H.M. Yin, Nonlinear parabolic equations with nonlinear functionals, J. Math. Anal. Appl., **168** (1992), 28-41.
16. M.M. Aptsiauri, T.A. Jangveladze and Z.V. Kiguradze, Asymptotic behavior of the solution of a system of nonlinear integro-differential equations. Differ. Uravn. **48** (2012), 70-78 (Russian). English translation: Differ. Equ. **48** (2012), 72-80.
17. T. Jangveladze, Z. Kiguradze, B. Neta and S. Reich, Finite element approximations of a nonlinear diffusion model with memory. Numer. Algor. **64** (2013), 127-155.
18. Z.J. Zhou, F.X. Chen and H.Z. Chen. Convergence analysis of an -Galerkin mixed finite element method for one nonlinear integro-differential equation. Applied Mathematics and Computation, **220** (2013), 783-791.
19. N. Sharma, M. Khebchareon, K. Sharma and A.K. Pani, Finite element Galerkin approximations to a class of nonlinear and nonlocal parabolic problems. Numer. Meth. PDEs **32** (2016), 1232-1264.
20. F. Hecht, T. Jangveladze, Z. Kiguradze and O. Pironneau, Finite difference scheme for one system of nonlinear partial integro-differential equations. Appl. Math. Comput. **328** (2018), 287-300.
21. T. Jangveladze, Z. Kiguradze and B. Neta, Numerical Solutions of Three Classes of Nonlinear Parabolic Integro-Differential Equations. Elsevier/Academic Press, Amsterdam, 2016.
22. T. Jangveladze, Investigation and Numerical Solution of Nonlinear Partial Differential and Integro-Differential Models Based on System of Maxwell Equations. Mem. Differential Equations Math. Phys., **76** 1-118, 2019.
23. J. Blechschmidt and O. G. Ernst, Three Ways to Solve Partial Differential Equations with Neural Networks A Review. GAMM-Mitteilungen, **44.2**, (2021), e2021000062021.
24. M. Raissi, P. Perdikaris and G. E. Karniadakis, Physics Informed Deep Learning (Part I): Datadriven Solutions of Nonlinear Partial Differential Equations. arXiv 1711.10561, (2017).

25. M. Raissi, P. Perdikaris and G. E. Karniadakis, Physics Informed Deep Learning (Part II): Datadriven Discovery of Nonlinear Partial Differential Equations. arXiv 1711.10566, (2017).
26. M. Raissi, P. Perdikaris and G. E. Karniadakis, Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. Journal of Computational Physics, **378** (2019) 686-707.
27. T. Chkhikvadze, On one nonlinear integro-differential parabolic equation. Rep. Enlarged Sess. Semin. I.Vekua Inst. Appl. Math. **35**, (2021) 19-22.
28. Z. Kiguradze, Approximate solution for heat equation applying Neural Network. International Workshop on the Qualitative Theory of Differential Equations, QUALITDE - 2022, December 17-19, 2022, Tbilisi, Georgia, 135-138.
29. Z. Kiguradze, Numerical Solution for One Nonlinear Integro-differential Equation Applying Deep Neural Network. International Workshop on the Qualitative Theory of Differential Equations, QUALITDE - 2023, December 9-11, 2023, Tbilisi, Georgia 108-111.
30. T. Jangveladze, Z. Kiguradze and T. Chkhikvadze, Application of Deep Neural Network for numerical approximation for averaged nonlinear integro-differential equation. Bulletin of TICMI, **28** (2024), no. 1, 3-10.

(Received ??.??.20??)

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