LOGARITHMS AND STIRLING NUMBERS ASSOCIATED WITH DELTA SERIES

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ABSTRACT. This paper investigates the Stirling numbers of the first and second kind associated with a delta series $f(t)$, denoted by $S_1(n,k;f(t))$ and $S_2(n,k;f(t))$. These numbers provide a robust framework that satisfies the orthogonality and inverse relations, often lacking in recent probabilistic Stirling and B-Stirling numbers. Key contributions include the definition and analysis of the logarithm associated with a delta series $f(t)$. We further establish a Schlömilch-type formula, which provides an explicit connection between the two kinds of Stirling numbers. Using this formula, we derive another expression for the associated logarithm in terms of $S_2(n,k;f(t))$. Finally, we provide fifteen concrete examples to illustrate the versatility of this framework, demonstrating how it unifies and extends several known results in combinatorial analysis.

1. INTRODUCTION

The Bernoulli polynomials of order α and the Bernoulli numbers of order α are respectively defined by

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad B_n^{(\alpha)} = B_n^{(\alpha)}(0), \tag{1}$$

for any complex number α.
When $\alpha = 1$, $B_n(x) = B_n^{(1)}(x)$ are called the Bernoulli polynomials.

The Stirling numbers of the second kind $S_2(n,k)$ are characterized by

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \tag{2}$$

while the Stirling numbers of the first kind are specified as

$$\frac{1}{k!}\big(\log(1+t)\big)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad (x)_n = \sum_{k=0}^{n} S_1(n,k)x^k, \tag{3}$$

where $(x)_n$ are the falling factorial sequence given by

$$(x)_0 = 1, \quad (x)_n = x(x-1)\cdots(x-n+1), \quad (n \geq 1). \tag{4}$$

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Throughout this paper, let λ denote any nonzero real number. The degenerate exponentials are introduced as

$$(5)\qquad \begin{aligned} e_{\lambda}^{x}(t) &= (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!},\\ e_{\lambda}(t) &= e_{\lambda}^{1}(t) = (1+\lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{t^n}{n!}, \end{aligned}$$

where the degenerate falling factorial sequence is given by

$$(6)\qquad (x)_{0,\lambda} = 1,\quad (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda),\quad (n\geq 1).$$

The degenerate Bernoulli polynomials of order α and the degenerate Bernoulli numbers of order α are respectively defined by

$$(7)\qquad \Big(\frac{t}{e_{\lambda}(t)-1}\Big)^{\alpha}e_{\lambda}^{x}(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!},\quad \beta_{n,\lambda}^{(\alpha)} = \beta_{n,\lambda}^{(\alpha)}(0),$$

for any complex number α.
When $\alpha = 1$, $\beta_{n,\lambda}^{(1)}(x) = \beta_{n,\lambda}(x)$ are called the degenerate Bernoulli polynomials.

The degenerate Stirling numbers of the second kind are specified as (see [14, 20])

$$(8)\qquad \begin{aligned} \frac{1}{k!}\big(e_{\lambda}(t)-1\big)^{k} &= \sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!},\quad (k\geq 0),\\ (x)_{n,\lambda} &= \sum_{k=0}^{n}S_{2,\lambda}(n,k)(x)_{k},\quad (n\geq 0), \end{aligned}$$

while the degenerate Stirling numbers of the first kind are defined as (see [12, 20])

$$(9)\qquad \begin{aligned} \frac{1}{k!}\big(\log_{\lambda}(1+t)\big)^{k} &= \sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!},\quad (k\geq 0),\\ (x)_{n} &= \sum_{k=0}^{n}S_{1,\lambda}(n,k)(x)_{k,\lambda},\quad (n\geq 0). \end{aligned}$$

Here $\log_{\lambda}(1+t)$ is called the degenerate logarithm, which is the compositional inverse of $e_{\lambda}(t)-1$, and given by

$$(10)\qquad \log_{\lambda}(1+t) = \frac{1}{\lambda}((1+t)^{\lambda}-1) = \sum_{n=1}^{\infty}(\lambda-1)_{n-1}\frac{t^n}{n!}.$$

Taking the limits $\lambda\to 0$, we find that (see (5), (6), (7), (8), (9), (10))

$$\begin{aligned} &(x)_{n,\lambda}\to x^n,\quad e_{\lambda}^{x}(t)\to e^{xt},\quad e_{\lambda}(t)\to e^{t},\quad \log_{\lambda}(1+t)\to\log(1+t),\\ &\beta_{n,\lambda}^{(\alpha)}(x)\to B_{n}^{(\alpha)}(x),\quad S_{1,\lambda}(n,k)\to S_{1}(n,k),\quad S_{2,\lambda}(n,k)\to S_{2}(n,k). \end{aligned}$$

A formal power series $f(t) = \sum_{n=0}^{\infty}p_n\frac{t^n}{n!}$ is a delta series if $f(0) = 0$ and $f'(0)\neq 0$. If $f(t)$ is a delta series, then it has a compositional inverse $\bar{f}(t)$, which is also a delta series, satisfying $f(\bar{f}(t)) = t = \bar{f}(f(t))$. The connection of delta series to umbral calculus or Sheffer sequences are as follows. If $g(t)$ is an invertible series, namely it is a formal power series with $g(0) \neq 0$, and $f(t)$ is a delta series, then $s_n(x)$ is the Sheffer sequence for the pair $\big(g(t), f(t)\big)$ if

$$\sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k = \frac{1}{g\big(\bar{f}(t)\big)} e^{x\bar{f}(t)}.$$

Sheffer sequences occupy the central position in umbral calculus. We let the reader refer to [25] for more details. Let the sequence of polynomials, $\{p_n(x)\}_{n=0}^{\infty}$, or $p_n(x)$ for short, be given by the generating function

$$\sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!} = e^{x\bar{f}(t)}. \tag{11}$$

Then $p_n(x)$ are called the sequence associated with $f(t)$, which is indicated by $p_n(x) \sim (1, f(t))$. So the associated sequences are special cases of Sheffer sequences with $g(t) = 1$. We denote the sequence of polynomials by $\mathbf{P} = \{p_n(x)\}_{n=0}^{\infty}$ or $\mathbf{P} = \{p_n(x)\}$, for brevity. We also let

$$\sum_{n=0}^{\infty} p_n \frac{t^n}{n!} = e^{\bar{f}(t)}, \quad p_n = p_n(1). \tag{12}$$

We recall the following three Lagrange inversion formulas: for any formal power series $g(t)$ and any delta series $f(t)$, they are stated as (see [5, 25]):

(A) $[t^n]\, g(\bar{f}(t)) = \dfrac{1}{n}[t^{n-1}]\, g'(t)\Big(\dfrac{t}{f(t)}\Big)^n,$

(B) $[t^n]\, \big(\bar{f}(t)\big)^k = \dfrac{k}{n}[t^{n-k}]\, \Big(\dfrac{t}{f(t)}\Big)^n,$

(C) $[t^n]\, \bar{f}(t) = \dfrac{1}{n}[t^{n-1}]\, \Big(\dfrac{t}{f(t)}\Big)^n,$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$, n is any nonnegative integer and k is any positive integer.

Applying the formula (C) to $f(t) = e_\lambda(t) - 1$, we arrive at

$$[t^n]\, \log_\lambda(1+t) = \frac{1}{n}[t^{n-1}]\, \Big(\frac{t}{e_\lambda(t) - 1}\Big)^n = \frac{1}{n!}\beta_{n-1,\lambda}^{(n)}. \tag{13}$$

We see from (9), (10), and (13) that

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty} S_{1,\lambda}(n,1)\frac{t^n}{n!} = \sum_{n=1}^{\infty} (\lambda - 1)_{n-1}\frac{t^n}{n!} = \sum_{n=1}^{\infty} \beta_{n-1,\lambda}^{(n)}\frac{t^n}{n!}. \tag{14}$$

In addition, applying the formula (B) to $f(t) = e_\lambda(t) - 1$ gives

$$[t^n]\, \big(\log_\lambda(1+t)\big)^k = \frac{k}{n}\beta_{n-k,\lambda}^{(n)} \frac{1}{(n-k)!}. \tag{15}$$

Thus it follows from (15) that

$$\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k = \sum_{n=k}^{\infty} \binom{n-1}{k-1} \beta_{n-k,\lambda}^{(n)} \frac{t^n}{n!}, \quad (k \geq 0). \tag{16}$$

Now, one observes from (9) and (16) that

$$S_{1,\lambda}(n,k) = \binom{n-1}{k-1}\beta_{n-k,\lambda}^{(n)}, \quad (n \geq k). \tag{17}$$

Taking the limits $\lambda \to 0$ in (14) and (17) yields

$$S_1(n,1) = (-1)^{n-1}(n-1)! = B_{n-1}^{(n)}, \quad S_1(n,k) = \binom{n-1}{k-1} B_{n-k}^{(n)}. \tag{18}$$

The partial Bell polynomials are characterized by

$$B_{n,k}(x_1,x_2,\dots,x_{n-k+1}) = \sum_{\substack{j_1+j_2+\cdots=k \\ j_1+2j_2+\cdots=n}} \frac{n!}{j_1!j_2!\cdots}\left(\frac{x_1}{1!}\right)^{j_1}\left(\frac{x_2}{2!}\right)^{j_2}\cdots,$$

where $j_1, j_2, \dots$ are nonnegative integers. The generating function of the partial Bell polynomials is specified as

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1,x_2,\dots,x_{n-k+1})\frac{t^n}{n!}, \quad (k \geq 0). \tag{19}$$

General references for this paper include [5, 6, 23, 24, 25, 26, 27].

Let $\mathbf{P} = \{p_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials such that $\deg p_n(x) = n$ and $p_0(x) = 1$. In [7], the authors introduced Stirling numbers of the first and second kind associated with $\mathbf{P}$, defined respectively by the relations:

$$(x)_n = \sum_{k=0}^{n} S_1(n,k;\mathbf{P})p_k(x), \quad p_n(x) = \sum_{k=0}^{n} S_2(n,k;\mathbf{P})(x)_k. \tag{20}$$

These numbers satisfy fundamental orthogonality and inverse relations, generalizing the classical Stirling numbers.

Let Y be a random variable whose moment generating function exists in some neighborhood of the origin. Recently, Adell and Lekuona introduced the probabilistic Stirling numbers of the second kind associated with Y, denoted by $S_2^Y(n,k)$ [3]. This was followed by Adell and Bényi's definition of the probabilistic Stirling numbers of the first kind, $s_Y(n,k)$, based on the cumulant generating function [1]. While these initial formulations provided a foundational framework, they lacked the standard orthogonality properties necessary for broader application in inversion formulas. In addition, the definition of $s_Y(n,k)$ was not designed to recover the classical Stirling numbers of the first kind, $S_1(n,k)$, in the deterministic case where $Y = 1$.

To extend the utility of these concepts, a refined version of the probabilistic Stirling numbers of the first kind, $S_1^Y(n,k)$, was proposed in [8, 28]. This updated definition ensures that $S_1^Y(n,k)$ and $S_2^Y(n,k)$ form a consistent pair satisfying the expected orthogonality and inverse relations, while also maintaining compatibility with the classical $S_1(n,k)$ when $Y = 1$. Building on this refined approach, their degenerate counterparts–the probabilistic degenerate Stirling numbers of the second kind $S_{2,\lambda}^Y(n,k)$ and first kind $S_{1,\lambda}^Y(n,k)$–were established in [8]. These variants similarly preserve the requisite orthogonality and inverse properties, successfully reducing to the degenerate Stirling numbers $S_{1,\lambda}(n,k)$ under the condition $Y = 1$. Additionally, Adell and Bényi [2] generalized these to B-Stirling numbers associated with potential polynomials; however, these variants still lack the desired orthogonality properties.

In this paper, we investigate the Stirling numbers associated with a delta series $f(t)$ (i.e., $f(0) = 0$ and $f'(0) \neq 0$), denoted by $S_2(n,k;f(t))$ and $S_1(n,k;f(t))$, which unify probabilistic Stirling and B-Stirling numbers. They correspond to the Stirling numbers associated with polynomial sequence $\mathbf{P} = \{p_n(x)\}_{n=0}^{\infty}$, defined by the generating function $e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see (20)). They also correspond to the B-Stirling numbers with 'B' given by $B(t) = e^{\bar{f}(t)}$ (see [2]). Furthermore, the probabilistic Stirling numbers are the Stirling numbers associated with the delta series $f(t)$, whose compositional inverse is given by $\bar{f}(t) = \log E[e_{\lambda}^{Y}(t)]$ (see Example (a) in Section 3). These Stirling numbers satisfy orthogonality and inverse relations. As $\mathbf{P} = \{p_n(x)\}_{n=0}^{\infty}$ determines $f(t)$ and vice versa, $S_2(n,k;f(t))$ and $S_1(n,k;f(t))$ are also denoted respectively by $S_2(n,k;\mathbf{P})$ and $S_1(n,k;\mathbf{P})$. This correspondence extends to all related quantities whenever $\mathbf{P}$ and $f(t)$ are related by the generating function $e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}$. We define the logarithm associated with $f(t)$, $\log_{f(t)}(1+t)$, which is given by

$$\log_{f(t)}(1+t) = f(\log(1+t)) = \sum_{n=1}^{\infty} S_1(n,1;f(t))\frac{t^n}{n!} = \sum_{n=1}^{\infty} B_{n-1,\bar{f}(t)}^{(n)}\frac{t^n}{n!},$$

where $B_{n,f(t)}^{(\alpha)}(x)$ denotes the Bernoulli polynomials of order α associated with $f(t)$, defined by

$$\left(\frac{t}{e^{f(t)}-1}\right)^{\alpha} e^{xf(t)} = \sum_{n=0}^{\infty} B_{n,f(t)}^{(\alpha)}(x)\frac{t^n}{n!}.$$

It is worth noting that the generating function of $S_1(n,k;f(t))$ takes the form

$$\frac{1}{k!}\left(\log_{f(t)}(1+t)\right)^k = \sum_{n=k}^{\infty} S_1(n,k;f(t))\frac{t^n}{n!}.$$

We derive a Schlömilch-type formula for these Stirling numbers. In turn, this gives an alternative expression for the logarithm associated with $f(t)$:

$$\log_{f(t)}(1+t) = \sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\binom{2n-1}{n-1-j}(-1)^j\left(\bar{f}'(0)\right)^{-n-j} S_2(n-1+j,j;f(t))\frac{t^n}{n!}.$$

In Section 3, we provide fifteen concrete examples to illustrate our results, drawing primarily from [7] and [9]. We mention here two of them.

• Let $\mathbf{P} = \left\{(x)_{n,\lambda}\right\}$ be the sequence of degenerate falling factorials (see (6), (14), (17)). Then $(x)_{n,\lambda} \sim \left(1, f(t) = \frac{1}{\lambda}(e^{\lambda t}-1)\right)$ (see [17, 19]). We show that

$$S_2(n,k;\mathbf{P}) = S_{2,\lambda}(n,k), \quad S_1(n,k;\mathbf{P}) = S_{1,\lambda}(n,k) = \binom{n-1}{k-1}\beta_{n-k,\lambda}^{(n)},$$
$$\log_{\mathbf{P}}(1+t) = \log_{\lambda}(1+t) = \sum_{n=1}^{\infty} S_{1,\lambda}(n,1)\frac{t^n}{n!} = \sum_{n=1}^{\infty}\beta_{n-1,\lambda}^{(n)}\frac{t^n}{n!}.$$

• Let $\mathbf{P} = \{\mathrm{Bel}_n(x)\}$ be the sequence of Bell polynomials. Then it is given by (see [5, 24, 25])

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty}\mathrm{Bel}_n(x)\frac{t^n}{n!}, \quad \mathrm{Bel}_n(x) = \sum_{k=0}^{\infty} S_2(n,k)x^k.$$

Then $\mathrm{Bel}_n(x) \sim (1, \log(1+t))$. We find that

$$S_2(n,k;\mathbf{P}) = \sum_{l=k}^{n} S_2(n,l)S_2(l,k), \quad S_1(n,k;\mathbf{P}) = \sum_{l=k}^{n} S_1(n,l)S_1(l,k),$$
$$\log_{\mathbf{P}}(1+t) = \sum_{n=1}^{\infty}\sum_{l=1}^{n} S_1(n,l)S_1(l,1)\frac{t^n}{n!} = \log\left(1+\log(1+t)\right).$$

2. Logarithms and Stirling numbers associated with delta series

Let $f(t)$ be a delta series. So $f(0)=0$, and $f'(0)\neq 0$. Here we introduce the logarithm $\log_{f(t)}(1+t)$, called the logarithm associated with $f(t)$. It is also denoted by $\log_{\mathbf{P}}(1+t)$ and called the logarithm associated with $\mathbf{P}$. Here $\mathbf{P}=\{p_n(x)\}_{n=0}^{\infty}$, with $p_n(x)\sim(1,f(t))$. Thus we have (see (11))

$$\sum_{n}^{\infty} p_n(x)\frac{t^n}{n!}=e^{x\bar{f}(t)}.$$

We also recall that (see (12))

$$\sum_{n}^{\infty} p_n\frac{t^n}{n!}=e^{\bar{f}(t)}, \quad \text{with } p_n=p_n(1). \tag{21}$$

Note here that the coefficient of the linear term of (21) is given by $p_1=\bar{f}'(0)$.

We define the *Bernoulli polynomials of order* α *associated with* $f(t)$ *or* $\mathbf{P}$ by

$$\left(\frac{t}{e^{f(t)}-1}\right)^{\alpha}e^{xf(t)}=\sum_{n=0}^{\infty}B_{n,f(t)}^{(\alpha)}(x)\frac{t^n}{n!}, \tag{22}$$

for any complex number α.
If $f(t)=t$, then $B_{n,f(t)}^{(\alpha)}(x)=B_n^{(\alpha)}(x)$ are the classical Bernoulli polynomials of order α (see (1)); if $f(t)=\frac{1}{\lambda}\log(1+\lambda t)$, then $B_{n,f(t)}^{(\alpha)}(x)=\beta_{n,\lambda}^{(\alpha)}(x)$ are the degenerate Bernoulli polynomials of order α (see (7)).
When $\alpha=1$, they are called the *Bernoulli polynomials associated with* $f(t)$ *or* $\mathbf{P}$ and given by

$$\frac{t}{e^{f(t)}-1}e^{xf(t)}=\sum_{n=0}^{\infty}B_{n,f(t)}(x)\frac{t^n}{n!}.$$

The Stirling numbers of the second kind associated with $f(t)$ or $\mathbf{P}$, denoted by either $S_2(n,k;f(t))$ or $S_2(n,k;\mathbf{P})$, are defined by

$$\frac{1}{k!}\left(e^{\bar{f}(t)}-1\right)^k=\sum_{n=k}^{\infty}S_2(n,k;f(t))\frac{t^n}{n!}=\sum_{n=k}^{\infty}S_2(n,k;\mathbf{P})\frac{t^n}{n!}. \tag{23}$$

If $f(t)=t$, then $S_2(n,k;f(t))=S_2(n,k)$ are the ordinary Stirling numbers of the second (see (2)); if $f(t)=\frac{1}{\lambda}(e^{\lambda t}-1)$, then $S_2(n,k;f(t))=S_{2,\lambda}(n,k)$ are the degenerate Stirling numbers of the second kind (see (8)).
The Bell polynomials associated with $f(t)$, denoted by $\mathrm{Bel}_{n,f(t)}(x)$, are defined by

$$e^{x(e^{\bar{f}(t)}-1)}=\sum_{n=0}^{\infty}\mathrm{Bel}_{n,f(t)}(x)\frac{t^n}{n!}, \quad \mathrm{Bel}_{n,f(t)}(x)=\sum_{n=0}^{n}S_2(n,k;f(t))x^k.$$

The Stirling numbers of the first kind associated with $f(t)$ or $\mathbf{P}$, denoted by either $S_1(n,k;f(t))$ or $S_1(n,k;\mathbf{P})$, are defined by

$$\frac{1}{k!}\left(\bar{e}_{f(t)}(t)\right)^k=\sum_{n=k}^{\infty}S_1(n,k;f(t))\frac{t^n}{n!}=\sum_{n=k}^{\infty}S_1(n,k;\mathbf{P})\frac{t^n}{n!}, \tag{24}$$

where $\bar{e}_{f(t)}(t)$ is the compositional inverse of $e_{f(t)}(t)=e^{\bar{f}(t)}-1$. Here we note that $e_{f(t)}(t)$ is a delta series, since $e_{f(t)}(0)=0$, $e'_{f(t)}(0)=\bar{f}'(0)\neq 0$.
If $f(t)=t$, then $S_1(n,k;f(t))=S_1(n,k)$ are the ordinary Stirling numbers of the first kind (see (3)); if $f(t)=\frac{1}{\lambda}(e^{\lambda t}-1)$, then $S_1(n,k;f(t))=S_{1,\lambda}(n,k)$ are the degenerate Stirling numbers of the first kind (see (9)).

It is shown in [7] that, for $p_n(x) \sim (1, f(t))$, they are equivalently characterized by (see (4))

$$p_n(x) = \sum_{k=0}^{n} S_2(n,k;f(t))(x)_k, \tag{25}$$

$$(x)_n = \sum_{k=0}^{n} S_1(n,k;f(t))p_n(x). \tag{26}$$

Among other things, from (25) and (26), we note that $S_1(n,k;f(t))$ and $S_2(n,k;f(t))$ satisfy the orthogonality and inverse relations.

Proposition 2.1. *The following orthogonality and inverse relations are valid for $S_1(n,k;f(t))$ and $S_2(n,k;f(t))$.*

$$(a)\ \sum_{k=l}^{n} S_2(n,k;f(t))S_1(k,l;f(t)) = \delta_{n,l}, \quad \sum_{k=l}^{n} S_1(n,k;f(t))S_2(k,l;f(t)) = \delta_{n,l},$$

$$(b)\ a_n = \sum_{k=0}^{n} S_2(n,k;f(t))b_k \iff b_n = \sum_{k=0}^{n} S_1(n,k;f(t))a_k,$$

$$(c)\ a_n = \sum_{k=n}^{m} S_2(k,n;f(t))b_k \iff b_n = \sum_{k=n}^{m} S_1(k,n;f(t))a_k.$$

By using the formula (C) and (22), we derive that

$$\begin{aligned}[t^n]\,\bar{e}_{f(t)}(t) &= \frac{1}{n}[t^{n-1}]\left(\frac{t}{e^{\bar{f}(t)}-1}\right)^n \\ &= \frac{1}{n}[t^{n-1}]\sum_{k=0}^{\infty} B^{(n)}_{k,\bar{f}(t)}\frac{t^k}{k!} = \frac{1}{n!}B^{(n)}_{n-1,\bar{f}(t)}.\end{aligned} \tag{27}$$

Consequently, from (27) we have

$$\bar{e}_{f(t)}(t) = \sum_{n=1}^{\infty} B^{(n)}_{n-1,\bar{f}(t)}\frac{t^n}{n!} \tag{28}$$

Now, we define the *logarithm associated with the delta seires $f(t)$ or the associated sequence* $\mathbf{P}$ by

$$\log_{f(t)}(1+t) = \log_{\mathbf{P}}(1+t) = f(\log(1+t)). \tag{29}$$

Observe that $f(\log(1+t))$ is the compositional inverse of $e_{f(t)}(t) = e^{\bar{f}(t)} - 1$. Thus we see from (24), (28), and (29) that

(30)

$$\log_{f(t)}(1+t) = f(\log(1+t) = \bar{e}_{f(t)}(t) = \sum_{n=1}^{\infty} S_1(n,1;f(t))\frac{t^n}{n!} = \sum_{n=1}^{\infty} B^{(n)}_{n-1,\bar{f}(t)}\frac{t^n}{n!}.$$

Theorem 2.2. *For any integers $n \ge k \ge 0$, we have*

$$\begin{aligned} S_1(n,k;f(t)) &= \binom{n-1}{k-1}B^{(n)}_{n-k,\bar{f}(t)} \\ &= B_{n,k}\big(S_1(1,1;f(t)), S_1(2,1;f(t)), \ldots, S_1(n-k+1,1;f(t))\big) \\ &= B_{n,k}\left(B^{(1)}_{0,\bar{f}(t)}, B^{(2)}_{1,\bar{f}(t)}, \ldots B^{(n-k+1)}_{n-k,\bar{f}(t)}\right).\end{aligned}$$

Proof. By using the formula (B), we find that

$$\begin{aligned}[t^n]\left(\bar{e}_{f(t)}(t)\right)^k &= \frac{k}{n}[t^{n-k}]\left(\frac{t}{e^{\bar{f}(t)}-1}\right)^n \\ &= \frac{k}{n}[t^{n-k}]\sum_{k=0}^{\infty}B_{k,\bar{f}(t)}^{(n)}\frac{t^k}{k!} = \frac{k}{n}\frac{1}{(n-k)!}B_{n-k,\bar{f}(t)}^{(n)}.\end{aligned} \tag{31}$$

Hence from (31) we arrive at

$$\frac{1}{k!}\left(\bar{e}_{f(t)}(t)\right)^k = \sum_{n=k}^{\infty}\binom{n-1}{k-1}B_{n-k,\bar{f}(t)}^{(n)}\frac{t^n}{n!}. \tag{32}$$

From (24) and (30), we note that

$$\begin{aligned}\frac{1}{k!}\left(\log_{f(t)}(1+t)\right)^k &= \sum_{n=k}^{\infty}S_1(n,k;f(t))\frac{t^n}{n!} \\ &= \frac{1}{k!}\left(\sum_{m=1}^{\infty}S_1(m,1;f(t))\frac{t^m}{m!}\right)^k \\ &= \frac{1}{k!}\left(\sum_{m=1}^{\infty}B_{m-1,\bar{f}(t)}^{(m)}\frac{t^m}{m!}\right)^k.\end{aligned} \tag{33}$$

Then, from (19), (32), and (33), we obtain the theorem. □

To derive yet another expression for the logarithm associated with $f(t)$, we need to prove a Schlömilch-type formula for the Stirling numbers associated with $f(t)$. The next two lemmas and Theorem 2.7 can be proved just as those in Lemma 2.1, Lemma 2.2 and Theorem 2.1 of [28]. However, we give brief sketches of the proofs. The details are left to the reader.

Lemma 2.3. *Let $f(t)$ be a delta series, with $e^{\bar{f}(t)} = \sum_{n=0}^{\infty}p_n\frac{t^n}{n!}$. For any integers $n \geq k \geq 0$, we have*

$$\begin{aligned}&B_{n,k}\left(\frac{p_2}{2},\frac{p_3}{3},\dots,\frac{p_{n-k+2}}{n-k+2}\right) \\ &\quad = \sum_{j=0}^{k}\binom{n+k}{k-j}\frac{n!}{(n+k)!}(-p_1)^{k-j}S_2(n+j,j;f(t)).\end{aligned}$$

Proof. On the one hand, we observe that

$$\begin{aligned}\frac{1}{k!}\left(e^{\bar{f}(t)}-1-p_1t\right)^k &= \frac{1}{k!}t^k\left(\sum_{n=1}^{\infty}\frac{p_{n+1}}{n+1}\frac{t^n}{n!}\right)^k \\ &= \sum_{n=k}^{\infty}\frac{t^{n+k}}{n!}B_{n,k}\left(\frac{p_2}{2},\frac{p_3}{3},\dots,\frac{p_{n-k+2}}{n-k+2}\right).\end{aligned}$$

On the other hand, we also find that

$$\begin{aligned}\frac{1}{k!}\left(e^{\bar{f}(t)}-1-p_1t\right)^k &= \sum_{j=0}^{k}\frac{1}{(k-j)!}(-p_1t)^{k-j}\frac{1}{j!}\left(e^{\bar{f}(t)}-1\right)^j\\ &= \sum_{j=0}^{k}\frac{1}{(k-j)!}(-p_1t)^{k-j}\sum_{n=0}^{\infty}S_2(n+j,j;f(t))\frac{t^{n+j}}{(n+j)!}\\ &= \sum_{n=0}^{\infty}\frac{t^{n+k}}{n!}\sum_{j=0}^{k}\binom{n+k}{k-j}\frac{n!}{(n+k)!}(-p_1)^{k-j}S_2(n+j,j;f(t)),\end{aligned}$$

where we used $\frac{n!}{(k-j)!(n+j)!}=\binom{n+k}{k-j}\frac{n!}{(n+k)!}$. □

Lemma 2.4. *Let $f(t)$ be a delta series, with $e^{\bar{f}(t)}=\sum_{n=0}^{\infty}p_n\frac{t^n}{n!}$. For any integer $n\geq 0$, we have*

$$B^{(\alpha)}_{n,\bar{f}(t)}=\sum_{k=0}^{n}(-\alpha)_k p_1^{-\alpha-k}B_{n,k}\left(\frac{p_2}{2},\frac{p_3}{3},\dots,\frac{p_{n-k+2}}{n-k+2}\right).$$

Proof. We proceed as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}B^{(\alpha)}_{n,\bar{f}(t)}\frac{t^n}{n!} &= \left(\frac{e^{\bar{f}(t)}-1}{t}\right)^{-\alpha}=p_1^{-\alpha}\left(1+\sum_{n=1}^{\infty}\frac{p_{n+1}}{p_1}\frac{t^n}{(n+1)!}\right)^{-\alpha}\\ &= \sum_{k=0}^{\infty}(-\alpha)_k p_1^{-\alpha-k}\frac{1}{k!}\left(\sum_{n=1}^{\infty}\frac{p_{n+1}}{n+1}\frac{t^n}{n!}\right)^k\\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{n}(-\alpha)_k p_1^{-\alpha-k}B_{n,k}\left(\frac{p_2}{2},\frac{p_3}{3},\dots,\frac{p_{n-k+2}}{n-k+2}\right).\end{aligned}$$

□

Theorem 2.5. *Let $f(t)$ be a delta series. For any integer $n\geq 0$, the following identity holds true.*

$$B^{(\alpha)}_{n,\bar{f}(t)}=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{\alpha+k-1}{k}\binom{k}{j}\binom{n+j}{j}^{-1}(-1)^j\left(\bar{f}'(0)\right)^{-\alpha-j}S_2(n+j,j;f(t)).$$

Proof. Combining Lemmas 2.3 and 2.4, recalling $p_1=\bar{f}'(0)$ and noting $\frac{n!k!}{(n+k)!}\binom{n+k}{k-j}=\binom{k}{j}\binom{n+j}{j}^{-1}$, we arrive at the desired result. □

Corollary 2.6. *Let $f(t)$ be a delta series. For any integer $n\geq 0$, the following identity holds true.*

$$B_{n,\bar{f}(t)}=\sum_{j=0}^{n}\binom{n+1}{j+1}\binom{n+j}{j}^{-1}(-1)^j\left(\bar{f}'(0)\right)^{-1-j}S_2(n+j,j;f(t)).$$

Proof. This follows from Theorem 2.5, for $\alpha=1$, . □

Again, the following Schlömilch-type formula for the Stirling numbers associated with $f(t)$ is obtained by proceeding just as in the proof of Theorem 3.1 of [28].

Nevertheless, we give the short proof. For this, we need the following identities:
(34)
$$\sum_{i=j}^{n-k}\binom{n+i-1}{i}\binom{i}{j}=\frac{n}{n+j}\binom{2n-k}{n}\binom{n-k}{j},$$
$$\binom{n-1}{k-1}\binom{2n-k}{n}\frac{n}{n+j}\binom{n-k}{j}\binom{n-k+j}{j}^{-1}=\binom{n+j-1}{n+j-k}\binom{2n-k}{n-k-j}.$$
The second identity in (34) is straightforward. The first identity in (34) follows by using $\sum_{i=j}^{m}\binom{i}{j}=\binom{m+1}{j+1}$ and noting that
$$\sum_{i=j}^{n-k}\frac{n+j}{n}\binom{n+i-1}{i}\binom{i}{j}=\binom{n+j}{j}\sum_{i=n+j-1}^{2n-k-1}\binom{i}{n+j-1}.$$

Theorem 2.7. *Let $f(t)$ be a delta series. For any integers $n\geq k\geq 0$, we have*
$$S_1(n,k;f(t))=\sum_{j=0}^{n-k}\binom{n+j-1}{n+j-k}\binom{2n-k}{n-k-j}(-1)^j\left(\bar{f}'(0)\right)^{-n-j}S_2(n-k+j,j;f(t)).$$

Proof. From Theorem 2.5, we see that
(35)
$$\begin{aligned}&B^{(n)}_{n-k,\bar{f}(t)}\\&=\sum_{i=0}^{n-k}\sum_{j=0}^{i}\binom{n+i-1}{i}\binom{i}{j}\binom{n-k+j}{j}^{-1}(-1)^j\left(\bar{f}'(0)\right)^{-n-j}S_2(n-k+j,j;f(t))\\&=\sum_{j=0}^{n-k}\binom{n-k+j}{j}^{-1}(-1)^j\left(\bar{f}'(0)\right)^{-n-j}S_2(n-k+j,j;f(t))\sum_{i=j}^{n-k}\binom{n+i-1}{i}\binom{i}{j}.\end{aligned}$$
From Theorem 2.2, (34), and (35), we note
$$\begin{aligned}S_1(n,k;f(t))&=\sum_{j=0}^{n-k}\binom{n-1}{k-1}\binom{2n-k}{n}\frac{n}{n+j}\binom{n-k}{j}\binom{n-k+j}{j}^{-1}\\&\quad\times(-1)^j\left(\bar{f}'(0)\right)^{-n-j}S_2(n-k+j,j;f(t))\\&=\sum_{j=0}^{n-k}\binom{n+j-1}{n+j-k}\binom{2n-k}{n-k-j}(-1)^j\left(\bar{f}'(0)\right)^{-n-j}S_2(n-k+j,j;f(t)).\end{aligned}$$
□

Theorem 2.8. *Let $f(t)$ be a delta series. Then we have*
$$\log_{f(t)}(1+t)=\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\binom{2n-1}{n-1-j}(-1)^j\left(\bar{f}'(0)\right)^{-n-j}S_2(n-1+j,j;f(t))\frac{t^n}{n!}.$$

Proof. This follows from (30) and Theorem 2.7. □

3. Examples

In (a)-(o) below, we provide explicit expressions for the following quantities (see (23), (24), (30)):
$$\begin{aligned}&S_2(n,k;f(t))=S_2(n,k;\mathbf{P}),\quad S_1(n,k;f(t))=S_1(n,k;\mathbf{P}),\\&\log_{f(t)}(1+t)=\log_{\mathbf{P}}(1+t),\end{aligned}$$

where $\mathbf{P}=\{p_n(x)\}_{n=0}^{\infty}$, with $p_n(x)\sim(1,f(t))$.
(a) In recent years, probabilistic versions of Stirling numbers of both kinds have been explored by several authors (see [1, 3, 8, 21, 28]). We assume that Y is a random variable such that the moment-generating function of Y

$$E[e^{tY}]=\sum_{n=0}^{\infty}E[Y^n]\frac{t^n}{n!},\quad (|t|<r)$$

exists for some $r>0$, where E is the mathematical expectation (see [1, 3, 26]). We assume further that $E[Y]\neq 0$.

We recall that the probabilisitc degenerate Stirling numbers of the second kind associated with Y, $S_{2,\lambda}^{Y}(n,k)$, the probabilisitc degenerate Stirling numbers of the first kind associated with Y, $S_{1,\lambda}^{Y}(n,k)$, the probabilistic degenerate logarithm associated with Y, $\log_{\lambda}^{Y}(1+t)$, and the probabilistic degenerate Bernoulli polynomials of order α associated with Y, $\beta_{n,\lambda}^{(Y,\alpha)}(x)$, are respectively given by (see [9, 15, 22])

$$\begin{aligned}
&\frac{1}{k!}(E[e_{\lambda}^{Y}(t)]-1)^k=\sum_{n=k}^{\infty}S_{2,\lambda}^{Y}(n,k)\frac{t^n}{n!},\quad (k\geq 0),\\
&\frac{1}{k!}\big(\bar{e}_{Y,\lambda}(t)\big)^k=\sum_{n=k}^{\infty}S_{1,\lambda}^{Y}(n,k)\frac{t^n}{n!},\quad (k\geq 0),\\
&\log_{\lambda}^{Y}(1+t)=\bar{e}_{Y,\lambda}(t)=\sum_{n=1}^{\infty}S_{1,\lambda}^{Y}(n,1)\frac{t^n}{n!},\\
&\left(\frac{t}{E[e_{\lambda}^{Y}(t)]-1}\right)^{\alpha}\Big(E[e_{\lambda}^{Y}(t)]\Big)^{x}=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(Y,\alpha)}(x)\frac{t^n}{n!},
\end{aligned}\tag{36}$$

where $e_{Y,\lambda}(t)=E[e_{\lambda}^{Y}(t)]-1=E[Y]t+\sum_{n=2}^{\infty}E[(Y)_{n,\lambda}]\frac{t^n}{n!}$ is a delta series. In addition, we observe that

$$\begin{aligned}
\log E[e_{\lambda}^{Y}(t)]&=\log(1+E[e_{\lambda}^{Y}(t)]-1)\\
&=\sum_{k=1}^{\infty}(-1)^{k-1}(k-1)!\frac{1}{k!}\big(E[e_{\lambda}^{Y}(t)]-1\big)^k\\
&=\sum_{k=1}^{\infty}(-1)^{k-1}(k-1)!\sum_{n=k}^{\infty}S_{2,\lambda}^{Y}(n,k)\frac{t^n}{n!}\\
&=E[Y]t+\sum_{n=2}^{\infty}\sum_{k=1}^{n}(-1)^{k-1}(k-1)!S_{2,\lambda}^{Y}(n,k)\frac{t^n}{n!}.
\end{aligned}$$

Thus $\log E[e_{\lambda}^{Y}(t)]$ is a delta series. Now, with $\bar{f}(t)=\log E[e_{\lambda}^{Y}(t)]$, we see that

$$e^{\bar{f}(t)}-1=E[e_{\lambda}^{Y}(t)]-1,\quad \bar{e}_{Y,\lambda}(t)=\bar{e}_{f(t)}(t).\tag{37}$$

Consequently, we see from (36) and (37) that

$$\begin{aligned}
&S_2(n,k;f(t))=S_{2,\lambda}^{Y}(n,k),\quad S_1(n,k;f(t))=S_{1,\lambda}^{Y}(n,k),\\
&log_{f(t)}(1+t)=\log_{\lambda}^{Y}(1+t),\quad B_{n,\bar{f}(t)}^{(\alpha)}(x)=\beta_{n,\lambda}^{(Y,\alpha)}(x).
\end{aligned}$$

In [9], some explicit expressions of $S_{2,\lambda}^{Y}(n,k)$, $S_{1,\lambda}^{Y}(n,k)$, and $\log_{\lambda}^{Y}(1+t)$ are given for several discrete and continuous random variables Y. For example, it is shown that, for the uniform random variable ($Y \sim U[0,1]$) (see [26]),

$$\log_\lambda^Y(1+t) = \sum_{n=1}^{\infty} S_{1,\lambda}^Y(n,1)\frac{t^n}{n!} = \sum_{n=1}^{\infty} 2^n \sum_{m=0}^{n-1}\binom{n-1}{m}$$
$$\times \sum_{j_1+j_2+\cdots+j_n=m}\binom{m}{j_1,j_2,\dots,j_n}A_{2,j_1}A_{2,j_2}\cdots A_{2,j_n}\lambda^{n-m-1}\frac{t^n}{n!},$$

where the numers $A_{2,n}$'s are given by (see [10])

$$\frac{\frac{1}{2!}t^2}{e^t-1-t} = \sum_{n=0}^{\infty} A_{2,n}\frac{t^n}{n!}.$$

(b) Let $\mathbf{P} = \{x^n\}$. Then $x^n \sim (1,t)$ (see (18), [5, 24, 25]). It is evident that

$$S_2(n,k;\mathbf{P}) = S_2(n,k), \quad S_1(n,k;\mathbf{P}) = S_1(n,k) = \binom{n-1}{k-1}B_{n-k}^{(n)},$$
$$\log_{\mathbf{P}}(1+t) = \log(1+t) = \sum_{n=1}^{\infty} S_1(n,1)\frac{t^n}{n!} = \sum_{n=1}^{\infty} B_{n-1}^{(n)}\frac{t^n}{n!}.$$

(c) Let $\mathbf{P} = \big\{(x)_{n,\lambda}\big\}$ be the sequence of degenerate falling factorials (see (6), (14), (17)). Then $(x)_{n,\lambda} \sim \big(1, f(t) = \frac{1}{\lambda}(e^{\lambda t}-1)\big)$ (see [17, 19]). It is easy to see that

$$S_2(n,k;\mathbf{P}) = S_{2,\lambda}(n,k), \quad S_1(n,k;\mathbf{P}) = S_{1,\lambda}(n,k) = \binom{n-1}{k-1}\beta_{n-k,\lambda}^{(n)},$$
$$\log_{\mathbf{P}}(1+t) = \log_\lambda(1+t) = \sum_{n=1}^{\infty} S_{1,\lambda}(n,1)\frac{t^n}{n!} = \sum_{n=1}^{\infty} \beta_{n-1,\lambda}^{(n)}\frac{t^n}{n!}.$$

(d) Let $\mathbf{P} = \{\langle x\rangle_n\}$ be the sequence of rising factorials which are given by (see [5, 11, 24, 25])

$$\langle x\rangle_0 = 1, \quad \langle x\rangle_n = x(x+1)\cdots(x+n-1), \quad (n \ge 1).$$

Then $\langle x\rangle_n \sim (1, 1-e^{-t})$. We show that

$$S_2(n,k;\mathbf{P}) = L(n,k), \quad S_1(n,k;\mathbf{P}) = (-1)^{n-k}L(n,k),$$
$$\log_{\mathbf{P}}(1+t) = \sum_{n=1}^{\infty}(-1)^{n-1}L(n,1)\frac{t^n}{n!} = \frac{t}{1+t},$$

where the (unsigned) Lah numbers are specified as

$$\frac{1}{k!}\Big(\frac{t}{1-t}\Big)^k = \sum_{n=k}^{\infty} L(n,k)\frac{t^n}{n!}, \quad (k \ge 0), \quad L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}, \quad (n \ge k).$$

(e) Let $\mathbf{P} = \big\{\langle x\rangle_{n,\lambda}\big\}$ be the sequence of degenerate rising factorials which are given by (see [13])

$$\langle x\rangle_{0,\lambda} = 1, \quad \langle x\rangle_{n,\lambda} = x(x+\lambda)\cdots(x+(n-1)\lambda), \quad (n \ge 1).$$

Then $\langle x\rangle_{n,\lambda} \sim \big(1, \frac{1}{\lambda}(1-e^{-\lambda t})\big)$. Let $L_\lambda(n,k)$ be the degenerate Lah numbers given by

$$\sum_{n=k}^{\infty} L_\lambda(n,k)\frac{t^n}{n!} = \frac{1}{k!}\big(e^{-\frac{1}{\lambda}\log(1-\lambda t)}-1\big)^k = \frac{1}{k!}\big(e_{-\lambda}(t)-1\big)^k.$$

We derive that

$$S_2(n,k;\mathbf{P}) = L_\lambda(n,k), \quad S_1(n,k;\mathbf{P}) = (-\lambda)^{n-k} L_{\frac{1}{\lambda}}(n,k) = S_{1,-\lambda}(n,k),$$
$$\log_{\mathbf{P}}(1+t) = \sum_{n=1}^{\infty} S_{1,-\lambda}(n,1)\frac{t^n}{n!} = \log_{-\lambda}(1+t).$$

(f) The central factorials $x^{[n]}$ are defined by (see [4, 5, 25])

$$x^{[n]} = x\big(x+\frac{1}{2}n-1\big)_{n-1}, \ (n \geq 1), \quad x^{[0]} = 1,$$

where $x^{[n]} \sim (1, f(t) = e^{\frac{t}{2}} - e^{-\frac{t}{2}})$. Here we note that

$$\bar{f}(t) = 2\log\left(\frac{t+\sqrt{t^2+4}}{2}\right).$$

Carlitz-Riordan and Riordan (see [3, 4, 11, 23]) discussed the numbers $T_1(n,k)$ and $T_2(n,k)$ which are respectively called the central factorial numbers of the first kind and the second kind, and defined by

$$\frac{1}{k!}\left(2\log\left(\frac{t+\sqrt{t^2+4}}{2}\right)\right)^k = \sum_{n=k}^{\infty} T_1(n,k)\frac{t^n}{n!}, \quad x^{[n]} = \sum_{k=0}^{n} T_1(n,k)x^k,$$
$$\frac{1}{k!}\big(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\big)^k = \sum_{n=k}^{\infty} T_2(n,k)\frac{t^n}{n!}, \quad x^n = \sum_{k=0}^{n} T_2(n,k)x^{[k]}.$$

Let $\mathbf{P} = \{x^{[n]}\}$ be the sequence of central factorials. We find that

$$S_2(n,k;\mathbf{P}) = \sum_{l=k}^{n} T_1(n,l)S_2(l,k), \quad S_1(n,k;\mathbf{P}) = \sum_{l=k}^{n} S_1(n,l)T_2(l,k),$$
$$\log_{\mathbf{P}}(1+t) = \sum_{n=1}^{\infty}\sum_{l=1}^{n} S_1(n,l)T_2(l,1)\frac{t^n}{n!} = (1+t)^{\frac{1}{2}} - (1+t)^{-\frac{1}{2}}.$$

(g) The central Bell polynomials $\mathrm{Bel}_n^{(c)}(x)$ are defined by (see [16])

$$e^{x(e^{\frac{t}{2}} - e^{-\frac{t}{2}})} = \sum_{n=0}^{\infty} \mathrm{Bel}_n^{(c)}(x)\frac{t^n}{n!}.$$

Then we have

$$\mathrm{Bel}_n^{(c)}(x) \sim \left(1, f(t) = 2\log\left(\frac{t+\sqrt{t^2+4}}{2}\right)\right), \quad \mathrm{Bel}_n^{(c)}(x) = \sum_{k=0}^{n} T_2(n,k)x^k.$$

Let $\mathbf{P} = \left\{\mathrm{Bel}_n^{(c)}(x)\right\}$ be the sequence of central Bell polynomials. We obtain that

$$S_2(n,k;\mathbf{P}) = \sum_{l=k}^{n} T_2(n,l)S_2(l,k), \quad S_1(n,k;\mathbf{P}) = \sum_{l=k}^{n} S_1(n,l)T_1(l,k),$$
$$\log_{\mathbf{P}}(1+t) = \sum_{n=1}^{\infty}\sum_{l=1}^{n} S_1(n,l)T_1(l,1)\frac{t^n}{n!} = 2\log\left(\frac{\log(1+t)+\sqrt{(\log(1+t))^2+4}}{2}\right).$$

(h) The degenerate central factorial numbers of the first kind, $T_{1,\lambda}(n,k)$, and of the second kind, $T_{2,\lambda}(n,k)$, are respectively defined by (see [17])

$$\frac{1}{k!}\left(\log_\lambda\left(\frac{t+\sqrt{t^2+4}}{2}\right)^2\right)^k=\sum_{n=k}^{\infty}T_{1,\lambda}(n,k)\frac{t^n}{n!},\quad x^{[n]}=\sum_{k=0}^{n}T_{1,\lambda}(n,k)(x)_{k,\lambda},$$
$$\frac{1}{k!}\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^k=\sum_{n=k}^{\infty}T_{2,\lambda}(n,k)\frac{t^n}{n!},\quad (x)_{n,\lambda}=\sum_{k=0}^{n}T_{2,\lambda}(n,k)x^{[k]}.$$

The degenerate central Bell polynomials $\mathrm{Bel}_{n,\lambda}^{(c)}(x)$ are defined by (see [17])

$$e^{x(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))}=\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^{(c)}(x)\frac{t^n}{n!}.$$

Then we see that

$$\mathrm{Bel}_{n,\lambda}^{(c)}(x)\sim\left(1,f(t)=\log_\lambda\left(\frac{t+\sqrt{t^2+4}}{2}\right)^2\right),\quad \mathrm{Bel}_{n,\lambda}^{(c)}(x)=\sum_{k=0}^{n}T_{2,\lambda}(n,k)x^k.$$

Let $\mathbf{P}=\left\{\mathrm{Bel}_{n,\lambda}^{(c)}(x)\right\}$ be the sequence of degenerate central Bell polynomials. We observe that

$$S_2(n,k;\mathbf{P})=\sum_{l=k}^{n}T_{2,\lambda}(n,l)S_2(l,k),\quad S_1(n,k;\mathbf{P})=\sum_{l=k}^{n}S_1(n,l)T_{1,\lambda}(l,k),$$
$$\log_{\mathbf{P}}(1+t)=\sum_{n=1}^{\infty}\sum_{l=1}^{n}S_1(n,l)T_{1,\lambda}(l,1)\frac{t^n}{n!}=\log_\lambda\left(\frac{\log(1+t)+\sqrt{(\log(1+t))^2+4}}{2}\right)^2.$$

(i) Let $\mathbf{P}=\{LB_n(x)\}$ be the sequence of Lah-Bell polynomials. Then it is given by (see [11])

$$e^{x(\frac{t}{1-t})}=\sum_{n=0}^{\infty}LB_n(x)\frac{t^n}{n!},\quad LB_n(x)=\sum_{k=0}^{n}L(n,k)x^k,$$

so that $LB_n(x)\sim(1,f(t)=\frac{t}{1+t})$. We get that

$$S_2(n,k;\mathbf{P})=\sum_{l=k}^{n}L(n,l)S_2(l,k),\quad S_1(n,k;\mathbf{P})=\sum_{l=k}^{n}(-1)^{l-k}S_1(n,l)L(l,k),$$
$$\log_{\mathbf{P}}(1+t)=\sum_{n=1}^{\infty}\sum_{l=1}^{n}(-1)^{l-1}S_1(n,l)L(l,1)\frac{t^n}{n!}=\frac{\log(1+t)}{1+\log(1+t)}.$$

(j) Let $\mathbf{P}=\left\{LB_{n,\lambda}(x)\right\}$ be the sequence of degenerate Lah-Bell polynomials. Then it is given by (see [13])

$$e_\lambda^x\left(\frac{t}{1-t}\right)=\sum_{n=0}^{\infty}LB_{n,\lambda}(x)\frac{t^n}{n!},\quad LB_{n,\lambda}(x)=\sum_{k=0}^{n}L(n,k)(x)_{k,\lambda}.$$

Then we see that $LB_{n,\lambda}(x)\sim(1,\frac{e^{\lambda t}-1}{\lambda+e^{\lambda t}-1})$. We have that

$$S_2(n,k;\mathbf{P})=\sum_{l=k}^{n}L(n,l)L_{-\lambda}(l,k),\quad S_1(n,k;\mathbf{P})=\sum_{l=k}^{n}(-1)^{l-k}S_{1,\lambda}(n,l)L(l,k),$$
$$\log_{\mathbf{P}}(1+t)=\sum_{n=1}^{\infty}\sum_{l=1}^{n}(-1)^{l-1}S_{1,\lambda}(n,l)L(l,1)\frac{t^n}{n!}=\frac{\log_\lambda(1+t)}{1+\log_\lambda(1+t)}.$$

(k) Let $\mathbf{P}=\{\mathrm{Bel}_n(x)\}$ be the sequence of Bell polynomials. Then it is given by (see [5, 24, 25])

$$e^{x(e^t-1)}=\sum_{n=0}^{\infty}\mathrm{Bel}_n(x)\frac{t^n}{n!},\quad \mathrm{Bel}_n(x)=\sum_{k=0}^{\infty}S_2(n,k)x^k.$$

Then we see that $\mathrm{Bel}_n(x)\sim(1,\log(1+t))$. We show that

$$S_2(n,k;\mathbf{P})=\sum_{l=k}^{n}S_2(n,l)S_2(l,k),\quad S_1(n,k;\mathbf{P})=\sum_{l=k}^{n}S_1(n,l)S_1(l,k),$$
$$\log_{\mathbf{P}}(1+t)=\sum_{n=1}^{\infty}\sum_{l=1}^{n}S_1(n,l)S_1(l,1)\frac{t^n}{n!}=\log\big(1+\log(1+t)\big).$$

(l) Let $\mathbf{P}=\big\{\mathrm{Bel}_{n,\lambda}(x)\big\}$ be the sequence of partially degenerate Bell polynomials. Then it given by (see [18])

$$e^{x(e_\lambda(t)-1)}=\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}(x)\frac{t^n}{n!},\quad \mathrm{Bel}_{n,\lambda}(x)=\sum_{k=0}^{n}S_{2,\lambda}(n,k)x^k.$$

Then $\mathrm{Bel}_{n,\lambda}(x)\sim(1,\log_\lambda(1+t))$. We derive that

$$S_2(n,k;\mathbf{P})=\sum_{l=k}^{n}S_{2,\lambda}(n,l)S_2(l,k),\quad S_1(n,k;\mathbf{P})=\sum_{l=k}^{n}S_1(n,l)S_{1,\lambda}(l,k),$$
$$\log_{\mathbf{P}}(1+t)=\sum_{n=1}^{\infty}\sum_{l=1}^{n}S_1(n,l)S_{1,\lambda}(l,1)\frac{t^n}{n!}=\log_\lambda\big(1+\log(1+t)\big).$$

(m) Let $\mathbf{P}=\big\{\phi_{n,\lambda}(x)\big\}$ be the sequence of fully degenerate Bell polynomials. Then it is given by (see [19])

$$e_\lambda^x(e_\lambda(t)-1)=\sum_{n=0}^{\infty}\phi_{n,\lambda}(x)\frac{t^n}{n!},\quad \phi_{n,\lambda}(x)=\sum_{k=0}^{n}S_{2,\lambda}(n,k)(x)_{k,\lambda}.$$

Then $\phi_{n,\lambda}(x)\sim\big(1,\log_\lambda(1+\frac{1}{\lambda}(e^{\lambda t}-1))\big)$. We find that

$$S_2(n,k;\mathbf{P})=\sum_{m=k}^{n}S_{2,\lambda}(n,m)L_{-\lambda}(m,k),\quad S_1(n,k;\mathbf{P})=\sum_{l=k}^{n}S_{1,\lambda}(n,l)S_{1,\lambda}(l,k),$$
$$\log_{\mathbf{P}}(1+t)=\sum_{n=1}^{\infty}\sum_{l=1}^{n}S_{1,\lambda}(n,l)S_{1,\lambda}(l,1)\frac{t^n}{n!}=\log_\lambda\big(1+\log_\lambda(1+t)\big).$$

(n) Let $\mathbf{P}=\{M_n(x)\}$ be the sequence of Mittag-Leffler polynomials. That is, $M_n(x)\sim(1,f(t)=\frac{e^t-1}{e^t+1})$, with $\bar{f}(t)=\log\big(\frac{1+t}{1-t}\big)$ (see [25]). We obtain that

$$S_2(n,k;\mathbf{P})=2^kL(n,k),\quad S_1(n,k;\mathbf{P})=(-1)^{n-k}L(n,k)\frac{1}{2^n},$$
$$\log_{\mathbf{P}}(1+t)=\sum_{n=1}^{\infty}(-1)^{n-1}L(n,1)\frac{1}{2^n}\frac{t^n}{n!}=\frac{t}{2+t}.$$

(o) Let $\mathbf{P} = \{L_n(x)\}$ be the sequence of Laguerre polynomials of order -1 (see [25]). That is, $L_n(x) \sim (1, f(t) = \frac{t}{t-1})$, with $\bar{f}(t) = \frac{t}{t-1}$. We observe that

$$S_2(n,k;\mathbf{P}) = \sum_{l=k}^{n} (-1)^l L(n,l) S_2(l,k), \quad S_1(n,k;\mathbf{P}) = (-1)^k \sum_{l=k}^{n} S_1(n,l) L(l,k),$$

$$\log_{\mathbf{P}}(1+t) = -\sum_{n=1}^{\infty} \sum_{l=1}^{n} S_1(n,l) L(l,1) \frac{t^n}{n!} = -\frac{\log(1+t)}{1-\log(1+t)}.$$

4. Conclusion

In this paper, we have developed a unified approach to Stirling numbers associated with a delta series $f(t)$, which corresponds to Stirling numbers associated with the sequence $p_n(x) \sim (1, f(t))$. We have successfully addressed the structural deficiencies found in recent probabilistic Stirling and B-Stirling number variants. Specifically, our redefined Stirling numbers $S_1(n,k;f(t))$ and $S_2(n,k;f(t))$ restore the fundamental orthogonality and inverse relations, ensuring that these sequences remain robust tools for inversion formulas and combinatorial identities. We established the Schlömilch-type formula for these generalized numbers. This formula not only provides a direct computational link between the two kinds of Stirling numbers but also allows for a novel expansion of the associated logarithm $\log_{f(t)}(1+t)$. The fifteen examples detailed in Section 3 demonstrate that our framework is not merely a theoretical exercise but a versatile system capable of recovering and extending a wide array of known results. Whether applied to degenerate cases or probabilistic extensions, the delta series approach provides a consistent and computationally efficient methodology. Future research may involve applying this approach to other special sequences or extending these results to the q-analogues.

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