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SOME RAGUSA-TYPE EMBEDDING THEOREMS IN GENERALIZED SOBOLEV–MORREY SPACE

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Dedicated to Professor Maria Alessandra Ragusa on the occasion of her 60th birthday

Abstract. In this paper, we define generalized Sobolev–Morrey spaces, and using the method of integral representations, we study some properties of functions from these spaces.

1. Introduction and Preliminary Notes

In this paper, we construct generalized Sobolev–Morrey spaces $W_{p,A}^{l}(G)$, and, with the integral representation method, we study some differential and differential-difference properties of functions from the constructed spaces.

Note that the spaces with parameters (later called Morrey spaces) were first introduced by C. B. Morrey [11,12], and then studied and developed by many mathematicians, including the works of V. P. Ilyin [6], I. Ross [27], Y. V. Netrusov [23], A. Mazzucato [10], V. M. Kokilashvili, A. Meskhi, H. Rafeiro [9], V. S. Guliyev [3–5] and others [1,3,7,8,13–22,24–26]. In addition, we note that unlike previous works, the norm introduced in this work, is generalized.

Definition 1.1. A generalized Sobolev–Morrey space, denoted by $W_{p,A}^{l}(G)$, is the space of locally summable functions f with weak derivatives $D_i^{l_i} f$ ($l_i > 0$ are integers, $i = 1, 2, \ldots, n$) on G with the finite norm

$$\|f\|_{W_{p,A}^{l}(G)} = \|f\|_{L_{p,A}(G)} + \sum_{i=1}^{n} \|D_i^{l_i} f\|_{L_{p,A}(G)},$$

where

$$\|f\|_{L_{p,A}(G)} = \sup_{B\subset G}\left(\int_B \left|A\left([|B|]_{h_0}, p, f(x)\right)\right|^p dx\right)^{\frac{1}{p}},$$

$[|B|]_{h_0} = \min\{h_0, |B|\}$, h_0 is a positive fixed number, $1 \le p < \infty$, $G \subset R^n$, $A : (0, h_0) \times [1, \infty) \times L_1^{loc}(G) \to R$ is a measurable function on $D = (0, h_0) \times [1, \infty) \times L_1^{loc}(G)$.

Let A be a differentiable function with respect to the argument f, and there exist the constants c_1, $c_2 > 0$ such that for all $B \subset G$ and $f \in L_1^{loc}(G)$ the following double inequality

$$c_1 \left\|A\left([|B|]_{h_0}, p, f^{(\nu)}\right)\right\|_{L_p(B)} \le \left\|A^{(\nu)}\left([|B|]_{h_0}, p, f\right)\right\|_{L_p(B)} \le c_2 \left\|A\left([|B|]_{h_0}, p, f^{(\nu)}\right)\right\|_{L_p(B)} \tag{1.1}$$

holds.

Moreover, there exists a constant $M = M\left([|B|]_{h_0}, p\right) > 0$ such that for all $f \in L_1^{loc}(G)$ the following inequality

$$\left|A\left([|B|]_{h_0}, p, f^{(\nu)}\right)\right| \ge M\,|f(x)|, \quad \text{a.e. } x \in G,$$

holds.

The space $L_{p,A}(G)$ in case $B \equiv G_{\psi(v)}(x) = G \cap \left\{y : |y_j - x_j| < \frac{1}{2}\psi_j(v)\right\}$, $x \in G$, $v > 0$ and $A(|B|, p, f) = \prod_{j=1}^{n} \left(\psi_j([v]_1)\right)^{-\frac{\beta_j}{p}} f$, coincides with the spaces $L_{p,\psi,\beta}(G)$ introduced and studied in [17],

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where $\beta_j \in [0,1]$, $j = 1,\dots,n$, $[v]_1 = \min\{1,v\}$, $\psi(v) = (\psi_1(v), \psi_2(v), \dots, \psi_n(v))$ are vector-functions, $\psi_j(v) > 0$ $(j = 1,2,\dots,n)$ are summable functions, and $\lim\limits_{v\to+0}\psi_j(v)$, $\lim\limits_{v\to+\infty}\psi_j(v) = k_j \le \infty$, $j = 1,2,\dots,n$. In case $B \equiv G_{v^\varkappa}(x)$ $(\varkappa \in (0,\infty)^n)$, $A\left([|B|]_{h_0}, p, f\right) \equiv [v]_1^{-\frac{|\varkappa|a}{p}} f$; $a \in [0,1]$, then the space $L_{p,A}(G)$ coincides with spaces $L_{p,\varkappa,a}(G)$ introduced and studied by V. P. Ilyin in [6].

Let us note some properties of the spaces $L_{p,A}(G)$ and $W^l_{p,A}(G)$.

1. The following embeddings:

$$L_{p,A}(G) \to L_p(G), \quad W^l_{p,A}(G) \to W^l_p(G)$$

hold, i.e.,

$$\|f\|_{L_p(G)} \le C\|f\|_{L_{p,A}(G)} \text{ and } \|f\|_{W^l_p(G)} \le C\|f\|_{W^l_{p,A}(G)}.$$

Indeed, we have

$$\|f\|^p_{L_{p,A}(G)} = \sup_{B\subset G}\left(\int\limits_B \left|A\left([|B|]_{h_0}, p, f(x)\right)\right|^p dx\right)$$

$$\ge \left(\int\limits_G \left|A\left([|B|]_{h_0}, p, f(x)\right)\right|^p dx\right) \ge C\left(\int\limits_G |f(x)|^p dx\right) = C\|f\|^p_{L_p(G)}.$$

We will use the integral representation for $f \in W^l_{p,A}(G)$, defined in domains satisfying the Φ-horn condition, obtained in the form

$$D^\nu A\left([|B|]_{h_0}, p, f\right)(x) = A^{(\nu)}_{\Phi(h)}\left([|B|]_{h_0}, p, f(x)\right)$$

$$+\sum_{i=1}^n \int\limits_0^h \int\limits_{R^n} D_i^{l_i} A\left([|B|]_{h_0}, p, f(x+y)\right) \tilde{L}_i^{(\nu)}\left(\frac{y}{\Phi(v)}\right) \prod_{j=1}^n (\Phi_j(v))^{-1-\nu_j} (\Phi_i(v))^{-1+l_i} \Phi_i'(v)\, dy\, dv, \quad (1.2)$$

$$A^{(\nu)}_{\psi(h)}\left([|B|]_{h_0}, p, f(x)\right) = \prod_{j=1}^n (\Phi_j(h))^{-1-\nu_j} \int\limits_{R^n} A\left([|B|]_{h_0}, p, f(x+y)\right) \Omega^{(\nu)}\left(\frac{y}{\Phi(h)}\right) dy, \quad (1.3)$$

where $0 < h \le h_0$, $\frac{y}{\Phi(v)} = \left(\frac{y_1}{\Phi_1(v)}, \frac{y_2}{\Phi_2(v)}, \dots, \frac{y_n}{\Phi_n(v)}\right)$, $\Phi_j(v)$, $j = 1,2,\dots,n$, are arbitrary differentiable positive nondecreasing functions defined for $v > 0$, and $\lim\limits_{v\to+0}\Phi_j(v) = 0$, $\lim\limits_{v\to+\infty}\Phi_i(v) = K_i \le \infty$, $i = 1,2,\dots,n$ and $L_i, \Omega \in C_0^\infty(R^n)$,

$$S(M) = \operatorname{supp} M = I_{\Phi(h_0)} = \left\{y : |y_j| < \frac{1}{2}\Phi_j(h_0), \ j = 1,2,\dots,n\right\},$$

and the Φ horn $x + V = x + \bigcup\limits_{0<h\le h_0}\left\{y : \frac{y}{\Phi(h)} \in S(\Omega)\right\}$ is the support of the representations (1.2), (1.3) and $\nu = (\nu_1, \nu_2, \dots, \nu_n)$, $\nu_j \ge 0$ are integers $(j = 1,2,\dots,n)$.

Suppose that U is an open set contained in the domain G, and $U + V \subset B \subset G$.

In the text, $\|\cdot\|_{q,U}$ is understood as $\|\cdot\|_{L_q(U)}$.

Lemma 1.1. *Let $1 \le p \le q \le r \le \infty$, $0 < \eta$, $v \le h \le h_0$, $\nu = (\nu_1, \nu_2, \dots, \nu_n)$, $\nu_j \ge 0$ be integers $(j = 1,2,\dots,n)$, $\varphi \in L_{p,A}(G)$ and*

$$E^i_\eta(x) = \int\limits_0^\eta \prod_{j=1}^n (\Phi_j(v))^{-1-\nu_j} (\Phi_i(v))^{-1+l_i} \Phi_i'(v) \int\limits_{R^n} \varphi(x+y) K\left(\frac{y}{\Phi(v)}\right) dy\, dv,$$

$$E^i_{\eta,h}(x) = \int\limits_\eta^h \prod_{j=1}^n (\Phi_j(v))^{-1-\nu_j} (\Phi_i(v))^{-1+l_i} \Phi_i'(v) \int\limits_{R^n} \varphi(x+y) K\left(\frac{y}{\Phi(v)}\right) dy\, dv,$$

$$R_{\eta}^{i}=\int\limits_{0}^{\eta}\prod_{j=1}^{n}\left(\Phi_{j}\left(v\right)\right)^{-\nu_{j}-\frac{1}{p}+\frac{1}{q}}\left(\Phi_{i}\left(v\right)\right)^{-1+l_{i}}\Phi_{i}'\left(v\right)dv. \tag{1.4}$$

Then

$$\left\|E_{\eta}^{i}\right\|_{q,U}\leq C_{1}\left|R_{\eta}^{i}\right|\,\|\varphi\|_{p,A,G}\quad\left(R_{\eta}^{i}<\infty\right), \tag{1.5}$$

$$\left\|E_{\eta,h}^{i}\right\|_{q,U}\leq C_{2}\left|R_{\eta,h}^{i}\right|\,\|\varphi\|_{p,A,G}\quad\left(R_{\eta,h}^{i}<\infty\right), \tag{1.6}$$

where

$$R_{\eta,h}^{i}\left(x\right)=\int\limits_{\eta}^{h}\prod_{j=1}^{n}\left(\Phi_{j}\left(v\right)\right)^{-\nu_{j}-\frac{1}{p}+\frac{1}{q}}\left(\Phi_{i}\left(v\right)\right)^{-1+l_{i}}\Phi_{i}'\left(v\right)dv.$$

Proof. Applying the generalized Minkowskii inequality, we obtain

$$\left\|E_{\eta}^{i}\right\|_{q,U}\leq\int\limits_{0}^{\eta}\prod_{j=1}^{n}\left(\Phi_{j}\left(v\right)\right)^{-1-\nu_{j}}\left(\Phi_{i}\left(v\right)\right)^{-1+l_{i}}\Phi_{i}'\left(v\right)\left\|F\left(\cdot,v\right)\right\|_{q,U}\,dv, \tag{1.7}$$

where

$$F\left(x,v\right)=\int\limits_{R^{n}}\varphi\left(x+y\right)K\left(\frac{y}{\Phi\left(v\right)}\right)dy. \tag{1.8}$$

Estimate the norm $\left\|F\left(\cdot,v\right)\right\|_{q,U}$. From Hölder's inequality $\left(q\leq r\right)$, we obtain

$$\left\|F\left(\cdot,v\right)\right\|_{q,U}\leq\left\|F\left(\cdot,v\right)\right\|_{r,U}\left|U\right|^{\frac{1}{q}-\frac{1}{r}}. \tag{1.9}$$

Let χ be the characteristic function $S\left(K\right)$, and let the integrand function φK, expressed in formula (1.8) be represented as follows:

$$\left|\varphi K\right|=\left(|\varphi|^{p}|K|^{s}\right)^{\frac{1}{r}}\left(|\varphi|^{p}\chi\right)^{\frac{1}{p}-\frac{1}{r}}\left(|K|^{s}\right)^{\frac{1}{s}-\frac{1}{r}},$$

where $\frac{1}{s}=1-\frac{1}{p}+\frac{1}{r}$. Applying Hölder's inequality $\left(\frac{1}{r}+\left(\frac{1}{p}-\frac{1}{r}\right)+\left(\frac{1}{s}-\frac{1}{r}\right)=1\right)$, we have

$$\begin{aligned}\left\|F\left(\cdot,v\right)\right\|_{r,U}&\leq\sup_{x\in U}\left(\int\limits_{R^{n}}\left|\varphi\left(x+y\right)\right|^{p}\chi\left(\frac{y}{\Phi\left(v\right)}\right)dy\right)^{\frac{1}{p}-\frac{1}{r}}\\&\times\sup_{y\in V}\left(\int\limits_{U}\left|\varphi\left(x+y\right)\right|^{p}dx\right)^{\frac{1}{r}}\left(\int\limits_{R^{n}}\left|K\left(\frac{y}{\Phi\left(v\right)}\right)\right|^{s}dy\right).\end{aligned} \tag{1.10}$$

For every $I_{\Phi(v)}\left(x\right)\equiv B$, $x\in U$, we get

$$\int\limits_{R^{n}}\left|\varphi\left(x+y\right)\right|^{p}\chi\left(\frac{y}{\Phi\left(v\right)}\right)dy\leq\int\limits_{I_{\Phi(v)}(x)}\left|\varphi\left(y\right)\right|^{p}dy\leq\|\varphi\|_{p,G}^{p}\leq C\|\varphi\|_{p,A,G}^{p}. \tag{1.11}$$

Also, for every $y\in V$, we obtain

$$\int\limits_{U}\left|\varphi\left(x+y\right)\right|^{p}dx\leq\|\varphi\|_{p,U}^{p}\leq\|\varphi\|_{p,G}^{p}\leq C\|\varphi\|_{p,A,G}, \tag{1.12}$$

$$\int\limits_{R^{n}}\left|K\left(\frac{y}{\Phi\left(v\right)}\right)\right|^{s}dy=\prod_{j=1}^{n}\Phi_{j}\left(v\right)\|k\|_{s}^{s}. \tag{1.13}$$

Let $r=q$. It follows from (1.9)–(1.13) that

$$\left\|F\left(\cdot,v\right)\right\|_{q,U}\leq\|\varphi\|_{p,A,G}\|K\|_{s}\prod_{j=1}^{n}\left(\Phi_{j}\left(v\right)\right)^{\frac{1}{s}}. \tag{1.14}$$

Taking into account inequality (1.14) in inequality (1.7), we get inequality (1.5). So, we have

$$\left\|E_\eta^i\right\|_{q,U} \leq \|\varphi\|_{p,A,G}\|K\|_s \int_0^\eta \prod_{j=1}^n \left(\Phi_j(v)\right)^{-1-\nu_j+1-\frac{1}{p}+\frac{1}{q}}$$
$$\times \left(\Phi_i(v)\right)^{-1+l_i} \Phi_i'(v)\, dv \leq C_1 \left|R_\eta^i\right| \|\varphi\|_{p,A,G} \quad \left(R_\eta^i < \infty\right).$$

Inequality (1.6) is proved similarly. □

2. Main Results

Now, we prove the following theorems on the properties of the functions from the spaces $W_{p,A}^l(G)$.

Theorem 2.1. *Let $G \subset R^n$ be an open set satisfying the Φ-horn condition, and let $1 \leq p \leq q \leq \infty$, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, $\nu_j \geq 0$ be integers $(j = 1, 2, \ldots, n)$, $R_h^i < \infty$ $(i = 1, 2, \ldots, n)$. Suppose that $f \in W_{p,A}^l(G)$.*

Then $D^\nu : W_{p,A}^l(G) \hookrightarrow L_q(G)$, i.e., for all $f \in W_{p,A}^l(G)$ on the domain G there exist weak mixed derivatives $D^\nu f \in L_q(G)$ and there are positive numbers C_1 and C_2 such that

$$\|D^\nu f\|_{L_{q,A}(G)} \leq C_1 \prod_{j=1}^n \left(\Phi_j(h)\right)^{1-\frac{1}{p}+\frac{1}{q}} \|f\|_{L_{p,A}(G)} + C_2 \sum_{i=1}^n \left|R_n^i\right| \left\|D_i^{l_i} f\right\|_{L_{p,A}(G)}. \tag{2.1}$$

In particular,

$$R_n^{i,0} = \int_0^h \prod_{j=1}^n \left(\Phi_j(v)\right)^{-\nu_j-\frac{1}{p}} \left(\Phi_i(v)\right)^{-1-l_i} \Phi_i'(v)\, dv < \infty, \quad i = 1, 2, \ldots, n,$$

then $D^\nu f(x)$ is continuous on G and

$$\sup_{x \in G} |D^\nu f(x)| \leq C_1 \prod_{j=1}^n \left(\Phi_j(h)\right)^{1-\frac{1}{p}} \|f\|_{L_{p,A}(G)} + C_2 \sum_{i=1}^n \left|R_h^{i,0}\right| \left\|D_i^{l_i} f\right\|_{L_{p,A}(G)},$$

where $0 < h \leq h_0$ C_1 and C_2 are the constants, independent of h and f.

Proof. First, we note that under the conditions of our theorem, there exist generalized derivatives $D^\nu f$ on G. Indeed, since $p \leq q$, $R_h^i < \infty$ $(i = 1, 2, \ldots, n)$, there exist weak mixed derivatives $D^\nu f$, and for them, the following integral representations (1.2), (1.3) are valid with the same kernels.

Applying the generalized Minkowskii inequality, we have

$$\|D^\nu A\|_{q,G} \leq \left\|A_{\Phi(h)}^{(\nu)}\right\|_{q,G} + \sum_{i=1}^n \left\|E_h^i\right\|_{q,G}. \tag{2.2}$$

By inequality (1.13), for $U \equiv G$, $\varphi = A\left([|B|]_{h_0}, p, f\right)$, we get

$$\left\|A_{\Phi(h)}^{(\nu)}\right\|_{q,G} \leq C_1 \left\|A\left([|B|]_{h_0}, p, f\right)\right\|_{q,G} \prod_{j=1}^n \left(\Phi_j(h)\right)^{1-\frac{1}{p}+\frac{1}{p}}. \tag{2.3}$$

Also, by inequality (1.4), for $U \equiv G$, $\varphi = D_i^{l_i} A\left([|B|]_{h_0}, p, f\right)$, we have

$$\left\|E_h^i\right\|_{q,G} \leq C_2 \left\|D_i^{l_i} A\left([|B|]_{h_0}, p, f\right)\right\|_{p,G} \left|R_h^i\right|.$$

Using inequalities (1.1), (2.2), (2.3) and taking into account the lower boundedness of the function A, we obtain inequality (2.1).

Showing that $D^\nu f$ is continuous on G. By (2.1) and (2.2), for $q = \infty$, we obtain

$$\left\|D^\nu f - f_{\Phi(h)}^{(\nu)}\right\|_{\infty,G} \leq \overline{C} \sum_{i=1}^n \left|R_h^i\right| \left\|D_i^{l_i} f\right\|_{L_{p,A}(G)}.$$

It follows that the left-hand part of the last inequality tends to zero as $h \to 0$. Since $f^{(\nu)}_{\Phi(h)}$ is continuous on G, the convergence in $L_\infty(G)$ coincides in our case with the uniform convergence; consequently, $D^\nu f$ is continuous on G.

Theorem 2.1 has been proven. □

Let γ be an n-dimensional vector.

Theorem 2.2. *Suppose that a domain G, the parameters p, q and the vector ν satisfy all the conditions of Theorem 2.1. If $R^i_h < \infty$ $(i = 1, 2, \dots, n)$, then $D^\nu f$ satisfies the Hölder condition on G in the metric of L_q, more exactly,*

$$\|\Delta(\gamma, G) D^\nu f\|_{q,G} \le \overline{C} \|f\|_{W^l_{p,A}(G)} \left|R^1_{\gamma,h}\right|. \tag{2.4}$$

If $R^{i,0}_h < \infty$ $(i = 1, 2, \dots, n)$, then

$$\sup_{x\in G} |\Delta(\gamma, G) D^\nu f(x)| \le \overline{C} \|f\|_{W^l_{p,A}(G)} \left|R^{1,0}_{\gamma,h}\right|,$$

where

$$R^1_{\gamma,h} = \max_i \left\{|\gamma|, |\gamma| \, |R^i_h|, |\gamma| \, |R^i_{\gamma,h}|\right\},$$
$$R^{1,0}_{\gamma,h} = \max_i \left\{|\gamma|, |\gamma| \, |R^{i,0}_h|, |\gamma| \, |R^{i,0}_{\gamma,h}|\right\}.$$

Proof. By Lemma 1.2 in [2], there is a domain $G_\delta \subset G$ $(G_\delta = \xi\rho(x), \xi > 0, \ \rho(x) = \operatorname{dist}(x, \partial G),$ $x \in G)$ and $|\gamma| < \delta$. Then for every $x \in G_\delta$, the segment joining the point x and $x + \gamma$ is contained in G. After some simple transformations, we have

$$\begin{aligned}
&\left|\Delta(\gamma, G) D^\nu A\left([|B|]_{h_0}, p, f(x)\right)\right| \\
&\le \prod_{j=1}^n (\Phi_j(h))^{-1-\nu_j} \int_{R^n} \left|A\left([|B|]_{h_0}, p, f(x+y)\right)\right| \left|\Omega^{(\nu)}\left(\frac{y-\gamma}{\Phi(h)} - \frac{y}{\Phi(h)}\right)\right| dy \\
&+ \sum_{i=1}^n \Bigg\{ \int_0^{|\gamma|} \prod_{j=1}^n (\Phi_j(v))^{-1-\nu_j} (\Phi_i(v))^{-1+l_i} \\
&\times \int_{R^n} \left|D^{l_i}_i A\left([|B|]_{h_0}, p, f(x+\gamma+y)\right)\right| + \left|D^{l_i}_i A\left([|B|]_{h_0}, p, f(x+y)\right)\right| \\
&\times \left|L^{(\nu)}_i\left(\frac{y}{\Phi(v)}\right)\right| \Phi_i'(v)\, dv\, dy \\
&+ \int_{|\gamma|}^h \prod_{j=1}^n (\Phi_j(v))^{-1-\nu_j} (\Phi_i(v))^{-1+l_i} \int_{R^n} \left|D^{l_i}_i A\left([|B|]_{h_0}, p, f(x+y)\right)\right| \\
&\times \left|L^{(\nu)}_i\left(\frac{y-\gamma}{\Phi(v)}\right) - L^{(\nu)}_i\left(\frac{y}{\Phi_i(v)}\right)\right| \Phi_i'(v)\, dv\, dy \Bigg\} = F(x,\gamma) + \sum_{i=1}^n \left(F^i_1(x,\gamma) + F^i_2(x,\gamma)\right),
\end{aligned} \tag{2.5}$$

where $0 < h \le h_0$, h_0 is a fixed positive number. We also assume that $|\gamma| < h$, hence $|\gamma| \le \min\{\delta, h\}$. If $x \in G \setminus G_\delta$, then by definition of $\Delta(\gamma, G), D^\nu f(x) = 0$. By (2.5), we get

$$\begin{aligned}
&\left\|\Delta(\gamma, G_\delta) D^\nu A\left([|B|]_{h_0}, p, f(x)\right)\right\|_{q,G} = \left\|\Delta(\gamma, G) D^\nu A\left([|B|]_{h_0}, p, f(x)\right)\right\|_{q,G_\delta} \\
&\le \|F(\cdot,\gamma)\|_{q,G_\delta} + \sum_{i=1}^n \left(\left\|F^i_1(\cdot,\gamma)\right\|_{q,G_\delta} + \left\|F^i_2(\cdot,\gamma)\right\|_{q,G_\delta}\right).
\end{aligned} \tag{2.6}$$

We notice that

$$\left|\Omega^{(\nu)}\left(\frac{y-\gamma}{\Phi(h)}\right)-\Omega^{(\nu)}\left(\frac{y}{\Phi(h)}\right)\right|\le\left|\int_0^{|\gamma|}\frac{d}{d\xi}\Omega^{(\nu)}\left(\left(y-\xi\frac{\gamma}{|\gamma|}:\Phi(h)\right)\right)d\xi\right|$$

$$\le\sum_{j=1}^{n}(\Phi(h))^{-1}\int_0^{|\gamma|}\left|D_j\Omega^{(\nu)}\left((y-\xi e_\gamma):\Phi(h)\right)\right|d\xi,\quad e_\gamma=\frac{\gamma}{|\gamma|}.$$

Therefore,

$$F(x,\gamma)\le\prod_{j=1}^{n}(\Phi_j(h))^{-1-\nu_j}\sum_{j=1}^{n}|\Phi_j(h)|^{-1}\int_0^{|\gamma|}d\xi\int_{R^n}|f(x+\xi e_\gamma+y)|\left|D_j\Omega^{(\nu)}\left(\frac{y}{\Phi(h)}\right)\right|dy,$$

Similarly, we get

$$F_2(x,\gamma)\le\sum_{j=1}^{n}(\varphi_j(v))^{-1}\int_0^{|\gamma|}d\xi\int_{|\gamma|}^{h}\prod_{j=1}^{n}(\Phi_j(v))^{-1-\nu_j}(\Phi_i(v))^{-1+l_i}(\Phi_i'(v))\,dv$$

$$\times\int_{R^n}\left|D_i^{l_i}f(x+\xi e_\gamma+y)\right|\left|D_jL_i^{(\nu)}\left(\frac{y}{\Phi(v)}\right)\right|dy. \tag{2.7}$$

Using inequality (2.7), for $U=G$, $\varphi=A\left([|B|]_{h_0},p,f\right)$, $K=\Omega^{(\nu)}$, we obtain

$$\|F(\cdot,\gamma)\|_{q,G}\le C_1|\gamma|\,\left\|A\left([|B|]_{h_0},p,f\right)\right\|_{p,G}, \tag{2.8}$$

and with the help of (1.5), for $U=G$, $\varphi\equiv D_i^{l_i}A\left([|B|]_{h_0},p,f\right)$, $\eta=|\gamma|$, $K=L_i^{(\nu)}$, we have

$$\|F_1(\cdot,\gamma)\|_{q,G}\le C_2|\gamma|\,\left\|D_i^{l_i}A\left([|B|]_{h_0},p,f\right)\right\|_{p,G}\left|R_h^i\right|. \tag{2.9}$$

Also, by inequality (1.6), for $U=G$, $\varphi\equiv D_i^{l_i}A\left([|B|]_{h_0},p,f\right)$, $\eta=|\gamma|$, $K=L_i^{(\nu)}$, we get

$$\|F_2(\cdot,\gamma)\|_{q,G}\le C_3|\gamma|\,\left\|D_i^{l_i}A\left([|B|]_{h_0},p,f\right)\right\|_{p,G}\left|R_{\gamma,h}\right|. \tag{2.10}$$

It follows from inequalities (2.6), (2.8)–(2.10) that

$$\left\|\Delta(\gamma,G)\,D^\nu A\left([|B|]_{h_0},p,f\right)\right\|_{q,G}\le C\left\|A\left([|B|]_{h_0},p,f\right)\right\|_{W_p^l(G)}\left|R_{\gamma,h}^1\right|,$$

where

$$R_{\gamma,h}^1=\max_i\left\{|\gamma|,|\gamma||R_h^i|,|\gamma||R_{\gamma,h}^i|\right\}.$$

Suppose that $|\gamma|\ge\{\delta,h\}$. Then

$$\left\|\Delta(\gamma,G)\,D^\nu A\left([|B|]_{h_0},p,G\right)\right\|_{q,G}\le 2\left\|D^\nu A\left([|B|]_{h_0},p,G\right)\right\|_{q,G}$$

$$\le C(\delta,h)\left\|D^\nu A\left([|B|]_{h_0},p,f\right)\right\|_{q,G}\left|R_{\gamma,h}^1\right|.$$

By inequality (1.1), estimating for $\left\|D^\nu A\left([|B|]_{h_0},p,q\right)\right\|_{q,G}$ and taking into account the lower boundedness of the function A, we obtain inequality (2.4).

This completes the proof of Theorem 2.2. □

3. Conflict of Interests

The authors declare that they have no conflict of interests.

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