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MONOTONICITY AND ABSOLUTE CONVEXITY OF TWO FUNCTIONS INVOLVING RIEMANN ZETA FUNCTION

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Abstract. Let $\rho>0$ be a constant, let $j\ge0$ be an integer, and let $\Gamma(z)$ denote the Euler gamma function. With the aid of the integral representation for the Riemann zeta function $\zeta(z)$, by virtue of a monotonicity rule, and by means of some properties of the function $\frac{1}{\mathrm{e}^t-1}$ and its derivatives, the authors discuss the increasing monotonicity of the function $t\mapsto\binom{t+\rho+j}{\rho}\frac{\zeta(t+\rho)}{\zeta(t)}$, study the absolute convexity and logarithmic convexity of the function $t\mapsto\Gamma(t+j)\zeta(t)$, and derive the increasing monotonicity and inequalities of some sequences involving the ratios $\bigl|\frac{B_{2n+2}}{B_{2n}}\bigr|$ of the Bernoulli numbers B_{2n}, where $\binom{z}{z}$ denotes the extended binomial coefficient.

1. A concise review

As usual and by convention, we use the following symbols

$$\mathbb{Z}=\{0,\pm1,\pm2,\dots\},\qquad \mathbb{N}=\{1,2,\dots\},$$
$$\mathbb{N}_0=\{0,1,2,\dots\},\qquad \mathbb{N}_-=\{-1,-2,\dots\}.$$

The classical Euler gamma function $\Gamma(z)$ can be defined [1, Chapter 6] by

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{\prod_{k=0}^n(z+k)},\quad z\in\mathbb{C}\setminus\{0,-1,-2,\dots\}.$$

See also [27, Chapter 3]. In [6, Fact 13.3], we find that, for $z\in\mathbb{C}$ such that $\Re(z)>1$, the Riemann zeta function $\zeta(z)$ is defined and satisfies

$$\zeta(z)=\sum_{k=1}^\infty\frac{1}{k^z}=\frac{1}{1-2^{-z}}\sum_{k=1}^\infty\frac{1}{(2k-1)^z}=\frac{1}{1-2^{1-z}}\sum_{k=1}^\infty(-1)^{k-1}\frac{1}{k^z}\tag{1.1}$$

and

$$\zeta(z)=\frac{1}{\Gamma(z)}\int_0^\infty\frac{t^{z-1}}{\mathrm{e}^t-1}\,\mathrm{d}t,\quad \Re(z)>1.\tag{1.2}$$

The last two equalities in (1.1) tell us some reasons why many mathematicians investigated the Dirichlet eta and lambda functions

$$\eta(z)=\biggl(1-\frac{1}{2^{z-1}}\biggr)\zeta(z)\quad\text{and}\quad\lambda(z)=\biggl(1-\frac{1}{2^z}\biggr)\zeta(z).$$

According to discussions in [27, Section 3.5, pp. 57–58], the zeta function $\zeta(z)$ has an analytic continuation which has a unique singular point $z=1$, which is a simple pole with residue 1, on the complex plane $\mathbb{C}$.

We now collect several known properties and applications of the Riemann zeta function $\zeta(z)$, the Dirichlet eta function $\eta(z)$, and the Dirichlet lambda function $\lambda(z)$ as follows.

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(1) In [29], Wang proved that the eta function $\eta(t)$ is logarithmically concave on $(0,\infty)$. In [15, 19], making use of Wang's result, the second author of this paper established the first double inequality for bounding the ratio $\frac{|B_{2(j+1)}|}{|B_{2j}|}$ of the Bernoulli numbers B_{2j} for $j\ge1$, where the Bernoulli numbers B_{2j} are generated by

$$\frac{z}{e^z-1}=\sum_{n=0}^\infty B_n\frac{z^n}{n!}=1-\frac{z}{2}+\sum_{j=1}^\infty B_{2j}\frac{z^{2j}}{(2j)!},\quad |z|<2\pi.$$

(2) In [7], Cerone and Dragomir proved that the reciprocal $\frac{1}{\zeta(t)}$ is concave on $(1,\infty)$.

(3) In [36], Zhu and Hua proved that the sequence $\lambda(j)$ for $j\in\mathbb{N}$ is decreasing. This result was applied in [15, 19] to establish the first double inequality for bounding the ratio $\frac{|B_{2(j+1)}|}{|B_{2j}|}$ for $j\in\mathbb{N}$. In [35], Zhu used this result once again to develop and sharpen Qi's first double inequality discovered in [15, 19].

(4) In 2015, Adell–Lekuona [2] and Alzer–Kwong [3] proved the concavity of $\eta(t)$ on $(0,\infty)$.

(5) In [12], Hu and Kim discovered many families of linear recurrent relations and convolution identities of $\lambda(2j)$ for $j\in\mathbb{N}$.

(6) In [33], Yang and Tian verified that the function

$$\frac{1}{2^t}\frac{\zeta(t)-2^{-q}\zeta(t+q)}{\zeta(t)-\zeta(t+q)}$$

is increasing from $(1,\infty)$ onto $\left(\frac12,1\right)$. By this, Yang and Tian [33] extended and sharpened Qi's first double inequality established in [15, 19].

(7) In [4], Qi and his two coauthors reviewed and surveyed some results developed in recent years about several functions involving the Riemann zeta function $\zeta(x)$ and about several sequences involving the ratio $\frac{|B_{2(j+1)}|}{|B_{2j}|}$.

As a continuation of the papers [13, 22, 23, 26], in what follows, we will consider the following two problems.

(1) What about the monotonicity of the function

$$t\mapsto\binom{t+\rho+j}{\rho}\frac{\zeta(t+\rho)}{\zeta(t)}\tag{1.3}$$

on $(1,\infty)$? where $\rho>0$ is a constant, $j\in\mathbb{N}_0$, and the extended binomial coefficient $\binom{z}{w}$ is defined by

$$\binom{z}{w}=\begin{cases}\dfrac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}, & z\notin\mathbb{N}_-,\quad w,z-w\notin\mathbb{N}_-\\ 0, & z\notin\mathbb{N}_-,\quad w\in\mathbb{N}_-\text{ or }z-w\in\mathbb{N}_-\\ \dfrac{\langle z\rangle_w}{w!}, & z\in\mathbb{N}_-,\quad w\in\mathbb{N}_0\\ \dfrac{\langle z\rangle_{z-w}}{(z-w)!}, & z,w\in\mathbb{N}_-,\quad z-w\in\mathbb{N}_0\\ 0, & z,w\in\mathbb{N}_-,\quad z-w\in\mathbb{N}_-\\ \infty, & z\in\mathbb{N}_-,\quad w\notin\mathbb{Z}\end{cases}$$

in terms of the gamma function $\Gamma(z)$ and the falling factorial

$$\langle\nu\rangle_m=\prod_{j=0}^{m-1}(\nu-j)=\begin{cases}\nu(\nu-1)\cdots(\nu-m+1), & m\in\mathbb{N}\\ 1, & m=0\end{cases}$$

for $\nu\in\mathbb{C}$.

(2) What about the convexity of the function $t\mapsto\Gamma(t+j)\zeta(t)$ on $(1,\infty)$ for $j\in\mathbb{N}$?

If $(-1)^j g^{(j)}(t) \ge 0$ for $j \ge 0$ holds on an interval $I \subseteq \mathbb{R}$, then we say that $g(t)$ is a completely monotonic function on I; see [14, Chapter XIII], [25, Chapter 1], and [31, Chapter IV]. If $(-1)^j[\ln g(t)]^{(j)} \ge 0$ for $j \in \mathbb{N}$ holds on $I \subseteq \mathbb{R}$, then we call $g(t)$ a logarithmically completely monotonic function on $I \subseteq \mathbb{R}$; see the papers [5], [9, Definition 1], and [20, Definition 1]. If $g^{(2j)}(t) \ge 0$ for $j \ge 0$ holds on an interval $I \subseteq \mathbb{R}$, then we say that $g(t)$ is an absolutely convex function on I; if $(-1)^j G^{(2j)}(t) \ge 0$ for $j \ge 0$ holds on an interval $I \subseteq \mathbb{R}$, then we say that $g(t)$ is a completely convex function on I; see [14, p. 375, Definition 3]. These and related functions are important in the theory of entire functions.

Our main results in this paper are as follows:

(1) the increasing monotonicity of the function defined in (1.3) are discussed;
(2) the absolute convexity and logarithmic convexity of the function $\Gamma(t+j)\zeta(t)$ are studied;
(3) the increasing monotonicity of a sequence involving the ratio $\frac{|B_{2n+4}|}{|B_{2n+2}|}$ and inequalities for the ratio $\frac{|B_{2n+2}|}{|B_{2n}|}$ are derived.

2. Preliminaries

For proving our main results, we prepare necessary lemmas below.

Lemma 1 ([17, Lemma 9] and [21, Remark 7.2]). *Let $U(y)$, $V(y)$, $W(y,t)$ be integrable in $y \in (a,b) \subseteq \mathbb{R}$ and satisfy $V(y) > 0$ and $W(y,t) > 0$.*

(1) *If the ratios $\frac{\partial W(y,t)/\partial t}{W(y,t)}$ and $\frac{U(y)}{V(y)}$ are both increasing or both decreasing in $y \in (a,b) \subseteq \mathbb{R}$, then the ratio*
$$R(t) = \frac{\int_a^b W(y,t)U(y)\,\mathrm{d}y}{\int_a^b W(y,t)V(y)\,\mathrm{d}y}$$
is increasing in t.
(2) *If one of the ratios $\frac{\partial W(y,t)/\partial t}{W(y,t)}$ and $\frac{U(y)}{V(y)}$ is increasing and another one of them is decreasing in $y \in (a,b) \subseteq \mathbb{R}$, then the ratio $R(t)$ is decreasing in t.*

Lemma 2 ([10, Theorem 1.3], [11, Theorems 2.1 and 2.2], [32, Theorems 3.1 and 3.2]). *Let $\varphi \ne 0$ and $\phi \ne 0$ be real constants and $j \in \mathbb{N}$. If $\varphi > 0$ and $x \ne -\frac{\ln\varphi}{\phi}$ or if $\varphi < 0$ and $x \in \mathbb{R}$, then*
$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left(\frac{1}{\varphi\,\mathrm{e}^{\phi x}-1}\right) = (-1)^j\phi^j\sum_{q=1}^{j+1}(q-1)!S(j+1,q)\left(\frac{1}{\varphi\,\mathrm{e}^{\phi x}-1}\right)^q, \tag{2.1}$$
where
$$S(j,q) = \frac{1}{q!}\sum_{\ell=1}^{q}(-1)^{q-\ell}\binom{q}{\ell}\ell^j, \quad 1 \le q \le j$$
are the second kind Stirling numbers.

About the second kind Stirling numbers $S(m,j)$ for $m \ge j \ge 0$, please see [1, Section 24.1.4], [27, Section 1.3], and the literature [16, 18, 24].

Lemma 3 ([8] and [28, p. 395]). *If $G(t)$ is not identically zero and is completely monotonic on $(0,\infty)$, then $G^{(j)}(t)$ for $j \in \mathbb{N}_0$ is impossibly equal to 0 on $(0,\infty)$.*

Lemma 4. *When $m = 0, 1$, the formula*
$$\Gamma(t+m)\zeta(t) = (-1)^m\int_0^\infty\left(\frac{1}{\mathrm{e}^x-1}\right)^{(m)}x^{t+m-1}\,\mathrm{d}x \tag{2.2}$$
is valid for $\Re(t) > 1$. When $m \ge 2$, the formula (2.2) is valid for $\Re(t) > 2$.

Proof. For $m = 0$, the formula (2.2) is just the one (1.2).

For $m=1$ and $\Re(t)>1$, we have

$$\Gamma(t+1)\zeta(t)=t\Gamma(t)\zeta(t)=t\int_0^\infty\frac{1}{\mathrm{e}^x-1}x^{t-1}\,\mathrm{d}x=\int_0^\infty\frac{1}{\mathrm{e}^x-1}\frac{\mathrm{d}x^t}{\mathrm{d}x}\,\mathrm{d}x$$
$$=\frac{x^t}{\mathrm{e}^x-1}\bigg|_{x=0}^{x=\infty}-\int_0^\infty\biggl(\frac{1}{\mathrm{e}^x-1}\biggr)'x^t\,\mathrm{d}x=-\int_0^\infty\biggl(\frac{1}{\mathrm{e}^x-1}\biggr)'x^t\,\mathrm{d}x.$$

Assume that the formula (2.2) is valid for some $m\ge2$ and $\Re(t)>2$. Then

$$\Gamma(t+m+1)\zeta(t)=(t+m)\Gamma(t+m)\zeta(t)$$
$$=(t+m)\int_0^\infty\biggl(\frac{1}{\mathrm{e}^x-1}\biggr)^{(m)}x^{t+m-1}\,\mathrm{d}x=(-1)^m\int_0^\infty\biggl(\frac{1}{\mathrm{e}^x-1}\biggr)^{(m)}\frac{\mathrm{d}x^{t+m}}{\mathrm{d}x}\,\mathrm{d}x$$
$$=(-1)^m\Biggl(\Biggl[\biggl(\frac{1}{\mathrm{e}^x-1}\biggr)^{(m+1)}x^{t+m}\Biggr]\Bigg|_{x=0}^{x=\infty}-\int_0^\infty\biggl(\frac{1}{\mathrm{e}^x-1}\biggr)^{(m+1)}x^{t+m}\,\mathrm{d}x\Biggr]\Biggr)$$
$$=(-1)^{m+1}\int_0^\infty\biggl(\frac{1}{\mathrm{e}^x-1}\biggr)^{(m+1)}x^{t+m}\,\mathrm{d}x,$$

where we used the fact that, by virtue of Lemma 2 for $\phi=\varphi=1$,

$$\Biggl[\biggl(\frac{1}{\mathrm{e}^x-1}\biggr)^{(m+1)}x^{t+m}\Biggr]\Bigg|_{x=0}^{x=\infty}=(-1)^{m+1}\Biggl[\Biggl(\sum_{q=1}^{m+2}(q-1)!S(m+2,q)\biggl(\frac{1}{\mathrm{e}^x-1}\biggr)^q\Biggr)x^{t+m}\Biggr]\Bigg|_{x=0}^{x=\infty}$$
$$=(-1)^{m+1}\sum_{q=1}^{m+2}(q-1)!S(m+2,q)\Biggl[\biggl(\frac{1}{\mathrm{e}^x-1}\biggr)^q x^{t+m}\Biggr]\Bigg|_{x=0}^{x=\infty}$$
$$=(-1)^{m+1}\sum_{q=1}^{m+2}(q-1)!S(m+2,q)\Biggl(\lim_{x\to\infty}\Biggl[\biggl(\frac{1}{\mathrm{e}^x-1}\biggr)^q x^{t+m}\Biggr]-\lim_{x\to0^+}\Biggl[\biggl(\frac{x}{\mathrm{e}^x-1}\biggr)^q x^{t+m-q}\Biggr]\Biggr)$$
$$=0$$

for $\Re(t)>2$. Consequently, by induction, we are sure that the formula (2.2) is valid for $m\ge2$ and $\Re(t)>2$. The proof of Theorem 4 is complete. □

Lemma 5 ([30, Theorem 1])**.** *For $j\in\mathbb{N}_0$, the function*

$$\mathscr{G}_j(t)=(-1)^j\biggl(\frac{1}{\mathrm{e}^t-1}\biggr)^{(j)}\tag{2.3}$$

is completely monotonic on $(0,\infty)$. In particular, the function $\mathscr{G}_0(t)$ is logarithmically completely monotonic on $(0,\infty)$.

3. Monotonicity result and absolute convexity

Our main results and their proofs are as follows.

Theorem 3.1. *Let $\rho>0$ be a scalar and let $j\in\mathbb{N}_0$. Then*

1. *the function in (1.3) for $j=0$ is an increasing function from $(1,\infty)$ onto $(0,\infty)$;*
2. *the function in (1.3) for $j\in\mathbb{N}$ is an increasing function from $(2,\infty)$ onto $(0,\infty)$;*
3. *the function $\Gamma(t+j)\zeta(t)$ for $j=0,1$ is an absolutely convex function in $t\in(1,\infty)$;*
4. *the function $\Gamma(t+j)\zeta(t)$ for given $j\ge2$ is an absolutely convex function in $t\in(2,\infty)$;*
5. *the function $\Gamma(t+j)\zeta(t)$ for given $j\in\mathbb{N}$ is a logarithmically convex function in $t\in(2,\infty)$.*

Proof. Making use of the formula (2.2) in Lemma 4, we obtain

$$\frac{\Gamma(t+\rho+1)\zeta(t+\rho)}{\Gamma(t+1)\zeta(t)}=\frac{-\int_0^\infty\bigl(\frac{1}{\mathrm{e}^x-1}\bigr)'x^{t+\rho}\,\mathrm{d}x}{-\int_0^\infty\bigl(\frac{1}{\mathrm{e}^x-1}\bigr)'x^{t}\,\mathrm{d}x}=\frac{\int_0^\infty\frac{\mathrm{e}^x}{(\mathrm{e}^x-1)^2}x^{t+\rho}\,\mathrm{d}x}{\int_0^\infty\frac{\mathrm{e}^x}{(\mathrm{e}^x-1)^2}x^{t}\,\mathrm{d}x},\quad \Re(t)>1.$$

Applying Lemma 1 to

$$U(x)=\frac{\mathrm{e}^x x^\rho}{(\mathrm{e}^x-1)^2},\quad V(x)=\frac{\mathrm{e}^x}{(\mathrm{e}^x-1)^2}>0,\quad W(x,t)=x^t>0,$$

and $(a,b)=(0,\infty)$, making use of the facts that both $\frac{U(x)}{V(x)}=x^\rho$ and

$$\frac{\partial W(x,t)/\partial t}{W(x,t)}=\ln x \tag{3.1}$$

are increasing on $(0,\infty)$, we conclude that the ratio

$$\frac{\int_0^\infty\frac{\mathrm{e}^x}{(\mathrm{e}^x-1)^2}x^{t+\rho}\,\mathrm{d}x}{\int_0^\infty\frac{\mathrm{e}^x}{(\mathrm{e}^x-1)^2}x^{t}\,\mathrm{d}x}=\frac{\Gamma(t+\rho+1)\zeta(t+\rho)}{\Gamma(t+1)\zeta(t)}=\Gamma(\rho+1)\binom{t+\rho}{\rho}\frac{\zeta(t+\rho)}{\zeta(t)}$$

is increasing in $t\in(1,\infty)$. Consequently, the function in (1.3) for $j=0$ is increasing in $t\in(1,\infty)$.

Once making use of the formula (2.2) in Lemma 4, for $j,m\ge2$, we obtain

$$\frac{\Gamma(t+\rho+j)\zeta(t+\rho)}{\Gamma(t+m)\zeta(t)}=(-1)^{j-m}\frac{\int_0^\infty\bigl(\frac{1}{\mathrm{e}^x-1}\bigr)^{(j)}x^{t+\rho+j-1}\,\mathrm{d}x}{\int_0^\infty\bigl(\frac{1}{\mathrm{e}^x-1}\bigr)^{(m)}x^{t+m-1}\,\mathrm{d}x}=\frac{\int_0^\infty\mathscr{G}_j(x)x^{t+\rho+j-1}\,\mathrm{d}x}{\int_0^\infty\mathscr{G}_m(x)x^{t+m-1}\,\mathrm{d}x}$$

for $\Re(t)\ge2$. By Lemmas 3 and 5, we see that the functions $\mathscr{G}_k(x)$ for $k\ge0$ are all positive on $(0,\infty)$.

Once applying Lemma 1 to

$$U(x)=\mathscr{G}_j(x)x^{\rho+j},\quad V(x)=\mathscr{G}_m(x)x^m>0,\quad W(x,t)=x^{t-1}>0,$$

and $(a,b)=(0,\infty)$, since $\frac{U(x)}{V(x)}=\frac{\mathscr{G}_j(x)}{\mathscr{G}_m(x)}x^{j-m+\rho}$ for $m=j$ and the partial derivative in (3.1) are both increasing on $(0,\infty)$, we acquire that the ratio

$$\frac{\Gamma(t+\rho+j)\zeta(t+\rho)}{\Gamma(t+j)\zeta(t)}=\Gamma(\rho+1)\binom{t+\rho+j-1}{\rho}\frac{\zeta(t+\rho)}{\zeta(t)}=\frac{\int_0^\infty\mathscr{G}_j(x)x^{t+\rho+j-1}\,\mathrm{d}x}{\int_0^\infty\mathscr{G}_j(x)x^{t+j-1}\,\mathrm{d}x}$$

for $j\ge2$ and $\rho>0$ is increasing in $t\in(2,\infty)$. Consequently, the function in (1.3) for $j\ge1$ and $\rho>0$ is increasing in $t\in(2,\infty)$.

Because the ratio $\frac{\Gamma(t+\rho+j)\zeta(t+\rho)}{\Gamma(t+j)\zeta(t)}$ for given $j\in\mathbb{N}$ is increasing in $t\in(2,\infty)$, its derivative

$$\left[\frac{\Gamma(t+\rho+j)\zeta(t+\rho)}{\Gamma(t+j)\zeta(t)}\right]'=\frac{[\Gamma(t+\rho+j)\zeta(t+\rho)]'[\Gamma(t+j)\zeta(t)]-[\Gamma(t+\rho+j)\zeta(t+\rho)][\Gamma(t+j)\zeta(t)]'}{[\Gamma(t+j)\zeta(t)]^2}$$

is nonnegative in $t\in(2,\infty)$. This means that

$$\frac{[\Gamma(t+\rho+j)\zeta(t+\rho)]'}{\Gamma(t+\rho+j)\zeta(t+\rho)}>\frac{[\Gamma(t+j)\zeta(t)]'}{[\Gamma(t+j)\zeta(t)]},\quad t\in(2,\infty),$$

that is, the logarithmic derivative

$$(\ln[\Gamma(t+j)\zeta(t)])'=\frac{[\Gamma(t+j)\zeta(t)]'}{[\Gamma(t+j)\zeta(t)]}$$

is increasing in $t\in(2,\infty)$. Consequently, for given $j\in\mathbb{N}$, the function $\Gamma(t+j)\zeta(t)$ is logarithmically convex in $t\in(2,\infty)$.

Differentiating $2k$ times for $k\in\mathbb{N}_0$ with respect to t on both sides of (2.2) yields

$$[\Gamma(t+m)\zeta(t)]^{(2k)}=\int_0^\infty(-1)^m\left(\frac{1}{\mathrm{e}^x-1}\right)^{(m)}x^{t+m-1}(\ln x)^{2k}\,\mathrm{d}x.$$

By virtue of Lemmas 3 and 5, we see that the completely monotonic function $\mathscr{G}_m(x)=(-1)^m\left(\frac{1}{e^x-1}\right)^{(m)}$ for $m\ge0$ are all positive on $(0,\infty)$. Hence, the derivatives $[\Gamma(t+m)\zeta(t)]^{(2k)}$ for $k\in\mathbb{N}_0$ are positive, that is, they are absolutely convex on their corresponding intervals. The proof of Theorem 3.1 is thus complete. □

4. Monotonicity and bounds for the ratio of Bernoulli numbers

Applying Theorem 3.1, we can derive the following results on the ratio $\left|\frac{B_{2n+2}}{B_{2n}}\right|$.

Theorem 4.1. *The sequences* $\left|\frac{B_{2n+2}}{B_{2n}}\right|$ *and*

$$\frac{(2n+j+3)(2n+j+4)}{(n+2)(2n+3)}\left|\frac{B_{2n+4}}{B_{2n+2}}\right|,\quad j\in\mathbb{N} \tag{4.1}$$

are both increasing in $n\in\mathbb{N}$.

The inequality

$$\left|\frac{B_{2n+2}}{B_{2n}}\right|\ge\frac{(n+1)(2n+1)}{21} \tag{4.2}$$

holds for $n\ge2$ *and the inequality*

$$\frac{B_{2n}B_{2n+4}}{B_{2n+2}^2}\ge\frac{(n+2)(2n+3)}{(n+1)(2n+1)} \tag{4.3}$$

holds for $n\in\mathbb{N}$.

Proof. In [1, pp. 807–808, Section 23.2] and [27, p. 5, (1.14)], we find the relation

$$B_{2n}=\frac{(-1)^{n+1}2(2n)!}{(2\pi)^{2n}}\zeta(2n),\quad n\in\mathbb{N}. \tag{4.4}$$

Hence, from Theorem 3.1, we obtain that the sequence $\binom{2n+2}{2}\frac{\zeta(2n+2)}{\zeta(2n)}$ is increasing in $n\in\mathbb{N}$, that is, the sequence

$$\binom{2n+2}{2}\frac{\frac{(2\pi)^{2n+2}}{(-1)^{n+2}2(2n+2)!}B_{2n+2}}{\frac{(2\pi)^{2n}}{(-1)^{n+1}2(2n)!}B_{2n}}=2\pi^2\left|\frac{B_{2n+2}}{B_{2n}}\right|$$

is increasing in $n\in\mathbb{N}$, that is, the ratio $\left|\frac{B_{2n+2}}{B_{2n}}\right|$ is increasing in $n\in\mathbb{N}$.

Employing the relation (4.4) and Theorem 3.1, we see that the sequence

$$\begin{aligned}\binom{2n+j+4}{2}\frac{\zeta(2n+4)}{\zeta(2n+2)}&=\binom{2n+j+4}{2}\frac{\frac{(2\pi)^{2n+4}}{(-1)^{n+3}2(2n+4)!}B_{2n+4}}{\frac{(2\pi)^{2n+2}}{(-1)^{n+2}2(2n+2)!}B_{2n+2}}\\&=\pi^2\frac{(2n+j+3)(2n+j+4)}{(n+2)(2n+3)}\left|\frac{B_{2n+4}}{B_{2n+2}}\right|\end{aligned}$$

for fixed $j\in\mathbb{N}$ is increasing in $n\in\mathbb{N}$. In [34, Lemma 1], see also [4, Theorem 6], among other things, the sequence

$$\frac{1}{(2n+1)(n+1)}\frac{2^{2n+2}-1}{2^{2n}-1}\left|\frac{B_{2n+2}}{B_{2n}}\right|$$

was proved to be increasing in $n\in\mathbb{N}$ and to tend to $\frac{2}{\pi^2}$ as $n\to\infty$. Accordingly, we arrive at

$$\begin{aligned}&\lim_{n\to\infty}\left[\frac{(2n+j+3)(2n+j+4)}{(n+2)(2n+3)}\left|\frac{B_{2n+4}}{B_{2n+2}}\right|\right]\\&=\lim_{n\to\infty}\left[\frac{(2n+j+3)(2n+j+4)}{(n+2)(2n+3)}(2n+3)(n+2)\frac{2^{2n+2}-1}{2^{2n+4}-1}\right]\\&\quad\times\lim_{n\to\infty}\left[\frac{1}{(2n+3)(n+2)}\frac{2^{2n+4}-1}{2^{2n+2}-1}\left|\frac{B_{2n+4}}{B_{2n+2}}\right|\right]\\&=\frac{2}{\pi^2}\lim_{n\to\infty}\left[(2n+j+3)(2n+j+4)\frac{2^{2n+2}-1}{2^{2n+4}-1}\right]\end{aligned}$$

$$= \infty, \quad j \in \mathbb{N}.$$

Therefore, we are only able to acquire two one-sided inequalities

$$\pi^2 \frac{(2n+j+3)(2n+j+4)}{(n+2)(2n+3)} \left| \frac{B_{2n+4}}{B_{2n+2}} \right| \ge \pi^2 \frac{(j+5)(j+6)}{15} \left| \frac{B_6}{B_4} \right|$$

and

$$\pi^2 \frac{(2n+j+5)(2n+j+6)}{(n+3)(2n+5)} \left| \frac{B_{2n+6}}{B_{2n+4}} \right| \ge \pi^2 \frac{(2n+j+3)(2n+j+4)}{(n+2)(2n+3)} \left| \frac{B_{2n+4}}{B_{2n+2}} \right|$$

for $j \in \mathbb{N}$ and $n \in \mathbb{N}$. That is,

$$\left| \frac{B_{2n+4}}{B_{2n+2}} \right| \ge \frac{(n+2)(2n+3)}{15} \frac{(j+5)(j+6)}{(2n+j+3)(2n+j+4)} \left| \frac{B_6}{B_4} \right|$$

and

$$\frac{B_{2n+2}B_{2n+6}}{B_{2n+4}^2} \ge \frac{(n+3)(2n+5)}{(n+2)(2n+3)} \frac{(2n+j+3)(2n+j+4)}{(2n+j+5)(2n+j+6)}$$

for $j \in \mathbb{N}$ and $n \in \mathbb{N}$. Since the sequences

$$\frac{(j+5)(j+6)}{(2n+j+3)(2n+j+4)} \quad \text{and} \quad \frac{(2n+j+3)(2n+j+4)}{(2n+j+5)(2n+j+6)}$$

are increasing in $j \in \mathbb{N}$ and tend to 1 as $j \to \infty$ for all fixed $n \in \mathbb{N}$ respectively, we acquire

$$\left| \frac{B_{2n+4}}{B_{2n+2}} \right| \ge \frac{(n+2)(2n+3)}{15} \left| \frac{B_6}{B_4} \right| = \frac{(n+2)(2n+3)}{21}$$

and

$$\frac{B_{2n+2}B_{2n+6}}{B_{2n+4}^2} \ge \frac{(n+3)(2n+5)}{(n+2)(2n+3)} \tag{4.5}$$

for $n \in \mathbb{N}$. Direct computation shows that the inequality (4.5) is also valid for $n = 0$. The proof of Theorem 4.1 is complete. □

Remark 1. The first conclusion in Theorem 4.1 that the ratio $\left|\frac{B_{2n+2}}{B_{2n}}\right|$ is increasing in $n \in \mathbb{N}$ is a recovery of [26, Theorem 1.1]. The increasing monotonicity of the sequence (4.1) is a recovery of [4, Theorem 5] and [26, Theorems 1.1 and 1.2].

The inequality (4.2) is better than

$$\left| \frac{B_{2n+2}}{B_{2n}} \right| \ge \frac{(n+1)(2n+1)}{30}, \quad n \in \mathbb{N}$$

obtained in [4, Remark 1].

The inequality (4.3) is stronger than the logarithmic convexity of the sequence B_{2n} for $n \in \mathbb{N}$, which was derived in [26, Theorem 1.1].

5. A short appendix

In this section, we slightly strengthen [30, Theorem 3] as follows.

Proposition 1. *For $j \in \mathbb{N}_0$, the ratio*

$$\mathfrak{G}_j(x) = \frac{\mathscr{G}_{j+1}(x)}{\mathscr{G}_j(x)} \tag{5.1}$$

is decreasing from $(0,\infty)$ onto $(1,\infty)$, where the function $\mathscr{G}_j(x)$ is defined by (2.3) *in Lemma 5.*

Proof. In [30, Theorem 3], the decreasing monotonicity of the ratio $\mathfrak{G}_j(x)$ in (5.1) and $\lim_{x\to\infty} \mathfrak{G}_j(x) = 1$ has been proved.

Employing the equation (2.1) in Lemma 2 for $\varphi = \phi = 1$ yields

$$\mathfrak{G}_j(x) = \frac{(-1)^{j+1}\left(\frac{1}{\mathrm{e}^x - 1}\right)^{(j+1)}}{(-1)^j\left(\frac{1}{\mathrm{e}^x - 1}\right)^{(j)}}$$

$$
\begin{aligned}
&= \frac{\sum_{q=1}^{j+2}(q-1)!S(j+2,q)\left(\frac{1}{e^x-1}\right)^q}{\sum_{q=1}^{j+1}(q-1)!S(j+1,q)\left(\frac{1}{e^x-1}\right)^q}\\
&= \frac{\sum_{q=1}^{j+2}(q-1)!S(j+2,q)\left(\frac{x}{e^x-1}\right)^q x^{j-q+1}}{\sum_{q=1}^{j+1}(q-1)!S(j+1,q)\left(\frac{x}{e^x-1}\right)^q x^{j-q+1}}\\
&\to \frac{j!S(j+2,j+1)+(j+1)!S(j+2,j+2)\lim_{x\to0^+}x^{-1}}{j!S(j+1,j+1)}, \quad x\to0^+\\
&= \infty.
\end{aligned}
$$

The proof of Proposition 1 is thus complete. □

Remark 2. This paper is a revised version of the electronic arXiv preprint arXiv:2201.06970 whose DOI site is `https://doi.org/10.48550/arXiv.2201.06970`.

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