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HIGHER ORDER VERSIONS OF THE k-JACOBSTHAL-LUCAS NUMBERS AND THEIR QUATERNION REPRESENTATIONS

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Abstract. In this paper, our research focuses on a new generalization that extends the k-Jacobsthal–Lucas numbers to a broader class. Higher order versions of the k-Jacobsthal–Lucas numbers together with their quaternion counterparts are explored. The study commences with the formulation of the higher order sequence and an examination of its principal characteristics. Recurrence relations, a Binet-type expression, generating functions, and a collection of noteworthy identities are derived. Subsequently, quaternion representations corresponding to these numbers are introduced, and their properties are addressed both within the framework of quaternion algebra and in the setting of number sequences.

1. Introduction

The study of integer sequences and their generalizations contributes significantly to different branches of mathematics, including combinatorics, number theory, and algebraic structures. To date, number sequences most notably the Fibonacci sequence and its generalizations have found applications in nearly all areas of science and art. Among these sequences, the Jacobsthal and Jacobsthal–Lucas numbers have attracted considerable attention due to their rich algebraic properties and diverse applications in coding theory, recurrence relations, and combinatorial identities [1, 2, 4–16].
The Fibonacci numbers are defined by the following relation [8]:

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

with $F_0 = 0$ and $F_1 = 1$.

In her master's thesis, Ozvatan defined higher-order Fibonacci numbers as a generalization of the Fibonacci numbers [22]. Thus, higher order Fibonacci numbers are defined by

$$F_n^{(s)} = \frac{F_{ns}}{F_s}.$$

The Jacobsthal numbers, first introduced by E. Jacobsthal in 1956, are defined by the second-order linear recurrence relation for $n \geqslant 2$ [1, 3]:

$$J_0 = 0, \quad J_1 = 1, \quad J_n = J_{n-1} + 2J_{n-2}.$$

Similarly, the Jacobsthal-Lucas numbers

$$j_0 = 2, \quad j_1 = 1, \quad j_n = j_{n-1} + 2j_{n-2}, \quad n \geq 2$$

[1].
As a novel generalization of the Jacobsthal–Lucas numbers, the k-Jacobsthal–Lucas numbers are defined by the following relation [10]:

$$s_{k,n} = s_{k,n-1} + 2ks_{k,n-2}, \quad n \geq 2$$

with $s_{k,0} = 2$, $s_{k,1} = 1$. According to this recurrence relation, the roots of the second-order characteristic equation are

$$a = \frac{1 + \sqrt{1 + 8k}}{2}$$

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and

$$b = \frac{1-\sqrt{1+8k}}{2}.$$

The properties of these roots are given as follows:

$$a+b=1, \quad ab=-2k, \quad a-b=\sqrt{1+8k}.$$

The Binet formula is

$$s_{k,n} = a^n + b^n.$$

Quaternions were introduced by Hamilton in 1843 as an extension of the complex number system. This number system builds on complex numbers. Hamilton found that there isn't a consistent way to multiply three components, but four components do work. He called this idea quaternions. They are a four-component algebra that comes after real and complex numbers, and their multiplication is non-commutative. A quaternion q can be written as follows [17]:

$$\left\{ q = q_0 + q_1\mathbb{i} + q_2\mathbb{j} + q_3\mathbb{k} \mid q_0, q_1, q_2, q_3 \in \mathbb{R} \right\}.$$

Quaternions are utilized in various scientific fields such as theoretical physics, mathematics, computer graphics and animation, aviation and robotics, video games, space science, machine learning, and simulation. They have also been linked to numerical sequences in mathematics, with numerous studies conducted on this topic [17–27]. Horadam first defined Fibonacci quaternions and gave several properties [18]:

$$Q_n = F_n + F_{n+1}\mathbb{i} + F_{n+2}\mathbb{j} + F_{n+3}\mathbb{k}.$$

Later, Jacobsthal quaternions were defined by Szynal-Liana, A et.al for the generalized of the Jacobsthal number sequences [20]:

$$JQ_n = J_n + J_{n+1}\mathbb{i} + J_{n+2}\mathbb{j} + J_{n+3}\mathbb{k}.$$

One of the significant recent works related to this topic is [23], focusing on the introduction of higher order Fibonacci quaternions:

$$Q_n^{(s)} = F_n^{(s)} + F_{n+1}^{(s)}\mathbb{i} + F_{n+2}^{(s)}\mathbb{j} + F_{n+3}^{(s)}\mathbb{k}.$$

Later in addition to these studies, the authors defined the higher order forms of Jacobsthal and Jacobsthal-Lucas numbers and examined features of these numbers [24, 25]:

$$i)\ J_n^{(s)} = \frac{J_{ns}}{J_s},$$

$$ii)\ j_n^{(s)} = \frac{j_{ns}}{j_s}.$$

In [24, 25], the authors also defined and investigated the quaternion counterparts of these numbers:

$$i)\ OJ_n^{(s)} = J_n^{(s)} + J_{n+1}^{(s)}\mathbb{i} + J_{n+2}^{(s)}\mathbb{j} + J_{n+3}^{(s)}\mathbb{k},$$

$$ii)\ Oj_n^{(s)} = j_n^{(s)} + j_{n+1}^{(s)}\mathbb{i} + j_{n+2}^{(s)}\mathbb{j} + j_{n+3}^{(s)}\mathbb{k}.$$

In this study, inspired by [22, 23], we define higher order versions of the k-Jacobsthal–Lucas numbers and their associated quaternions.

2. The Higher Order versions of the k-Jacobsthal-Lucas Numbers

In this section, we define the generalized versions of the k-Jacobsthal-Lucas numbers of higher order and investigate their fundamental properties. We also derive the recurrence relation, Binet formula, generating functions, summation formulas, and several related identities.

Definition 2.1. The higher order versions of the k-Jacobsthal-Lucas numbers are characterized by a specific equation:

$$s_{k,n}^{(t)} = \frac{s_{k,nt}}{s_{k,t}},$$

where $s_{k,t}$ denotes the k-Jacobsthal-Lucas numbers.

The Binet formula is a closed-form expression that typically allows us to compute each term of a number sequence (such as the Fibonacci, Balancing, or Pell sequence) directly, without using the recurrence relation. Terms of the the generalized versions of the k-Jacobsthal-Lucas numbers of higher order take the following form.

- For $t = 1$, k-Jacobsthal–Lucas numbers are obtained:
$$s_{k,n}^{(1)} = \frac{a^n + b^n}{a + b} = s_{k,n}.$$
- For $t = 1$ and $k = 1$ Jacobsthal numbers are obtained.
- For $t = 2, n = 1$,
$$s_{k,1}^{(2)} = \frac{s_{k,2}}{s_{k,2}} = 1.$$
- For $t = 2, n = 2$,
$$s_{k,2}^{(2)} = \frac{s_{k,4}}{s_{k,2}} = \frac{8k^2 + 8k + 1}{4k + 1}.$$
- For $t = 2, n = 3$,
$$s_{k,3}^{(2)} = \frac{s_{k,6}}{s_{k,2}} = \frac{16k^3 + 36k^2 + 12k + 1}{4k + 1}.$$
- For $t = 2, n = 4$,
$$s_{k,4}^{(2)} = \frac{s_{k,8}}{s_{k,2}} = \frac{88k^4 + 128k^3 + 80k^2 + 16k + 1}{4k + 1}.$$

For $k = 0$, the constant sequence $\{1, 1, 1, 1, \ldots\}$ is obtained.

- For $t = 3$, higher order versions of the k–Jacobsthal–Lucas numbers $s_{k,n}^{(3)}$ are
$$s_{k,n}^{(3)} = \frac{s_{k,3n}}{s_{k,3}}.$$

Now, we present the recurrence-like relation of the higher order versions of the $k-$Jacobsthal–Lucas numbers as follows.

Theorem 2.1. *For $n \geq 1$, the higher order versions of the k-Jacobsthal-Lucas numbers satisfy the recurrence-like relation defined by the following equation:*
$$s_{k,n+1}^{(t)} = s_{k,t} s_{k,n}^{(t)} - (-2k)^t s_{k,n-1}^{(t)}.$$

Proof.
$$\begin{aligned}
s_{k,n+1}^{(t)} &= \frac{a^{tn+t} + b^{tn+t}}{a^t + b^t} \\
&= \frac{a^{tn+t} + b^{tn}a^t + b^{tn}a^t - b^{tn}a^t}{a^t + b^t} \\
&= a^t s_{k,n}^{(t)} + \frac{b^{tn}b^s - b^{tn}a^t}{a^t + b^t} \\
&= (a^t + b^t)s_{k,n}^{(t)} - b^t s_{k,n}^{(t)} + \frac{b^{tn}b^t - b^{tn}a^t}{a^t + b^t} \\
&= s_{k,t}s_{k,n}^{(t)} - \frac{(ab)^t}{a^t + b^t}\left(a^{tn-t} + b^{tn-t}\right) \\
&= s_{k,t}s_{k,n}^{(t)} - (ab)^t\left(\frac{a^{s(n-1)} + b^{s(n-1)}}{a^t + b^t}\right).
\end{aligned}$$

Since
$$ab = -2k$$

and

$$s_{k,n-1}^{(t)}=\frac{a^{t(n-1)}+b^{t(n-1)}}{a^{t}+b^{t}},$$

we have

$$\mathbf{s}_{k,n+1}^{(t)}=s_{k,t}\mathbf{s}_{k,n}^{(t)}-(-2k)^{t}\mathbf{s}_{k,n-1}^{(t)}.$$

□

Theorem 2.2. *For $m=0,1,\ldots,r-1$, the k-Jacobsthal-Lucas numbers satisfy the generalized recurrence-like relation:*

$$s_{k,(n+1)r+m}=s_{k,r}s_{k,nr+m}-(-2k)^{t}s_{k,r(n-1)+m}.$$

Proof.

$$\begin{aligned}
s_{k,(n+1)r+m}&=a^{(n+1)r+m}+b^{(n+1)r+m}\\
&=a^{nr+m}a^{r}+b^{nr+m}b^{r}+b^{nr+m}a^{r}-b^{nr+m}b^{r}\\
&=a^{r}\left(a^{nr+m}+b^{nr+m}\right)+b^{nr+m}b^{r}-b^{nr+m}a^{r}\\
&=a^{r}s_{k,nr+m}+b^{nr+m}b^{r}-b^{nr+m}a^{r}\\
&=s_{k,r}s_{k,nr+m}+b^{nr+m}b^{r}-b^{nr+m}a^{r}-a^{nr+m}b^{r}\\
&\quad -b^{nr+m}b^{r}\\
&=s_{k,r}s_{k,nr+m}-(ab)^{r}\left(b^{nr+m-r}+a^{nr+m-r}\right).
\end{aligned}$$

Since $ab=-2k$ and $s_{k,n}=a^{n}+b^{n}$, we have

$$s_{k,(n+1)r+m}=s_{k,r}s_{k,nr+m}-(2k)^{r}s_{k,r(n-1)+m}.$$

As a result, the statement has been successfully established. □

Proposition 2.1. *The higher order versions of the k–Jacobsthal–Lucas numbers satisfy the following relations for negative value n and t. For $n,s\geq 0$, we have*

$$\begin{aligned}
&i)\quad s_{k,-n}^{(t)}=-(-2k)^{-tn}\,s_{k,n}^{(t)},\\
&ii)\quad s_{k,-n}^{(-t)}=(-2k)^{t}\,s_{k,n}^{(t)},\\
&iii)\quad s_{k,n}^{(-t)}=(-2)^{t(1-n)}\,s_{k,n}^{(t)}.
\end{aligned}$$

Proof.

$$\begin{aligned}
s_{k,-n}^{(t)}&=\frac{(a^{t})^{-n}+(b^{t})^{-n}}{a^{t}+b^{t}}\\
&=\frac{\frac{1}{a^{tn}}+\frac{1}{b^{tn}}}{a^{t}+b^{t}}\\
&=\left(\frac{b^{tn}+a^{tn}}{(ab)^{tn}}\right)\left(\frac{1}{a^{t}+b^{t}}\right)\\
&=-(ab)^{-tn}\left(\frac{a^{tn}+b^{tn}}{a^{t}+b^{t}}\right)
\end{aligned}$$

Since $ab=-2k$ and

$$s_{k,n}^{(t)}=\frac{a^{tn}+b^{tn}}{a^{t}+b^{t}},$$

we have

$$s_{k,-n}^{(t)}=-(-2k)^{-tn}s_{k,n}^{(t)}.$$

□

As a result, the statement has been successfully established. The proofs of the remaining parts are obtained similarly.

Theorem 2.3. *The limit value of ratio between successive terms of the higher order versions of the k-Jacobsthal-Lucas numbers is*

$$\lim_{n\to\infty}\frac{s_{k,n+1}^{(t)}}{s_{k,n}^{(t)}}=a^t.$$

Proof.

$$\begin{aligned}\lim_{n\to\infty}\frac{s_{k,n+1}^{(t)}}{s_{k,n}^{(t)}}&=\lim_{n\to\infty}\frac{a^{tn+t}+b^{tn+t}}{a^t+b^t}\frac{a^t+b^t}{a^{tn}+b^{tn}}\\&=\lim_{n\to\infty}\frac{a^{tn+t}+b^{tn+t}}{a^t+b^t}\\&=\lim_{n\to\infty}\frac{(a^t)^{n+1}\left(1-\left(\frac{b}{a}\right)^{tn+t}\right)}{(a^t)^n\left(1-\left(\frac{b}{a}\right)^{tn}\right)}\end{aligned}$$

Since $|b|<a$, we have

$$\lim_{n\to\infty}\frac{s_{k,n+1}^{(t)}}{s_{k,n}^{(t)}}=\lim_{n\to\infty}\frac{(a^t)^{n+1}}{(a^t)^n}=a^t.$$

□

Theorem 2.4. *For the higher order versions of the k-Jacobsthal-Lucas numbers, the generating function can be expressed as:*

$$\sum_{n=0}^{\infty}s_{k,n}^{(t)}x^n=\frac{\frac{2}{s_{k,t}}-x}{1-s_{k,t}x+(-2k)^tx^2}.$$

Proof. Let

$$G=\sum_{n=0}^{\infty}s_{k,n}^{(t)}x^n$$

$$G=s_{k,0}^{(t)}+s_{k,1}^{(t)}x+s_{k,2}^{(t)}x^2+s_{k,3}^{(t)}x^3+\cdots+s_{k,n}^{(t)}x^n+\cdots \tag{2.1}$$

$$-s_{k,t}x\,G=-s_{k,t}x\,s_{k,0}^{(t)}-s_{k,t}x^2s_{k,1}^{(t)}-s_{k,t}x^3s_{k,2}^{(t)}-s_{k,t}x^4s_{k,3}^{(t)}-\cdots-s_{k,t}x^{n+1}s_{k,n}^{(t)}-\cdots \tag{2.2}$$

$$(-2k)^tx^2G=(-2k)^tx^2s_{k,0}^{(t)}+(-2k)^tx^3s_{k,1}^{(t)}+(-2k)^tx^4s_{k,2}^{(t)}+\cdots+(-2k)^tx^{n+2}s_{k,n}^{(t)}+\cdots \tag{2.3}$$

By applying the necessary operations to 2.1, 2.2 and 2.3, we obtain:

$$G\big(1-s_{k,t}x+(-2k)^tx^2\big)=s_{k,0}^{(t)}+s_{k,1}^{(t)}x-xs_{k,0}^{(t)}s_{k,t}.$$

Since $s_{k,1}^{(t)}=1$ and $s_{k,0}^{(t)}=\frac{2}{s_{k,t}}$, we have

$$G=\frac{\frac{2}{s_{k,t}}-x}{1-s_{k,t}x+(-2k)^tx^2}.$$

□

Theorem 2.5. *For the higher order versions of the k-Jacobsthal–Lucas numbers, the exponential generating function can be expressed as:*

$$\sum_{n=0}^{\infty}s_{k,n}^{(t)}\frac{x^n}{n!}=\frac{e^{a^tx}+e^{b^tx}}{a^t+b^t}.$$

Proof. Let

$$H=\sum_{n=0}^{\infty}s_{k,n}^{(t)}\frac{x^n}{n!}.$$

$$\begin{aligned} H &= \sum_{n=0}^{\infty} \left(\frac{a^{tn} + b^{tn}}{a^t + b^t} \right) \frac{x^n}{n!} \\ &= \frac{1}{a^t + b^t} \left[\sum_{n=0}^{\infty} \frac{(a^{tn} x^n)}{n!} + \sum_{n=0}^{\infty} \frac{(b^{tn} x^n)}{n!} \right] \\ &= \frac{1}{a^t + b^t} \left[\sum_{n=0}^{\infty} \frac{(a^t x)^n}{n!} + \sum_{n=0}^{\infty} \frac{(b^t x)^n}{n!} \right]. \end{aligned}$$

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have

$$\sum_{n=0}^{\infty} s_{k,n}^{(t)} \frac{x^n}{n!} = \frac{e^{a^t x} + e^{b^t x}}{a^t + b^t}.$$

□

Theorem 2.6. *Derivatives of the generating and exponential generating functions of higher order k-Jacobsthal-Lucas numbers are given by*

$$i) \quad G'(x) = \frac{1 + (-2k)^t x \left(x - \frac{4}{s_{k,t}} \right)}{\left(1 - s_{k,t} x + (-2k)^t x^2\right)^2},$$

$$ii) \quad H'(x) = \frac{e^{a^t x} a^t + e^{b^t x} b^t}{a^t + b^t}.$$

Proof.

$$i) \quad G(x) = \frac{\frac{2}{s_{k,t}} - x}{1 - s_{k,t} x + (-2k)^t x^2}.$$

Hence,

$$\begin{aligned} G'(x) &= \frac{-\left(1 - s_{k,t} x + (-2k)^t x^2\right) - \left(\frac{2}{s_{k,t}} - x \right) \left(-s_{k,t} + 2(-2k)^t x \right)}{\left(1 - s_{k,t} x + (-2k)^t x^2\right)^2} \\ &= \frac{1 + (-2k)^t x \left(x - \frac{4}{s_{k,t}} \right)}{\left(1 - s_{k,t} x + (-2k)^t x^2\right)^2}. \end{aligned}$$

ii) can be obtained similarly □

Now, we compute the Cassini, Catalan, d'Ocagne, Vajda and Honsberger identities for the higher order versions of the k-Jacobsthal-Lucas numbers. In the proofs of all these identities, we use the Binet formula of the higher order versions of the k-Jacobsthal-Lucas numbers.

Theorem 2.7. *(Cassini Identity) The following relation holds true for each $n \geq 1$:*

$$s_{k,n-1}^{(t)} \, s_{k,n+1}^{(t)} - \left(s_{k,n}^{(t)} \right)^2 = (-2k)^{t(n-1)} \, \left(a_1^t - a_2^t \right)^2 \, s_{k,t}^{-2}.$$

Proof.

$$\begin{aligned} s_{k,n-1}^{(t)} \, s_{k,n+1}^{(t)} - \left(s_{k,n}^{(t)} \right)^2 &= \frac{a^{tn-t} \, b^{tn+t} + b^{tn-t} \, a^{tn+t} - 2 a^{tn} \, b^{tn}}{\left(a^t + b^t\right)^2} \\ &= \frac{\frac{(ab)^{tn} \, (a^t - b^t)^2}{(ab)^t}}{\left(a^t + b^t\right)^2}. \end{aligned}$$

Since $ab = -2k$ and $a^t + b^t = s_{k,t}$, we have

$$s_{k,n-1}^{(t)} \, s_{k,n+1}^{(t)} - \left(s_{k,n}^{(t)} \right)^2 = (-2k)^{t(n-1)} \, (a^t - b^t)^2 \, s_{k,t}^{-2}.$$

Accordingly, the intended form is achieved. □

Theorem 2.8. *(Catalan Identity) The following relation holds true for each $n \geq 1$:*

$$s_{k,n-r}^{(t)}\, s_{k,n+r}^{(t)} - \left(s_{k,n}^{(t)}\right)^2 = (-2k)^{t(n-r)}\, \left(a^{sr} - b^{sr}\right)^2\, s_{k,t}^{-2}.$$

Proof.

$$\begin{aligned} & s_{k,n-r}^{(t)}\, s_{k,n+r}^{(t)} - \left(s_{k,n}^{(t)}\right)^2 \\ &= \frac{a^{tn-tr}\, b^{tn+tr} + b^{tn-tr}\, a^{tn+tr} - 2a^{tn}\, b^{tn}}{\left(a^t + b^t\right)^2} \\ &= \frac{\frac{(ab)^{tn}\,(a^{tr}-b^{tr})^2}{(ab)^{tr}}}{\left(a^t + b^t\right)^2}. \end{aligned}$$

Since $ab = -2k$ and $a^t + b^t = s_{k,t}$, we have

$$s_{k,n-r}^{(t)}\, s_{k,n+r}^{(t)} - \left(s_{k,n}^{(t)}\right)^2 = (-2k)^{t(n-r)}\, \left(a^{tr} - b^{tr}\right)^2\, s_{k,t}^{-2}.$$

Accordingly, the intended form is achieved. In particular, for $r = 1$, the identity becomes the Cassini identity. □

Theorem 2.9. *(d'Ocagne Identity) The following relation holds true for each integers n, m :*

$$s_{k,m}^{(t)}\, s_{k,n+1}^{(t)} - s_{k,m+1}^{(t)}\, s_{k,n}^{(t)} = \frac{-(-2k)^{tn}\, \left(a^t - b^t\right) \left(a^{t(m-n)} - b^{t(m-n)}\right)}{\left(s_{k,t}\right)^2}.$$

Proof. We have

$$\begin{aligned} s_{k,m}^{(t)}\, s_{k,n+1}^{(t)} - s_{k,m+1}^{(t)}\, s_{k,n}^{(t)} &= \frac{a^{tm}\, b^{tn+t} + b^{tm}\, a^{tn+t} - a^{tm+t}\, b^{tn} - b^{tm+t}\, a^{tn}}{(a^t + b^t)^2} \\ &= \frac{(a^t - b^t)(a^{tn}b^{tm} - a^{tm}b^{tn})}{(a^t + b^t)^2} \\ &= \frac{(a^t - b^t)(ab)^{tn}(b^{t(m-n)} - a^{t(m-n)})}{(a^t + b^t)^2} \\ &= \frac{-(-2k)^{tn}(a^t - b^t)(a^{t(m-n)} - b^{t(m-n)})}{(s_{k,t})^2}. \end{aligned}$$

Accordingly, the intended form is achieved. □

Theorem 2.10. *(Vajda Identity) The following relation holds true for each integers n, i, j :*

$$s_{k,n+i}^{(t)}\, s_{k,n+j}^{(t)} - s_{k,n}^{(t)}\, s_{k,n+i+j}^{(t)} = -\frac{(-2k)^{tn}\, \left(a^{tj} - b^{tj}\right) \left(a^{ti} - b^{ti}\right)}{\left(s_{k,t}\right)^2}.$$

Proof.

$$\begin{aligned} s_{k,n+i}^{(t)}\, s_{k,n+j}^{(t)} - s_{k,n}^{(t)}\, s_{k,n+i+j}^{(t)} &= \frac{a^{tn+ti}\, b^{tn+tj} + b^{tn+ti}\, a^{tn+tj} - a^{tn}\, b^{tn+ti+tj} - b^{tn}\, a^{tn+ti+tj}}{(a^t + b^t)^2} \\ &= \frac{(a^{tj} - b^{tj})(a^{tn}b^{tn+ti} - a^{tn+ti}b^{tn})}{(a^t + b^t)^2} \\ &= \frac{-(a^{tj} - b^{tj})\, a^{tn}\, b^{tn}(a^{ti} - b^{ti})}{(a^t + b^t)^2} \\ &= -\frac{(-2k)^{tn}(a^{tj} - b^{tj})(a^{ti} - b^{ti})}{(s_{k,t})^2}. \end{aligned}$$

Accordingly, the intended form is achieved. □

Theorem 2.11. *(Honsberger Identity) The following relation holds true for integers m, n:*

$$s^{(t)}_{k,m-1}\, s^{(t)}_{k,n} + s^{(t)}_{k,m}\, s^{(t)}_{k,n+1} = \frac{a^{t(m+n-1)}(1+a^{2t}) + b^{t(m+n-1)}(1+b^{2t}) + (1+(-2k)^t)(a^{t(m-1)}b^{tn} + a^{tn}b^{t(m-1)})}{(a^t+b^t)^2}.$$

Proof.

$$\begin{aligned} & s^{(t)}_{k,m-1}\, s^{(t)}_{k,n} + s^{(t)}_{k,m}\, s^{(t)}_{k,n+1} \\ &= \frac{a^{tm-t+tn} + a^{tm-t}b^{tn} + b^{tm-t}a^{tn} + b^{tm-t+tn}}{(a^t+b^t)^2} \\ &\quad + \frac{a^{tm+tn+t} + a^{tm}b^{tn+t} + b^{tm}a^{tn+t} + b^{tm+tn+t}}{(a^t+b^t)^2} \\ &= \frac{a^{t(m+n-1)}(1+a^{2t}) + b^{t(m+n-1)}(1+b^{2t})}{(a^t+b^t)^2} \\ &\quad + \frac{a^{t(m-1)}b^{tn}(1+a^tb^t) + a^{tn}b^{t(m-1)}(1+a^tb^t)}{(a^t+b^t)^2} \\ &= \frac{a^{t(m+n-1)}(1+a^{2t}) + b^{t(m+n-1)}(1+b^{2t}) + (1+(-2k)^t)(a^{t(m-1)}b^{tn} + a^{tn}b^{t(m-1)})}{(a^t+b^t)^2}. \end{aligned}$$

□

3. Quaternion Representations of the Higher Order versions of the k-Jacobsthal-Lucas Numbers

In this section, we introduce quaternions constructed from the higher order versions k-Jacobsthal–Lucas numbers as their components. We begin by presenting their basic quaternionic properties. Next, we define a recurrence relation for these quaternions as a sequence and analyze their related characteristics.

Definition 3.1. Quaternions of the higher order versions of the k-Jacobsthal-Lucas numbers are symbolized by $Os^{(t)}_{k,n}$ and defined as

$$Os^{(t)}_{k,n} = s^{(t)}_{k,n} + s^{(t)}_{k,n+1}\mathbb{i} + s^{(t)}_{k,n+2}\mathbb{j} + s^{(t)}_{k,n+3}\mathbb{k}, \tag{3.1}$$

where $\mathbb{i}, \mathbb{j}, \mathbb{k}$ are quaternionic units and $s^{(t)}_{k,n}$ denotes the higher order versions of the k-Jacobsthal-Lucas numbers.

Note that, choosing $k = 1$, the higher order Jacobsthal-Lucas quaternions are obtained. Furthermore, choosing $t = 1$ and $k = 1$, the classical Jacobsthal-Lucas quaternions are obtained.

Definition 3.2. The real part of $Os^{(t)}_{k,n}$ given in equation 3.1 is symbolized by $Re(Os^{(t)}_{k,n})$ and is given by

$$Re(Os^{(t)}_{k,n}) = s^{(t)}_{k,n}.$$

Similarly, the imaginary part of $Os^{(t)}_{k,n}$ is given by

$$Im(Os^{(t)}_{k,n}) = v = s^{(t)}_{k,n+1}\mathbb{i} + s^{(t)}_{k,n+2}\mathbb{j} + s^{(t)}_{k,n+3}\mathbb{k}.$$

Thus, we have

$$Os^{(t)}_{k,n} = s^{(t)}_{k,n} + v.$$

Definition 3.3. The conjugate of the quaternions of the higher order versions of the k-Jacobsthal-Lucas numbers is symbolized by $\left(Os^{(t)}_{k,n}\right)^*$ and is given by

$$\left(Os^{(t)}_{k,n}\right)^* = s^{(t)}_{k,n} - s^{(t)}_{k,n+1}\mathbb{i} - s^{(t)}_{k,n+2}\mathbb{j} - s^{(t)}_{k,n+3}\mathbb{k}. \tag{3.2}$$

Theorem 3.1. *The norm value of* $Os_{k,n}^{(t)}$ *is*

$$N\left(Os_{k,n}^{(t)}\right)=\sqrt{\left(s_{k,n}^{(t)}\right)^2+\left(s_{k,n+1}^{(t)}\right)^2+\left(s_{k,n+2}^{(t)}\right)^2+\left(s_{k,n+3}^{(t)}\right)^2}.$$

Proof.

$$\begin{aligned}
N(Os_{k,n}^{(t)})^2 &= \left(Os_{k,n}^{(t)}\right)^* Os_{k,n}^{(t)} \\
&= \left(s_{k,n}^{(t)}\right)^2 + s_{k,n+1}^{(t)}\,\mathbb{i} + s_{k,n}^{(t)}s_{k,n+2}^{(t)}\,\mathbb{j} + s_{k,n}^{(t)}s_{k,n+3}^{(t)}\,\mathbb{k} \\
&\quad - s_{k,n+1}^{(t)}s_{k,n}^{(t)}\,\mathbb{i} + \left(s_{k,n+1}^{(t)}\right)^2 - s_{k,n+1}^{(t)}s_{k,n+2}^{(t)}\,\mathbb{k} + s_{k,n+1}^{(t)}s_{k,n+3}^{(t)}\,\mathbb{j} \\
&\quad - s_{k,n+2}^{(t)}s_{k,n}^{(t)}\,\mathbb{j} + s_{k,n+2}^{(t)}s_{k,n+1}^{(t)}\,\mathbb{k} + \left(s_{k,n+2}^{(t)}\right)^2 - s_{k,n+2}^{(t)}s_{k,n+3}^{(t)}\,\mathbb{i} \\
&\quad - s_{k,n+3}^{(t)}s_{k,n}^{(t)}\,\mathbb{k} - s_{k,n+3}^{(t)}s_{k,n+1}^{(t)}\,\mathbb{j} + s_{k,n+3}^{(t)}s_{k,n+2}^{(t)}\,\mathbb{i} + \left(s_{k,n+3}^{(t)}\right)^2 \\
&= \left(s_{k,n}^{(t)}\right)^2 + \left(s_{k,n+1}^{(t)}\right)^2 + \left(s_{k,n+2}^{(t)}\right)^2 + \left(s_{k,n+3}^{(t)}\right)^2. \\
N(Os_{k,n}^{(t)}) &= \sqrt{\left(s_{k,n}^{(t)}\right)^2+\left(s_{k,n+1}^{(t)}\right)^2+\left(s_{k,n+2}^{(t)}\right)^2+\left(s_{k,n+3}^{(t)}\right)^2}.
\end{aligned}$$

□

Thus, the proof is completed. Quaternions have three distinct quaternionic units. Based on each of these units, one can define distinct the conjugation-like operations. Accordingly:

Definition 3.4. The Conjugate-like form with respect to $\mathbb{i}$ of $Os_{k,n}^{(t)}$ is symbolized by $Os_{k,n}^{(t)^{*,\mathbb{i}}}$ and is defined as follows:

$$Os_{k,n}^{(t)^{*,\mathbb{i}}} = s_{k,n}^{(t)} - s_{k,n+1}^{(t)}\mathbb{i} + s_{k,n+2}^{(t)}\mathbb{j} + s_{k,n+3}^{(t)}\mathbb{k}. \tag{3.3}$$

The *Conjugate-like* form with respect to $\mathbb{j}$ of $Os_{k,n}^{(t)}$ is symbolized by $Os_{k,n}^{(t)^{*,\mathbb{j}}}$ and is defined as follows:

$$Os_{k,n}^{(t)^{*,\mathbb{j}}} = s_{k,n}^{(t)} + s_{k,n+1}^{(t)}\mathbb{i} - s_{k,n+2}^{(t)}\mathbb{j} + s_{k,n+3}^{(t)}\mathbb{k}. \tag{3.4}$$

The *Conjugate-like* form with respect to $\mathbb{k}$ of $Os_{k,n}^{(t)}$ is symbolized by $Os_{k,n}^{(t)^{*,\mathbb{k}}}$ and is defined as follows:

$$Os_{k,n}^{(t)^{*,\mathbb{k}}} = s_{k,n}^{(t)} + s_{k,n+1}^{(t)}\mathbb{i} + s_{k,n+2}^{(t)}\mathbb{j} - s_{k,n+3}^{(t)}\mathbb{k}. \tag{3.5}$$

Thus, based on these conjugate-like definitions, we also give the norm-like forms of these numbers.

Theorem 3.2. *The norm-like values of* $Os_{k,n}^{(t)}$ *are given by:*

$$\begin{aligned}
i)\quad \left(N^{\mathbb{i}}(Os_{k,n}^{(t)})\right)^2 &= \left(s_{k,n}^{(t)}\right)^2 + \left(s_{k,n+1}^{(t)}\right)^2 - \left(s_{k,n+2}^{(t)}\right)^2 - \left(s_{k,n+3}^{(t)}\right)^2 \\
&\quad + 2\mathbb{j}\left(s_{k,n}^{(t)}s_{k,n+2}^{(t)} + s_{k,n+1}^{(t)}s_{k,n+3}^{(t)}\right) \\
&\quad + 2\mathbb{k}\left(s_{k,n}^{(t)}s_{k,n+3}^{(t)} + s_{k,n+1}^{(t)}s_{k,n+2}^{(t)}\right), \\
ii)\quad \left(N^{\mathbb{j}}(Os_{k,n}^{(t)})\right)^2 &= \left(s_{k,n}^{(t)}\right)^2 - \left(s_{k,n+1}^{(t)}\right)^2 + \left(s_{k,n+2}^{(t)}\right)^2 - \left(s_{k,n+3}^{(t)}\right)^2 \\
&\quad + 2\mathbb{i}\left(s_{k,n}^{(t)}s_{k,n+1}^{(t)} - s_{k,n+1}^{(t)}s_{k,n+3}^{(t)}\right) \\
&\quad + 2\mathbb{k}\left(s_{k,n}^{(t)}s_{k,n+3}^{(t)} + s_{k,n+1}^{(t)}s_{k,n+2}^{(t)}\right), \\
iii)\quad \left(N^{\mathbb{k}}(Os_{k,n}^{(t)})\right)^2 &= \left(s_{k,n}^{(t)}\right)^2 - \left(s_{k,n+1}^{(t)}\right)^2 - \left(s_{k,n+2}^{(t)}\right)^2 + \left(s_{k,n+3}^{(t)}\right)^2 \\
&\quad + 2\mathbb{i}\left(s_{k,n}^{(t)}s_{k,n+1}^{(t)} - s_{k,n+2}^{(t)}s_{k,n+3}^{(t)}\right) \\
&\quad + 2\mathbb{j}\left(s_{k,n}^{(t)}s_{k,n+2}^{(t)} + s_{k,n+1}^{(t)}s_{k,n+3}^{(t)}\right).
\end{aligned}$$

Proof.

$$
\begin{aligned}
i)\ \left(N^{\mathbb{i}}(Os_{k,n}^{(t)})\right)^2 &= s_{k,n}^{(t)2} + s_{k,n}^{(t)}s_{k,n+1}^{(t)}\mathbb{i} - s_{k,n}^{(t)}s_{k,n+2}^{(t)}\mathbb{j} + s_{k,n}^{(t)}s_{k,n+3}^{(t)}\mathbb{k} \\
&+ s_{k,n+1}^{(t)}s_{k,n}^{(t)}\mathbb{i} + s_{k,n+1}^{(t)2} - s_{k,n+1}^{(t)}s_{k,n+2}^{(t)}\mathbb{k} - s_{k,n+1}^{(t)}s_{k,n+3}^{(t)}\mathbb{j} \\
&+ s_{k,n+2}^{(t)}s_{k,n}^{(t)}\mathbb{j} - s_{k,n+2}^{(t)}s_{k,n+1}^{(t)}\mathbb{k} - s_{k,n+2}^{(t)2} + s_{k,n+2}^{(t)}s_{k,n+3}^{(t)}\mathbb{i} \\
&+ s_{k,n+3}^{(t)}s_{k,n}^{(t)}\mathbb{k} - s_{k,n+3}^{(t)}s_{k,n+1}^{(t)}\mathbb{j} - s_{k,n+3}^{(t)}s_{k,n+2}^{(t)}\mathbb{i} - s_{k,n+3}^{(t)2} \\
&= s_{k,n}^{(t)2} + s_{k,n+1}^{(t)2} - s_{k,n+2}^{(t)2} - s_{k,n+3}^{(t)2} \\
&+ 2\mathbb{j}\left(s_{k,n}^{(t)}s_{k,n+2}^{(t)} + s_{k,n+2}^{(t)}s_{k,n+3}^{(t)}\right) \\
&+ 2\mathbb{k}\left(s_{k,n}^{(t)}s_{k,n+3}^{(t)} + s_{k,n+1}^{(t)}s_{k,n+2}^{(t)}\right).
\end{aligned}
$$

Thus, the proof is completed. The other cases are obtained analogously to *i*). □

Corollary 3.1. *The equalities below hold true.*

$$
\begin{aligned}
i)&\quad Os_{k,n}^{(t)} + Os_{k,n}^{(t)^{*,\mathbb{i}}} = 2\big(s_{k,n}^{(t)} + s_{k,n+2}^{(t)}\mathbb{j} + s_{k,n+3}^{(t)}\mathbb{k}\big),\\
ii)&\quad Os_{k,n}^{(t)} + Os_{k,n}^{(t)^{*,\mathbb{j}}} = 2\big(s_{k,n}^{(t)} + s_{k,n+1}^{(t)}\mathbb{i} + s_{k,n+3}^{(t)}\mathbb{k}\big),\\
iii)&\quad Os_{k,n}^{(t)} + Os_{k,n}^{(t)^{*,\mathbb{k}}} = 2\big(s_{k,n}^{(t)} + s_{k,n+1}^{(t)}\mathbb{i} + s_{k,n+2}^{(t)}\mathbb{j}\big),\\
iv)&\quad Os_{k,n}^{(t)^{*,\mathbb{i}}} + Os_{k,n}^{(t)^{*,\mathbb{j}}} = 2\big(s_{k,n}^{(t)} + s_{k,n+3}^{(t)}\mathbb{k}\big),\\
v)&\quad Os_{k,n}^{(t)^{*,\mathbb{i}}} + Os_{k,n}^{(t)^{*,\mathbb{k}}} = 2\big(s_{k,n}^{(t)} + s_{k,n+3}^{(t)}\mathbb{j}\big),\\
vi)&\quad Os_{k,n}^{(t)^{*,\mathbb{j}}} + Os_{k,n}^{(t)^{*,\mathbb{k}}} = 2\big(s_{k,n}^{(t)} + s_{k,n+3}^{(t)}\mathbb{i}\big).
\end{aligned}
$$

Proof. The proof of the corollary is obtained by using equalities (3.3), (3.4) and (3.5). □

Corollary 3.2. *The equalities below hold true for* $Os_{k,n}^{(t)}$.

$$Os_{k,n}^{(t)} + \left(Os_{k,n}^{(t)}\right)^* = 2s_{k,n}^{(t)}.$$

Proof. The proof of the stated result is achieved through the application of equality (3.2).

$$
\begin{aligned}
Os_{k,n}^{(t)} + \left(Os_{k,n}^{(t)}\right)^* &= s_{k,n}^{(t)} + s_{k,n+1}^{(t)}i + s_{k,n+2}^{(t)}j + s_{k,n+3}^{(t)}k \\
&+ s_{k,n}^{(t)} - s_{k,n+1}^{(t)}i - s_{k,n+2}^{(t)}j - s_{k,n+3}^{(t)}k \\
&= 2s_{k,n}^{(t)} = 2\mathrm{Re}(Os_{k,n}^{(t)}).
\end{aligned}
$$

□

An essential formula for computing the terms of quaternions of higher order versions k-Jacobsthal–Lucas is introduced in the following theorem.

Theorem 3.3. *The Binet formula of* $Os_{k,n}^{(t)}$ *is presented below:*

$$Os_{k,n}^{(t)} = \frac{(a^t)^n\hat{a} + (b^t)^n\hat{b}}{a^t + b^t}, \tag{3.6}$$

where

$$\hat{a} = 1 + (a^t)^n\mathbb{i} + (a^t)^{n+1}\mathbb{j} + (a^t)^{n+2}\mathbb{k}, \quad \hat{b} = 1 + (b^t)^n\mathbb{i} + (b^t)^{n+1}\mathbb{j} + (b^t)^{n+2}\mathbb{k}.$$

Proof. The proof is achieved by using the Binet formula of $s_{k,n}^{(t)}$.

$$\begin{aligned}
Os_{k,n}^{(t)} &= s_{k,n}^{(t)} + s_{k,n+1}^{(t)}\mathbb{i} + s_{k,n+2}^{(t)}\mathbb{j} + s_{k,n+3}^{(t)}\mathbb{k} \\
&= \frac{(a^t)^n + (b^t)^n}{a^t + b^t} + \frac{(a^t)^{n+1} + (b^t)^{n+1}}{a^t + b^t}\mathbb{i} + \frac{(a^t)^{n+2} + (b^t)^{n+2}}{a^t + b^t}\mathbb{j} + \frac{(a^t)^{n+3} + (b^t)^{n+3}}{a^t + b^t}\mathbb{k} \\
&= \frac{(a^t)^n}{a^t + b^t}\Big(1 + (a^t)^n\mathbb{i} + (a^t)^{n+1}\mathbb{j} + (a^t)^{n+2}\mathbb{k}\Big) \\
&\quad + \frac{(b^t)^n}{a^t + b^t}\Big(1 + (b^t)^n\mathbb{i} + (b^t)^{n+1}\mathbb{j} + (b^t)^{n+2}\mathbb{k}\Big) \\
&= \frac{(a^t)^n\hat{a} + (b^t)^n\hat{b}}{a^t + b^t}.
\end{aligned}$$

Accordingly, the intended form is achieved. □

Theorem 3.4. *The recurrence-like relation is*

$$Os_{k,n+1}^{(t)} = s_{k,t}\, Os_{k,n}^{(t)} - (-2k)^t\, Os_{k,n-1}^{(t)}.$$

Proof. We use the Binet formula of $Os_{k,n}^{(t)}$:

$$\begin{aligned}
Os_{k,n+1}^{(t)} &= \frac{(a^t)^n a^t\hat{a} + (b^t)^n b^t\hat{b}}{a^t + b^t} \\
&= \frac{(a^t)^n a^t\hat{a} + (b^t)^n b^t\hat{b} + (b^t)^n a^t\hat{b} - (b^t)^n a^t\hat{b}}{a^t + b^t} \\
&= a^t\, Os_{k,n}^{(t)} + \frac{(b^t)^n b^t\hat{b} - (b^t)^n a^t\hat{b}}{a^t + b^t} \\
&= (a^t + b^t)Os_{k,n}^{(t)} - \frac{b^t Os_{k,n}^{(t)}}{} + \frac{(b^t)^n b^t\hat{b} - (b^t)^n a^t\hat{b}}{a^t + b^t} \\
&= s_{k,t}\, Os_{k,n}^{(t)} - \frac{(ab)^t}{a^t + b^t}\Big((a^t)^{n-1}\hat{a} + (b^t)^{n-1}\hat{b}\Big) \\
&= s_{k,t}\, Os_{k,n}^{(t)} - (-2k)^t\, Os_{k,n-1}^{(t)}.
\end{aligned}$$

Accordingly, the intended form is achieved. □

Theorem 3.5. *The quaternions $Os_{k,n}^{(t)}$, associated with the higher order versions of k-Jacobsthal–Lucas numbers, fulfill the identities below for negative indices n and t:*

$$\begin{aligned}
i)\ Os_{k,-n}^{(t)} &= -(-2k)^{-tn}\frac{(b^t)^n\hat{a} + (a^t)^n\hat{b}}{a^t + b^t}, \\
ii)\ Os_{k,-n}^{(-t)} &= (-2k)^t\, Os_{k,n}^{(t)}, \\
iii)\ Os_{k,n}^{(-t)} &= -Os_{k,-n}^{(t)}(-2k)^{t(2n-1)}.
\end{aligned}$$

Proof. $i)$ Using the Binet formula, we have

$$\begin{aligned}
Os_{k,-n}^{(t)} &= \frac{a^{-tn}\hat{a} + b^{-tn}\hat{b}}{a^t + b^t} \\
&= \frac{(b^t)^n\hat{a} + (a^t)^n\hat{b}}{(ab)^{tn}(a^t + b^t)}.
\end{aligned}$$

Since $ab = -2k$, we have

$$Os_{k,-n}^{(t)} = -(-2k)^{-tn}\frac{(b^t)^n\hat{a} + (a^t)^n\hat{b}}{a^t + b^t}.$$

The other cases $ii)$ and $iii)$ are obtained analogously. □

Theorem 3.6. *The generating function of the quaternions $Os^{(t)}_{k,n}$, associated with the higher order versions of k-Jacobsthal–Lucas numbers is*

$$\sum_{n=0}^{\infty} Os^{(t)}_{k,n} x^n = \frac{\frac{2}{s_{k,t}} - x}{1 - s_{k,t}x + (-2k)^t x^2}.$$

Proof. Let

$$G = \sum_{n=0}^{\infty} Os^{(t)}_{k,n} x^n.$$

Then,

$$G = Os^{(t)}_{k,0} + Os^{(t)}_{k,1}x + Os^{(t)}_{k,2}x^2 + Os^{(t)}_{k,3}x^3 + \cdots + Os^{(t)}_{k,n}x^n + \cdots \tag{3.7}$$

$$-s_{k,t}xG = -s_{k,t}xOs^{(t)}_{k,0} - s_{k,t}x^2Os^{(t)}_{k,1} - s_{k,t}x^3Os^{(t)}_{k,2} - \cdots - s_{k,t}x^{n+1}Os^{(t)}_{k,n} - \cdots \tag{3.8}$$

$$(-2k)^t x^2 G = (-2k)^t x^2 Os^{(t)}_{k,0} + (-2k)^t x^3 Os^{(t)}_{k,1} + \cdots + (-2k)^t x^{n+2} Os^{(t)}_{k,n} + \cdots \tag{3.9}$$

By adding equations (3.7), (3.8), and (3.9), we get

$$G - s_{k,t}xG + (-2k)^t x^2 G = Os^{(t)}_{k,0} + Os^{(t)}_{k,1}x - s_{k,t}xOs^{(t)}_{k,0},$$

$$G = \frac{\dfrac{2}{s_{k,t}} - x}{1 - s_{k,t}x + (-2k)^t x^2}.$$

Consequently, the proof is obtained. □

Theorem 3.7. *The exponential generating function of the quaternions $Os^{(t)}_{k,n}$ is*

$$\sum_{n=0}^{\infty} Os^{(t)}_{k,n} \frac{x^n}{n!} = \frac{\hat{a}e^{a^t x} + \hat{b}e^{b^t x}}{a^t + b^t}.$$

Proof. By the Binet formula, we have

$$\begin{aligned}\sum_{n=0}^{\infty} Os^{(t)}_{k,n} \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \frac{(a^t)^n \hat{a} + (b^t)^n \hat{b}}{a^t + b^t} \frac{x^n}{n!} \\ &= \frac{\hat{a}_1}{a^t + b^t} \sum_{n=0}^{\infty} \frac{(a^t x)^n}{n!} + \frac{\hat{a}_2}{a^t + b^t} \sum_{n=0}^{\infty} \frac{(b^t x)^n}{n!} \\ &= \frac{\hat{a}_1 e^{a^t x} + \hat{b}e^{b^t x}}{a^t + b^t}.\end{aligned}$$

Consequently, the proof is obtained. □

Theorem 3.8. *The following relation holds true for each $n, m \in \mathbb{Z}$:*

$$\sum_{n=0}^{\infty} Os^{(t)}_{k,n+m} x^n = \frac{Os^{(t)}_{k,m} - (-2)^t Os^{(t)}_{k,t(m-1)} x}{1 - s_{k,t}x + (-2)^t x^2}.$$

Proof. By the Binet formula, we have

$$\begin{aligned}\sum_{n=0}^{\infty} Os_{k,n+m}^{(t)} x^n &= \sum_{n=0}^{\infty} \frac{(a^{t(n+m)}\hat{a} + b^{t(n+m)}\hat{b})}{a^t+b^t} x^n \\ &= \frac{\hat{a}a^{tm}}{a^t+b^t}\sum_{n=0}^{\infty}(a^t x)^n + \frac{\hat{b}b^{tm}}{a^t+b^t}\sum_{n=0}^{\infty}(b^t x)^n \\ &= \frac{1}{a^t+b^t}\left[\frac{\hat{a}a^{tm}+\hat{b}b^{tm}}{1-(a^t+b^t)x+(ab)^t x^2}\right. \\ &\quad \left. -\frac{(ab)^t(\hat{a}_1(a^t)^{m-1}+\hat{b}(b^t)^{m-1})x}{1-(a^t+b^t)x+(ab)^t x^2}\right] \\ &= \frac{Os_{k,m}^{(t)} - (-2)^t Os_{k,t(m-1)}^{(t)} x}{1-s_{k,t}x+(-2)^t x^2}.\end{aligned}$$

□

Lemma 3.1. *The following relations hold true:*

$$\hat{a}\hat{b} = \sigma - \upsilon\phi, \tag{3.10}$$

$$\hat{b}\hat{a} = \sigma + \upsilon\phi \tag{3.11}$$

where

$$\begin{aligned}\sigma &= 1-(-2k)^{tn}-(-2k)^{tn+t}-(-2k)^{tn+2t}+s_{k,tn}\mathbb{i}+s_{k,tn+t}\mathbb{j}+s_{k,tn+2t}\mathbb{k}, \\ \upsilon &= (-2)^{t+tn}\mathbb{i}-(-2)^{tn}s_{k,t}\mathbb{j}+(-2)^{tn}\mathbb{k}, \\ \phi &= a^t-b^t.\end{aligned}$$

Proof. We compute

$$\begin{aligned}\hat{a}\hat{b} &= (1+(a^t)^n\mathbb{i}+(a^t)^{n+1}\mathbb{j}+(a^t)^{n+2}\mathbb{k})(1+(b^t)^n\mathbb{i}+(b^t)^{n+1}\mathbb{j}+(b^t)^{n+2}\mathbb{k}) \\ &= 1-(-2k)^{tn}-(-2k)^{tn+t}-(-2k)^{tn+2t}+s_{k,tn}\mathbb{i}+s_{k,tn+t}\mathbb{j}+s_{k,tn+2t}\mathbb{k} \\ &\quad -(a^t-b^t)((-2)^{t+tn}\mathbb{i}-(-2)^{tn}s_{k,t}\mathbb{j}+(-2)^{tn}\mathbb{k}) \\ &= \sigma-\upsilon\phi.\end{aligned}$$

The relation for $\hat{b}\hat{a}$ is obtained analogously.Now, we proceed to compute several important identities, each of which is verified by means of the Binet formula for $Os_{k,n}^{(t)}$. □

Theorem 3.9. *(Vajda Identity) For each $n, m, r \in \mathbb{Z}$, the following relation holds:*

$$Os_{k,n+m}^{(t)} Os_{k,n+r}^{(t)} - Os_{k,n}^{(t)} Os_{k,n+m+r}^{(t)} = \frac{(-2k)^{tn}(a^{tm}-b^{tm})((b^{tr}-a^{tr})\sigma - s_{k,t}Os_{k,r}^{(t)}\upsilon\phi)}{(a^t+b^t)^2}.$$

Proof. By the Binet formula for $Os_{k,n}^{(t)}$, we have

$$\begin{aligned}&Os_{k,n+m}^{(t)} Os_{k,n+r}^{(t)} - Os_{k,n}^{(t)} Os_{k,n+m+r}^{(t)} \\ &= \frac{a^{t(n+m)}\hat{a}\, a^{t(n+r)}\hat{b} + b^{t(n+m)}\hat{b}\, b^{t(n+r)}\hat{a} - a^{tn}\hat{a}\, b^{t(n+m+r)}\hat{b} - b^{tn}\hat{b}\, a^{t(n+m+r)}\hat{a}}{(a^t+b^t)^2} \\ &= \frac{a^{tn}b^{t(n+r)}\hat{a}\hat{b}(a^{tm}-b^{tm}) + a^{t(n+r)}b^{tn}\hat{b}\hat{a}(b^{tm}-a^{tm})}{(a^t+b^t)^2} \\ &= \frac{a^{tn}b^{tn}(a^{tm}-b^{tm})\big(b^{tr}\hat{a}\hat{b}-a^{tr}\hat{b}\hat{a}\big)}{(a^t+b^t)^2}.\end{aligned}$$

From Lemma 3.1, we have

$$\begin{aligned} Os_{k,n+m}^{(t)} Os_{k,n+r}^{(t)} - Os_{k,n}^{(t)} Os_{k,n+m+r}^{(t)} &= \frac{(-2k)^{tn}(a^{tm}-b^{tm})\big(b^{tr}(\sigma-\upsilon\phi)-a^{tr}(\sigma+\upsilon\phi)\big)}{(a^t+b^t)^2} \\ &= \frac{(-2k)^{tn}(a^{tm}-b^{tm})\big((b^{tr}-a^{tr})\sigma-(a^{tr}+b^{tr})\upsilon\phi\big)}{(a^t+b^t)^2} \\ &= \frac{(-2k)^{tn}(a^{tm}-b^{tm})\big((b^{tr}-a^{tr})\sigma-s_{k,t}\,Os_{k,r}^{(t)}\,\upsilon\phi\big)}{(a^t+b^t)^2}. \end{aligned}$$

Accordingly, the intended form is achieved. □

Corollary 3.3. *(Cassini Identity) For each $n \in \mathbb{Z}$, we have*

$$Os_{k,n-1}^{(t)} Os_{k,n+1}^{(t)} - \big(Os_{k,n}^{(t)}\big)^2 = \frac{(-2k)^{tn-t}(b^t-a^t)\big((b^t-a^t)\sigma-s_{k,t}\,Os_{k,1}^{(t)}\,\upsilon\phi\big)}{(a^t+b^t)^2}.$$

Proof. Choosing $r=1$ and $m=-1$ in the Vajda Identity yields

$$Os_{k,n-1}^{(t)} Os_{k,n+1}^{(t)} - \big(Os_{k,n}^{(t)}\big)^2 = \frac{(-2k)^{tn-t}(b^t-a^t)\big((b^t-a^t)\sigma-s_{k,t}\,Os_{k,1}^{(t)}\,\upsilon\phi\big)}{(a^t+b^t)^2}.$$

is obtained. □

Corollary 3.4. *(Catalan Identity) For each $n > l$, we have*

$$Os_{k,n-l}^{(t)} Os_{k,n+l}^{(t)} - \big(Os_{k,n}^{(t)}\big)^2 = \frac{(-2k)^{t(n-l)}(b^{tl}-a^{tl})\big((b^{tl}-a^{tl})\sigma-s_{k,t}\,Os_{k,l}^{(t)}\,\upsilon\phi\big)}{(a^t+b^t)^2}.$$

Proof. Choosing $m=-l$ and $r=l$ in the Vajda Identity yields

$$Os_{k,n-l}^{(t)} Os_{k,n+l}^{(t)} - \big(Os_{k,n}^{(t)}\big)^2 = \frac{(-2k)^{t(n-l)}(b^{tl}-a^{tl})\big((b^{tl}-a^{tl})\sigma-s_{k,t}\,Os_{k,l}^{(t)}\,\upsilon\phi\big)}{(a^t+b^t)^2}.$$

is obtained. □

Corollary 3.5. *(d' Ocagne Identity) For each $n, l \in \mathbb{Z}$, we have*

$$Os_{k,l}^{(t)} Os_{k,n+1}^{(t)} - Os_{k,n}^{(t)} Os_{k,l+1}^{(t)} = \frac{(-2k)^{tn}\big((a^t)^{l-n}-(b^t)^{l-n}\big)\big((b^t-a^t)\sigma-s_{k,t}\,Os_{k,r}^{(t)}\,\upsilon\phi\big)}{(a^t+b^t)^2}.$$

Proof. Choosing $m+n=l$ and $r=1$ in the Vajda Identity yields

$$Os_{k,l}^{(t)} Os_{k,n+1}^{(t)} - Os_{k,n}^{(t)} Os_{k,l+1}^{(t)} = \frac{(-2k)^{tn}\big((a^t)^{(l-n)}-(b^t)^{(l-n)}\big)\big((b^t-a^t)\sigma-s_{k,t}\,Os_{k,1}^{(t)}\,\upsilon\phi\big)}{(a^t+b^t)^2}.$$

is obtained. □

4. Conclusion

In this study, a new higher order generalization of the k-Jacobsthal–Lucas numbers and their quaternionic representations has been developed. Initially, the higher order versions of the k-Jacobsthal–Lucas numbers were formulated, and their fundamental structural properties were systematically investigated within the quaternionic framework. In particular, several analytical tools associated with these sequences, including recurrence relations, closed-form (Binet-type) expressions, generating functions, and a collection of non-trivial identities, were derived and examined in detail.

In the latter part of the study, quaternion representations constructed from these higher order sequences were introduced. Their algebraic and analytical properties were analyzed both from the perspective of quaternion algebra and within the context of discrete number sequences. The results obtained in this work not only extend several known properties of classical Jacobsthal–Lucas numbers and their generalizations but also contribute to the growing literature on the interaction between number sequences and hypercomplex algebraic structures. It is expected that the findings presented here may provide a useful foundation for further investigations in related areas, such as higher dimensional algebraic systems and generalized recurrence sequences.

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