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ON ROBIN BOUNDARY VALUE PROBLEMS FOR LINEAR AND NONLINEAR EQUATIONS IN THE UNIT DISC

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Abstract. In this paper, we study the Robin boundary value problem for higher-order linear and nonlinear complex partial differential equations in the unit disc. We introduce integral representation formulas based on higher-order Pompeiu operators, transform the boundary value problems into singular integral equations, and establish solvability conditions using Fredholm theory and the Banach fixed point theorem. Explicit solution representations and operator estimates are provided, highlighting both linear and nonlinear cases.

1. Introduction

Boundary value problems (BVPs) for complex partial differential equations play a fundamental role in both pure and applied mathematics, with applications ranging from mathematical physics to engineering. Among these, Robin-type problems for higher-order equations involving polyanalytic operators have attracted significant attention due to their rich mathematical structure and potential applications [3,6]. These problems generalize classical Dirichlet and Neumann problems, incorporating linear combinations of function values and their derivatives on the boundary, which naturally arise in physical models such as elasticity, fluid mechanics, and electromagnetic theory.

Boundary value problems for higher-order complex partial differential equations are central to complex analysis and mathematical physics [1,5,7–18,20]. Early research focused on classical polyanalytic equations in standard domains, such as the unit disc and the upper half-plane, establishing integral representation formulas, hierarchical operator methods, and explicit solution techniques [1,7,8].

In this article, we extend these classical results by investigating Robin boundary value problems for higher-order elliptic linear and nonlinear complex PDEs in the unit disc. Here, the polyanalytic operator constitutes the main part of the differential operator, allowing a systematic treatment of both linear and nonlinear formulations. The linear problems are addressed via their equivalent singular integral equations, which provide explicit solution formulas under suitable solvability conditions. Nonlinear problems, on the other hand, are reduced to systems of integro-differential equations using a model equation approach, offering a unified framework to analyze existence, uniqueness, and explicit representations of solutions.

2. Integral Operators Arising in the Robin Problem on the Unit Disc

Integral representation formulas play a central role in the study of boundary value problems. In particular, such formulas provide explicit solutions and allow one to investigate the analytic and functional-analytic properties of the solution operators. Throughout the paper we denote by

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

the unit disc.

The solution w given in [3]

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial\mathbb{D}} (\bar{\varsigma} f(\varsigma) - \gamma(\varsigma)) \frac{\ln(1 - z\bar{\varsigma})}{z} \, d\varsigma - \frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{f(\varsigma)}{\varsigma - z} \, d\xi d\eta \tag{2.1}$$

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is the unique solution to the Robin problem for the inhomogeneous Cauchy-Riemann equation

$$w_{\bar{z}} = f(z), \quad z \in \mathbb{D}, \qquad w + \partial_\nu w = \gamma \quad \text{on } \partial\mathbb{D} \tag{2.2}$$

with the solvability condition

$$\frac{-1}{2\pi i}\int_{\partial\mathbb{D}} (\bar{\varsigma}f(\varsigma) - \gamma(\varsigma))\frac{d\varsigma}{1-\bar{z}\varsigma} + \frac{1}{\pi}\iint_{\mathbb{D}} f(\varsigma)\frac{\bar{z}\varsigma}{(1-\bar{z}\varsigma)^2}\,d\xi d\eta = 0, \tag{2.3}$$

where $f \in C^\alpha(\overline{\mathbb{D}};\mathbb{C}),\ 0<\alpha<1,\ \gamma \in C(\partial\mathbb{D};\mathbb{C})$. Here ∂_ν denotes the outward normal derivative, i.e.

$$\partial_\nu \omega = \nabla\omega \cdot \nu,$$

where ν is the outward unit normal vector on $\partial\mathbb{D}$.

The domain integral appearing on the right-hand side of (2.1) is the well-known *Pompeiu operator* (or T-operator), see [4, 19]. This operator plays an essential role in the theory of generalized analytic functions.

By iteration, one defines higher order Pompeiu operators according to the rule

$$T_{0,n}f(z) = T_{0,1}\big(T_{0,n-1}f(z)\big), \tag{2.4}$$

which leads to the explicit representation

$$T_{0,n}f(z) := \frac{1}{\pi}\iint_{\mathbb{D}} f(\varsigma)\,\frac{(-1)^n}{(n-1)!}\,\frac{\overline{(\varsigma-z)}^{n-1}}{\varsigma - z}\,d\xi d\eta, \tag{2.5}$$

for $n \in \mathbb{N}$, with the convention $T_{0,0}f(z) = f(z)$.

2.1. **Differential and Functional Properties of Higher Order Pompeiu Operators.** The differentiability and functional-analytic properties of these operators are described in detail in [4]. Here we summarize the main structural relations. The higher order Pompeiu integral operators satisfy the recursive differentiation formulas

$$\frac{\partial^l}{\partial\bar{z}^l}T_{m,n}f(z) = T_{m,n-l}f(z), \qquad 1 \le l \le n, \tag{2.6}$$

and satisfies the property

$$T_{0,0}f(z) := f(z). \tag{2.7}$$

Moreover, differentiation with respect to z yields

$$\partial_z T_{m,n}f(z) = T_{m-1,n}f(z). \tag{2.8}$$

On the other hand, the operators $\dfrac{\partial^l}{\partial\bar{z}^l}T_{m,n}$ are weakly singular integral operators for $0 \le l \le n-1$. The n-th order operator

$$\Pi_n f(z) := \frac{\partial^n}{\partial z^n}T_{0,n}f(z) = T_{-n,n}f(z) = \frac{n}{\pi}\iint_{\mathbb{D}} f(\varsigma)\,(-1)^n\,\frac{\overline{(\varsigma-z)}^{n-1}}{(\varsigma-z)^{n+1}}\,d\xi d\eta, \tag{2.9}$$

is a Calderón–Zygmund type singular integral operator, which is bounded on L^p spaces.

The following boundedness and continuity results are classical (see [4], [19]).

Theorem 2.1 ([4])**.** *Let D be a bounded domain, suppose $m+n \ge 1$ and assume $f \in L^p(D)$. Then*

$$|\, T_{m,n,D}f(z) \,| \le M\, \|f\|_{L^p(D)}. \tag{2.10}$$

This theorem ensures that the Pompeiu operators are uniformly bounded in L^p, which guarantees the stability of the solution operators.

Theorem 2.2 ([4]). *Suppose $m+n \geq 1$ and $mn \leq 0$, and let D be a bounded domain in $\mathbb{C}$ and assume $f \in L^p(D)$. Then for $z_1, z_2 \in \mathbb{C}$,*

$$| T_{m,n,D} f(z_1) - T_{m,n,D} f(z_2) | \leq M \, \|f\|_{L^p(D)} \begin{cases} |z_1 - z_2|, & m+n \geq 2, \\ |z_1 - z_2|^{\frac{p-2}{p}}, & m+n = 1, \end{cases} \tag{2.11}$$

where $M = M(m,n,p)$.

This result shows that the operators are not only bounded but also Lipschitz or Hölder continuous, depending on the order of the operator.

Theorem 2.3 ([4]). *Assume $m+n = 0$, $(m,n) \neq (0,0)$ and let f be a complex-valued function in $L^p(D)$, where $1 < p < \infty$. Then $T_{m,n} f$ also belongs to $L^p(D)$, and*

$$\|T_{m,n} f\|_{L^p(D)} \leq M(p) \|f\|_{L^p(D)}. \tag{2.12}$$

In this case the operators act as Calderón–Zygmund singular integrals, and hence preserve L^p-integrability of the function space.

3. Linear Higher-Order Equations with Homogeneous Robin Boundary Conditions

In this section, we investigate the solvability of higher-order elliptic partial differential equations subject to homogeneous Robin boundary conditions. More precisely, we consider the following problem:

Find $w \in W^{p,n}(\mathbb{D})$ such that it satisfies the n-th order complex differential equation

$$\frac{\partial^n w}{\partial \bar{z}^n} + \sum_{j=1}^{n} q_{1j}(z) \frac{\partial^n w}{\partial z^j \partial \bar{z}^{\,n-j}} + \sum_{j=1}^{n} q_{2j}(z) \frac{\partial^n \overline{w}}{\partial z^{\,n-j} \partial \bar{z}^j}$$
$$+ \sum_{l=0}^{n-1} \sum_{m=0}^{l} \left(a_{ml}(z) \frac{\partial^l w}{\partial z^m \partial \bar{z}^{\,l-m}} + b_{ml}(z) \frac{\partial^l \overline{w}}{\partial z^{\,l-m} \partial \bar{z}^m} \right) = f(z), \quad z \in \mathbb{D}. \tag{3.1}$$

The solution is required to satisfy the Robin boundary condition

$$\partial_{\bar{z}}^{\alpha-1} w + z \partial_{\bar{z}}^{\alpha-1} \partial_z w + \bar{z} \partial_{\bar{z}}^{\alpha} w = 0 \quad \text{on } \partial\mathbb{D}, \ \alpha = 1, ..., n. \tag{3.2}$$

Here the coefficients are assumed to satisfy

$$a_{ml}, \, b_{ml} \in L^p(\mathbb{D}), \quad f \in L^p(\mathbb{D}),$$

and q_{1j}, q_{2j} for $j = 1, \dots, n$ are bounded measurable functions such that

$$\sum_{j=1}^{n} \left| q_{1j}(z) \right| + \left| q_{2j}(z) \right| \leq q_0 < 1. \tag{3.3}$$

Condition (3.3) ensures that the leading part of the operator is elliptic, and guaranteeing well-posedness.

Under these assumptions, the problem admits a solution if and only if the following solvability condition holds:

$$\frac{1}{\pi} \iint_{\mathbb{D}} f(\varsigma) \frac{\bar{z}\varsigma}{(1 - \bar{z}\varsigma)^2} d\xi d\eta = 0, \tag{3.4}$$

$$\frac{1}{\pi} \frac{1}{(n-\alpha-1)!} \iint_{\mathbb{D}} f(\varsigma) \frac{\bar{z}\varsigma (\overline{z - \varsigma})^{n-\alpha}}{(n-\alpha)(1 - \bar{z}\varsigma)^2} d\xi d\eta = 0, \quad \alpha = 1, \dots, n-1. \tag{3.5}$$

Remark 1. The inequality (3.3) on q_{1j} and q_{2j} guarantees the ellipticity of the differential equation, which is essential for applying integral operator methods.

To proceed further, we reduce the boundary value problem (3.1)–(3.2) to an equivalent operator equation by means of integral representations. This allows us to reformulate the Robin problem in terms of singular integral operators, which is stated in the following lemma.

Lemma 1. *The Robin problem* (3.1)–(3.2) *is equivalent to the singular integral equation*

$$(I + \hat{\Pi} + \hat{K})g = f, \tag{3.6}$$

where $w = T_{0,n}g$*, and the operators* $\hat{\Pi}$ *and* $\hat{K}$ *are defined as*

$$\hat{\Pi}g = \sum_{j=1}^{n} \left(q_{1j}(z)\Pi_j g + q_{2j}(z)\overline{\Pi_j g} \right), \tag{3.7}$$

$$\hat{K}g = \sum_{l=0}^{n-1} \sum_{m=0}^{l} \left(a_{ml}(z)T_{-m,n-l+m}g + b_{ml}(z)\overline{T_{-m,n-l+m}g} \right). \tag{3.8}$$

Proof. According to [1], the function $w = T_{0,n}g$ is the unique solution of

$$\frac{\partial^n w}{\partial \bar{z}^n} = g, \tag{3.9}$$

satisfying the Robin condition (3.2), provided that the integral solvability conditions are fulfilled.

Using the differentiation properties of the Pompeiu operators (see previous section), we have

$$\frac{\partial^n w}{\partial z^j \partial \bar{z}^{\,n-j}} = T_{-j,j}g = \Pi_j g, \qquad j = 1, \dots, n,$$

$$\frac{\partial^n \overline{w}}{\partial z^{\,n-j} \partial \bar{z}^j} = \overline{T_{-j,j}g} = \overline{\Pi_j g}. \tag{3.10}$$

Similarly, for $0 \le m \le l \le n-1$,

$$\frac{\partial^l w}{\partial z^m \partial \bar{z}^{\,l-m}} = T_{-m,n-l+m}g, \quad \frac{\partial^l \overline{w}}{\partial z^{\,l-m} \partial \bar{z}^m} = \overline{T_{-m,n-l+m}g}. \tag{3.11}$$

Substituting these relations into (3.1), we obtain precisely the operator forms given in (3.7)–(3.8). Hence, g satisfies the singular integral equation (3.6) if and only if $w = T_{0,n}g$ solves the Robin problem (3.1)–(3.2) subject to the solvability conditions. □

Thus, Equation (3.1) with the boundary conditions (3.2) is transformed into a singular integral equation of the form (3.6). Now, we investigate the solvability of this singular integral equation using Fredholm theory.

Lemma 2. *If*

$$q_0 \max_{1 \le j \le n} \|\Pi_j\|_{L^p(\mathbb{D})} < 1, \tag{3.12}$$

then the operator $I + \hat{\Pi}$ *is invertible on* $L^p(\mathbb{D})$ *for* $p > 1$.

Proof. By definition,

$$\hat{\Pi}g = \sum_{j=1}^{n} \left(q_{1j}(z)\,\Pi_j g + q_{2j}(z)\,\overline{\Pi_j g} \right).$$

Hence, its operator norm satisfies

$$\|\hat{\Pi}\|_{L^p(\mathbb{D})} \le \sup_{z \in \mathbb{D}} \left(\sum_{j=1}^{n} \left(|q_{1j}(z)| + |q_{2j}(z)| \right) \right) \max_{1 \le j \le n} \|\Pi_j\|_{L^p(\mathbb{D})}. \tag{3.13}$$

From condition (3.3), we have

$$\sum_{j=1}^{n} \left(|q_{1j}(z)| + |q_{2j}(z)| \right) \le q_0 < 1,$$

so that

$$\|\hat{\Pi}\|_{L^p(\mathbb{D})} \le q_0 \max_{1 \le j \le n} \|\Pi_j\|_{L^p(\mathbb{D})}. \tag{3.14}$$

If condition (3.12) holds, then $\|\hat{\Pi}\|_{L^p(\mathbb{D})} < 1$. Therefore the operator $I + \hat{\Pi}$ is invertible on $L^p(\mathbb{D})$ via the Neumann series

$$(I + \hat{\Pi})^{-1} = \sum_{k=0}^{\infty} (-\hat{\Pi})^k,$$

which converges in the operator norm. □

Having established the invertibility of $I + \hat{\Pi}$ at hand, we next analyze the operator $\hat{K}$, which corresponds to the lower-order terms. We establish that $\hat{K}$ is compact in $L^p(\mathbb{D})$.

Theorem 3.1. *If $a_{ml}, b_{ml} \in L^p(\mathbb{D})$, $\quad f \in L^p(\mathbb{D})$, then $\hat{K}$ is a compact operator in $L^p(\mathbb{D})$ for $p > 2$.*

Proof. By Theorems 2.1 and 2.2, the operators $T_{-m,n-l+m}$ are bounded and equicontinuous for $0 \le l \le n-1$ on $\mathbb{D}$. Since $a_{ml}, b_{ml} \in L^p(\mathbb{D})$, the integral operators with such kernels map bounded sets into relatively compact sets. By the Arzelà–Ascoli theorem, $\hat{K}$ is compact on $L^p(\mathbb{D})$ for $p > 2$. □

The compactness of $\hat{K}$ together with the invertibility of $I + \hat{\Pi}$ allows us to apply the Fredholm alternative. This leads directly to the solvability criterion for the Robin problem, stated in the following theorem.

Theorem 3.2. *If condition (3.12) is satisfied, then Equation* (3.1) *with the boundary conditions* (3.2) *admits a solution if and only if the following solvability conditions hold:*

$$\frac{1}{\pi} \iint_{\mathbb{D}} f(\varsigma) \frac{\bar{z}\varsigma}{(1-\bar{z}\varsigma)^2} d\xi d\eta = 0, \tag{3.15}$$

$$\frac{1}{\pi} \frac{1}{(n-\alpha-1)!} \iint_{\mathbb{D}} f(\varsigma) \frac{\bar{z}\varsigma(\overline{z-\varsigma})^{n-\alpha}}{(n-\alpha)(1-\bar{z}\varsigma)^2} d\xi d\eta = 0, \quad \alpha = 1, \ldots, n-1. \tag{3.16}$$

In this case, the solution has the form $w = T_{0,n}g$ where $g \in L^p(\mathbb{D})$, $p > 2$ is a solution of the singular integral Equation (3.6).

Proof. Under condition (3.12), the operator $I + \hat{\Pi}$ is invertible on $L^p(\mathbb{D})$. Then we can write

$$I + \hat{\Pi} + \hat{K} = (I + \hat{\Pi})\big(I + (I + \hat{\Pi})^{-1}\hat{K}\big).$$

Since $(I+\hat{\Pi})^{-1}$ is bounded and $\hat{K}$ is compact on $L^p(\mathbb{D})$ for $p > 2$, the operator $(I+\hat{\Pi})^{-1}\hat{K}$ is compact. Thus, $I + \hat{\Pi} + \hat{K}$ is a Fredholm operator of index zero.

By the Fredholm alternative, Equation (3.6) has a solution precisely when f satisfies the solvability conditions stated in the theorem. In that case, the solution of (3.1)–(3.2) is given by $w = T_{0,n}g$, where $g \in L^p(\mathbb{D})$ solves (3.6).

□

4. Robin Problem for Nonlinear Higher-Order Equations

In the previous section, we established the solvability of linear higher-order elliptic equations with homogeneous Robin boundary conditions in the unit disc. Building upon these results, we now extend our analysis to the nonlinear setting, where the right-hand side of the equation may depend nonlinearly on w and its derivatives. This extension allows us to investigate a broader class of problems while still employing integral operator techniques adapted to the nonlinear context.

We consider the elliptic equation

$$\partial_{\bar{z}}^n w = F(z, w, D^{\alpha_1}w, D^{\alpha_2}w, \ldots, D^{\alpha_n}w), \tag{4.1}$$

of order n in $\mathbb{D}$. Following the notation of [2], we define

$$D = (\partial_z, \partial_{\bar{z}}), \quad \alpha_j = (k, l), \quad |\alpha_j| = k + l = j, \quad j = 1, 2, \ldots, n, \tag{4.2}$$

with the restriction $(k, l) \neq (0, n)$.

For simplicity, we denote

$$D^{\alpha_1}w, D^{\alpha_2}w, \ldots, D^{\alpha_n}w \quad \text{by} \quad w_{k,l}.$$

With this notation, Equation (4.1) can be equivalently written as

$$\partial_{\bar{z}}^{n} w = F(z, w_{k,l}). \tag{4.3}$$

Assumptions on F. We impose the following assumptions on the nonlinear function F to ensure well-posedness of the problem:

(1) $F(z, w_{k,l})$ is continuous in all its arguments, ensuring smooth dependence on z and the derivatives of w.
(2) If $w_{k,l} \in L^p(\mathbb{D})$ with $p > 2$, then $F(z, w_{k,l})$ also belongs to $L^p(\mathbb{D})$, guaranteeing integrability.
(3) $F(z, w_{k,l})$ satisfies a Lipschitz-type condition:

$$|F(z, w_{k,l}) - F(z, \tilde{w}_{k,l})| \le \sum_{k+l\le n} L_{kl} |w_{k,l} - \tilde{w}_{k,l}| \le L \sum_{k+l\le n} |w_{k,l} - \tilde{w}_{k,l}|, \tag{4.4}$$

where $L = \max_{k+l\le n} L_{kl}$, which controls the growth of F with respect to the derivatives of w.

Before formulating the Robin boundary value problem, we note that using the Pompeiu operators, the nonlinear differential equation can be reduced to an equivalent integro-differential equation. This reduction is essential for analyzing existence and uniqueness of solutions.

Robin Problem. Find a solution $w \in W^{n,p}(\mathbb{D})$, $p > 2$, of Equation (4.1) satisfying the Robin boundary conditions

$$\partial_{\bar{z}}^{\alpha-1} w + z\, \partial_{\bar{z}}^{\alpha-1} \partial_z w + \bar{z}\, \partial_{\bar{z}}^{\alpha} w = 0, \quad \text{on } \partial\mathbb{D},\ \alpha = 1, \dots, n. \tag{4.5}$$

Using the Pompeiu operators introduced in the previous section, the problem can be rewritten as the system

$$w(z) = \psi(z) + T_{0,n} F(z, w_{k,l}), \tag{4.6}$$

$$w_{k,l}(z) = \frac{\partial^{k+l}\psi(z)}{\partial z^k \partial \bar{z}^l} + \partial_z^k T_{0,n-l} F(z, w_{k,l}), \tag{4.7}$$

for $0 \le k + l \le n$, $(k, l) \ne (0, n)$, where

$$\psi(z) = -\frac{1}{2\pi i} \sum_{k=1}^{n} \frac{1}{(k-1)!} \int_{\partial\mathbb{D}} \gamma_k(\varsigma) (\overline{z - \varsigma})^{k-1} \frac{\ln(1 - z\bar{\varsigma})}{z}\, d\varsigma + \frac{1}{2\pi i} \frac{1}{(n-1)!} \int_{\partial\mathbb{D}} f(\varsigma) \bar{\varsigma} (\overline{z - \varsigma})^{n-1} \frac{\ln(1 - z\bar{\varsigma})}{z}\, d\varsigma. \tag{4.8}$$

4.1. **Solvability.** Define the Banach space

$$H^p(\mathbb{D}) = \{w_{k,l} \in L^p(\mathbb{D}) : 0 \le k + l \le n,\ (k, l) \ne (0, n)\}, \quad \|w_{k,l}\|_{H^p} = \max_{0\le k+l\le n} \|w_{k,l}\|_{L^p}. \tag{4.9}$$

We rewrite the Robin problem in operator form using the Pompeiu operators:

$$W(z) = \psi(z) + T_{0,n} F(z, w_{k,l}), \quad W_{k,l}(z) = \frac{\partial^{k+l}\psi(z)}{\partial z^k \partial \bar{z}^l} + \partial_z^k T_{0,n-l} F(z, w_{k,l}). \tag{4.10}$$

Define the operator $Q : H^p(\mathbb{D}) \to H^p(\mathbb{D})$ by

$$Q(w_{k,l}) = (W_{k,l}). \tag{4.11}$$

Now, for two elements $w_{k,l}, \tilde{w}_{k,l} \in H^p(\mathbb{D})$, we estimate

$$\begin{aligned} \|Q(w_{k,l}) - Q(\tilde{w}_{k,l})\|_{H^p} &= \max_{0\le k+l\le n} \|W_{k,l} - \tilde{W}_{k,l}\|_{L^p} \\ &\le \max_{0\le k+l\le n} \Big(\|T_{0,n}(F(z, w_{k,l}) - F(z, \tilde{w}_{k,l}))\|_{L^p} \\ &\quad + \|\partial_z^k T_{0,n-l}(F(z, w_{k,l}) - F(z, \tilde{w}_{k,l}))\|_{L^p} \Big). \end{aligned} \tag{4.12}$$

Using the Lipschitz property of F, we have

$$\|F(z, w_{k,l}) - F(z, \tilde{w}_{k,l})\|_{L^p} \le \sum_{k+l\le n} L_{kl} \|w_{k,l} - \tilde{w}_{k,l}\|_{L^p} \le L \|w_{k,l} - \tilde{w}_{k,l}\|_{H^p}, \tag{4.13}$$

where

$$L = \max_{k+l\leq n} L_{kl}. \tag{4.14}$$

Combining the boundedness of the Pompeiu operators with the Lipschitz estimate, we obtain

$$\|Q(w_{k,l}) - Q(\tilde{w}_{k,l})\|_{H^p} \leq L \max\left\{\|T_{0,n}\|, \|\partial_z^k T_{0,n-l}\|, \|\Pi_{n,\mathbb{D}}\|\right\} \|w_{k,l} - \tilde{w}_{k,l}\|_{H^p}. \tag{4.15}$$

If

$$L \max\left\{\|T_{0,n}\|, \|\partial_z^k T_{0,n-l}\|, \|\Pi_{n,\mathbb{D}}\|\right\} < 1, \tag{4.16}$$

then Q is a contraction. By the Banach fixed point theorem, Q admits a unique fixed point in $H^p(\mathbb{D})$, which corresponds to the unique solution of the Robin boundary value problem (4.1)–(4.5).

5. Conclusion

In this paper, we studied Robin boundary value problems for higher-order linear and nonlinear complex partial differential equations in the unit disc. By employing higher-order Pompeiu operators, we transformed these problems into singular integral equations and established solvability conditions. For the nonlinear case, we demonstrated existence and uniqueness of solutions using a fixed-point approach, highlighting the effectiveness of integral operator methods in handling complex boundary value problems.

Conflict of Interest

The author declares no conflict of interest.

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