# Mapping Properties Of Certain Hardy-Type Operator Over Cones In $\mathbb{R}^N$

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**Abstract.** Mapping properties of the operator

$$(Kf)(x) = u_1(x) \int_{b(|x|)S \setminus a(|x|)S} v_1(t)f(t)dt + u_2(x) \int_{d(|x|)S \setminus c(|x|)S} v_2(t)f(t)dt,$$

where $u_i, v_i, i = 1, 2$, are certain general measurable functions on $E$ (not necessarily non negative), between weighted Lebesgue spaces $L^p(E, w_0)$ and $L^q(E, w_1)$, $E$ and $\alpha S$ being certain cones in $\mathbb{R}^N$, $1 < p$, $q < \infty$, are studied in terms of the corresponding one dimensional operator and the precise weight conditions for the same are obtained.

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## 1 Introduction

In [18] (see also [12]), P.A. Zharov studied the $L^p - L^q$ boundedness of the operator

$$(Af)(s) = U_1(s) \int_0^s V_1(t)f(t)dt + U_2(s) \int_s^\infty V_2(t)f(t)dt,$$

where $U_i, V_i, i = 1, 2$, are certain general measurable functions (not necessarily non negative). This operator is more general than the Hardy operator

$$(Hf)(s) = \int_0^s f(t)dt$$

and its conjugate operator

$$(H^*f)(s) = \int_s^\infty f(t)dt.$$

A complete study of these operators can be found in [3, 11, 12, 13, 16, 17] and the references therein. In [7], Jain and Gupta considered a variant of the operator $A$ i.e.

$$(Lf)(s) = U_1(s) \int_{a(s)}^{b(s)} V_1(t)f(t)dt + U_2(s) \int_{c(s)}^{d(s)} V_2(t)f(t)dt, \qquad (1.1)$$

where (and throughout the paper) $a, b, c, d$ are strictly increasing differentiable functions on $[0, \infty]$ satisfying

$$\begin{aligned} a(s) &< b(s) \leq c(s) < d(s), \qquad s \in (0, \infty) \\ a(0) &= b(0) = c(0) = d(0) = 0 \\ a(\infty) &= b(\infty) = c(\infty) = d(\infty) = \infty, \end{aligned}$$

$U_i, V_i,\ i = 1, 2$ are general measurable functions (not necessarily non-negative) on $(0, \infty)$ and characterized the boundedness as well as compactness of $L$ between weighted Lebesgue spaces. This operator is derived from the Steklov operator:

$$(Tf)(s) = \int_{a(s)}^{b(s)} f(t)dt.$$

The boundedness of $T$ between weighted Lebesgue spaces was studied by Heinig and Sinnamon [4] and its compactness by Jain and Gupta [6].

The aim of the present paper is to study the operator

$$\begin{aligned} (Kf)(x) = u_1(x) &\int_{b(|x|)S \backslash a(|x|)S} v_1(t) f(t) dt \\ &+ u_2(x) \int_{d(|x|)S \backslash c(|x|)S} v_2(t) f(t) dt,\ x \in E \end{aligned} \tag{1.2}$$

where $E$ and $\alpha S$ are certain cones in $\mathbb{R}^N$ (defined below), $u_i, v_i,\ i = 1, 2$ are general measurable functions (not necessarily non-negative) on $E$ and $a, b, c, d$ are as defined above. This operator is a higher dimensional analogue of the operator $L$.

In Section 2, we characterize the boundedness and compactness of $K$ in terms of the corresponding properties of the operator $L$ and then in Section 3, using the weight characterization for the boundedness and compactness of the operator $L$, the precise conditions for the case of operator $K$ are obtained. Such reduction to one dimension for higher dimensional Steklov operator:

$$(T_E f)(x) = \int_{b(|x|)S \backslash a(|x|)S} f(y) dy,$$

was given by Sinnamon [15] for the case of boundedness and by Jain, Jain and Gupta [9] for the case of compactness. For a study of higher dimensional operators [8, 10, 14] may be referred. In [15], Sinnamon has studied the boundedness of Steklov operator over the smoothly star shaped regions in $\mathbb{R}^N$ which includes the class of cones in $\mathbb{R}^N$. The same notations $E$ and $\alpha S$ for cones in $\mathbb{R}^N$ are used in this paper as done by Sinnamon for star shaped regions [15] without any ambiguity.

Let $I \subset \mathbb{R}^N$, $w$ be a weight function on $I$ and $1 < p < \infty$. $L^p(I, w)$ denotes the weighted Lebesgue space

$$L^p(I, w) = \left\{ f : \left( \int_I |f(x)|^p \, w(x) dx \right)^{1/p} < \infty \right\}.$$

Let $\sum$ be the unit sphere in $\mathbb{R}^N$ i.e. $\sum = \{x \in \mathbb{R}^N : |x| = 1\}$, where $|x| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$ is the Euclidean norm of the vector $x \in \mathbb{R}^N$. Let $B$ be a measurable subset of $\sum$, we denote by $E$, a cone in $\mathbb{R}^N$ defined by

$$E = \{x \in \mathbb{R}^N : \ x = s\sigma; \ 0 \le s < \infty, \ \sigma \in B\}$$

and $\alpha S, \ \alpha > 0$, denote the part of $E$ with radius $\le \alpha$, i.e.,

$$\alpha S = \{y \in \mathbb{R}^N : \ y = s\sigma; \ 0 \le s \le \alpha, \ \sigma \in B\}.$$

In this light,

$$b(|x|)S \setminus a(|x|)S = \{y \in \mathbb{R}^N : \ y = s\sigma; \ a(|x|) \le s \le b(|x|), \ \sigma \in B\}.$$

Let $Y$ denote a normed linear space and $Y^*$ be its conjugate space. A sequence $\{y_n\}$ in $Y$ is said to be norm convergent (or convergent) to some $y \in Y$, denoted by $y_n \to y$ if $\|y_n - y\|_Y \to 0$ as $n \to \infty$. Sequence $\{y_n\}$ is said to be weakly convergent to $y \in Y$ if $y^*(y_n) \to y^*(y)$ for every $y^* \in Y^*$. A sequence $\{y_n^*\} \subset Y^*$ is said to be weak* convergent to some $y^* \in Y^*$ if $y_n^*(y) \to y^*(y)$ for every $y \in Y$. Every norm convergent sequence is known to be weakly convergent and every weakly convergent sequence to be weak* convergent. The reverse of these implications do not hold in general. In case $Y$ is a reflexive space weak* convergence implies weak convergence.

## 2 Boundedness and Compactness of the operator $K$

In this Section we shall study the boundedness and compactness of the operator $K$ given by (1.2) in terms of the operator $L$ given by (1.1). For the proofs of the subsequent theorems we shall need certain well known results which are stated below:

**Theorem A.** *Let $X$ and $Y$ be Banach spaces.*

(i) *A bounded linear operator $A : X \to Y$ is compact if and only if its conjugate $A^* : Y^* \to X^*$ is compact.*

(ii) *A compact operator maps every weakly convergent sequence to a norm convergent sequence.*

(iii) *An operator $A : X \to Y$ is compact if $A^* : Y^* \to X^*$ is weak*-norm sequentially continuous, i.e., for each sequence $\{f_n\}$ in $Y^*$ with $f_n \xrightarrow{w^*} f$ for some $f$ in $Y^*$, we have $A^*(f_n) \to A^*f$.*

These results can be found in any standard book on functional analysis e.g., [2], [5]. In particular, Theorem A(iii) is taken from ([2], p. 15).

For weight functions $w_0, \ w_1$ on $E$, let us set

$$W_1(t) = \int_B w_1(t\tau)t^{N-1}d\tau \tag{2.1}$$

and

$$W_0(t) = \left( \int_B w_0^{1-p'}(t\tau) t^{N-1} d\tau \right)^{1-p}, \tag{2.2}$$

$t \in (0, \infty)$. Now, we prove below our first main result wherein the boundedness of the operator $K : L^p(E, w_0) \to L^q(E, w_1)$ is studied in terms of the boundedness of the operator $L : L^p((0,\infty), W_0) \to L^q((0,\infty), W_1)$.

**Theorem 2.1.** *Let $E$ be a cone in $\mathbb{R}^N$, $1 < p, q < \infty$, $w_0, w_1$ be weight functions defined on $E$ and $u_i, v_i$, $i = 1,2$ be certain general measurable functions on $E$ depending only on the radial component. Then the inequality*

$$\left( \int_E |(Kf)(x)|^q \, w_1(x) dx \right)^{1/q} \leq C \left( \int_E |f(x)|^p \, w_0(x) dx \right)^{1/p} \tag{2.3}$$

*holds for all locally integrable functions $f : E \to \mathbb{R}$ if and only if*

$$\left( \int_0^\infty |(LF)(s)|^q \, W_1(s) ds \right)^{1/q} \leq C \left( \int_0^\infty |F(s)|^p \, W_0(s) ds \right)^{1/p} \tag{2.4}$$

*holds for all locally integrable functions $F : (0,\infty) \to \mathbb{R}$, where*

$$U_i(t) = u_i(t\tau), \; V_i(t) = v_i(t\tau), \; \tau \in B, \; i = 1, 2, \tag{2.5}$$

*and $W_0, W_1$ are given by (2.1) and (2.2). Moreover the best constants in (2.3) and (2.4) coincide.*

*Proof.* Suppose (2.4) holds and fix a locally integrable function $f : E \to \mathbb{R}$. Set

$$F(t) = \int_B f(t\tau) t^{N-1} d\tau. \tag{2.6}$$

Making change of variables and using (2.4), (2.5), (2.6), we have

$$
\begin{aligned}
&\left(\int_E |(Kf)(x)|^q\, w_1(x)dx\right)^{1/q} \\
&= \left(\int_E \left| u_1(x)\int_{b(|x|)S\setminus a(|x|)S} v_1(y)f(y)dy + u_2(x) \right.\right. \\
&\quad \times \left.\left. \int_{d(|x|)S\setminus c(|x|)S} v_2(y)f(y)dy\right|^q w_1(x)dx\right)^{1/q} \\
&= \left(\int_E \left| u_1(x)\int_{a(|x|)}^{b(|x|)}\int_B v_1(s\sigma)f(s\sigma)s^{N-1}d\sigma ds + u_2(x) \right.\right. \\
&\quad \times \left.\left. \int_{c(|x|)}^{d(|x|)}\int_B v_2(s\sigma)f(s\sigma)s^{N-1}d\sigma ds\right|^q w_1(x)dx\right)^{1/q} \\
&= \left(\int_E \left| u_1(x)\int_{a(|x|)}^{b(|x|)} V_1(s)F(s)ds + u_2(x)\int_{c(|x|)}^{d(|x|)} V_2(s)F(s)ds\right|^q w_1(x)dx\right)^{1/q} \\
&= \left(\int_0^\infty\int_B \left| u_1(t\tau)\int_{a(t)}^{b(t)} V_1(s)F(s)ds + u_2(t\tau) \right.\right. \\
&\quad \times \left.\left. \int_{c(t)}^{d(t)} V_2(s)F(s)ds\right|^q w_1(t\tau)t^{N-1}d\tau dt\right)^{1/q} \\
&= \left(\int_0^\infty \left| U_1(t)\int_{a(t)}^{b(t)} V_1(s)F(s)ds + U_2(t)\int_{c(t)}^{d(t)} V_2(s)F(s)ds\right|^q W_1(t)dt\right)^{1/q} \\
&\le C\left(\int_0^\infty |F(t)|^p\, W_0(t)dt\right)^{1/p},
\end{aligned}
$$

which is the R.H.S of (2.4), now following the proof of Theorem 1.1 in [15], it can be shown to be less than or equal to

$$
C\left(\int_E |f(x)|^p\, w_0(x)dx\right)^{1/p}.
$$

Thus the inequality (2.3) holds. Conversely, suppose that the inequality (2.3) holds and fix a locally integrable function $F:(0,\infty)\to\mathbb{R}$. Define $f:E\to\mathbb{R}$ by

$$
f(t\tau) = F(t)W_0^{p'-1}(t)w_0^{1-p'}(t\tau).
$$

It can be easily shown that the functions $f$ and $F$ also satisfy (2.6), and as above we get

$$
\left(\int_0^\infty |(LF)(t)|^q\, W_1(t)dt\right)^{1/q} = \left(\int_E |(Kf)(x)|^q\, w_1(x)dx\right)^{1/q}.
$$

Now using inequality (2.3) and following the proof of Theorem 1.1 in [15], we get the inequality (2.4). □

Next we study the compactness of the operator $K$ in terms of the operator $L$.

**Theorem 2.2.** *Let $E$ be a cone in $\mathbb{R}^N$, $1 < p, q < \infty$, $w_0, w_1$ be weight functions defined on $E$ and $u_i, v_i$, $i = 1, 2$ be certain general measurable functions on $E$ depending only on the radial component. Then the operator $K : L^p(E, w_0) \to L^q(E, w_1)$ is compact if and only if the corresponding one dimensional operator $L : L^p((0,\infty), W_0) \to L^q((0,\infty), W_1)$ is compact where*

$$U_i(t) = u_i(t\tau),\ V_i(t) = v_i(t\tau),\ \tau \in B,\ i = 1, 2, \tag{2.7}$$

*and $W_0, W_1$ are as given by (2.1) and (2.2).*

*Proof.* Let us first assume that $L : L^p((0,\infty), W_0) \to L^q((0,\infty), W_1)$ is compact. In order to show that the operator $K$ is compact, in view of Theorem A, it is sufficient to show that $K^* : L^{q'}(E, w_1^{1-q'}) \to L^{p'}(E, w_0^{1-p'})$ is $w^*-$ norm sequentially continuous. Note that

$$L^* : L^{q'}((0,\infty), W_1^{1-q'}) \to L^{p'}((0,\infty), W_0^{1-p'}),$$

is defined as

$$L^*(G(t)) = V_1(t)\int_{b^{-1}(t)}^{a^{-1}(t)} U_1(s)G(s)ds + V_2(t)\int_{d^{-1}(t)}^{c^{-1}(t)} U_2(s)G(s)ds,$$

since, by Fubini's Theorem, for any $G \in L^{q'}((0,\infty), W_1^{1-q'})$ and $F \in L^p((0,\infty), W_0)$, we have

$$\begin{aligned}
&< LF, G > \\
&= \int_0^\infty \left( U_1(s)\int_{a(s)}^{b(s)} V_1(t)F(t)dt + U_2(s)\int_{c(s)}^{d(s)} V_2(t)F(t)dt \right) G(s)ds \\
&= \int_0^\infty U_1(s)\left(\int_{a(s)}^{b(s)} V_1(t)F(t)dt\right) G(s)ds \\
&\quad + \int_0^\infty U_2(s)\left(\int_{c(s)}^{d(s)} V_2(t)F(t)dt\right) G(s)ds \\
&= \int_0^\infty \left( V_1(t)\int_{b^{-1}(t)}^{a^{-1}(t)} U_1(s)G(s)ds \right) F(t)dt \\
&\quad + \int_0^\infty V_2(t)\left(\int_{d^{-1}(t)}^{c^{-1}(t)} U_2(s)G(s)ds\right) F(t)dt \\
&= \int_0^\infty \left( V_1(t)\int_{b^{-1}(t)}^{a^{-1}(t)} U_1(s)G(s)ds + V_2(t)\int_{d^{-1}(t)}^{c^{-1}(t)} U_2(s)G(s)ds \right) F(t)dt \\
&= < F, L^*G > .
\end{aligned}$$

Similarly,

$$K^* : L^{q'}(E, w_1^{1-q'}) \to L^{p'}(E, w_0^{1-p'})$$

is given by

$$K^*(g(x)) = v_1(x) \int_{a^{-1}(|x|)S \setminus b^{-1}(|x|)S} u_1(y) g(y) dy$$
$$+ v_2(x) \int_{c^{-1}(|x|)S \setminus d^{-1}(|x|)S} u_2(y) g(y) dy.$$

Let us take a weak*-null sequence $\{f_n\}$ in $L^{q'}(E, w_1^{1-q'})$ and define

$$F_n(t) = \int_B f_n(t\tau) t^{N-1} d\tau, \qquad n \in N, \ t \in (0, \infty). \tag{2.8}$$

Following the proof of Theorem 3.1 in [9], we get that $F_n \in L^{q'}((0,\infty), W_1^{1-q'})$, $n \in N$ and $\{F_n\}$ is weakly convergent to 0. Using the compactness of $L$ and Theorem A, we get

$$\|L^* F_n\|_{L^{p'}((0,\infty), W_0^{1-p'})} \to 0.$$

Finally, by changing variables $x = t\tau$, $y = s\sigma$ and using (2.7), (2.8), we get

$$\begin{aligned}
&\|K^* f_n\|_{L^{p'}(E, w_0^{1-p'})} \\
&= \Bigg( \int_E \Bigg( v_1(x) \int_{a^{-1}(|x|)S \setminus b^{-1}(|x|)S} u_1(y) f_n(y) dy + v_2(x) \\
&\quad \times \int_{c^{-1}(|x|)S \setminus d^{-1}(|x|)S} u_2(y) f_n(y) dy \Bigg)^{p'} w_0^{1-p'}(x) dx \Bigg)^{1/p'} \\
&= \Bigg( \int_0^\infty \int_B \Bigg( v_1(t\tau) \int_{b^{-1}(t)}^{a^{-1}(t)} \int_B u_1(s\sigma) f_n(s\sigma) s^{N-1} d\sigma ds + v_2(t\tau) \\
&\quad \times \int_{d^{-1}(t)}^{c^{-1}(t)} \int_B u_2(s\sigma) f_n(s\sigma) s^{N-1} d\sigma ds \Bigg)^{p'} w_0^{1-p'}(t\tau) t^{N-1} d\tau dt \Bigg)^{1/p'} \\
&= \Bigg( \int_0^\infty \Bigg( V_1(t) \int_{b^{-1}(t)}^{a^{-1}(t)} U_1(s) F_n(s) ds + V_2(t) \int_{d^{-1}(t)}^{c^{-1}(t)} U_2(s) F_n(s) ds \Bigg)^{p'} \\
&\quad \times W_0^{1-p'}(t) dt \Bigg)^{1/p'} \\
&= \|L^* F_n\|_{L^{p'}((0,\infty), W_0^{1-p'})}.
\end{aligned}$$

Thus

$$\|K^* f_n\|_{L^{p'}(E, w_0^{1-p'})} \to 0.$$

This proves that $K$ is compact. Conversely, assume that $K : L^p(E, w_0) \to L^q(E, w_1)$ is compact. To show that $L$ is compact, we take a weak*-null sequence

$\{F_n\}$ in $L^{q'}((0,\infty), W_1^{1-q'})$. Next, define

$$f_n(t\tau) = F_n(t) w_1(t\tau) W_1^{-1}(t), \qquad n \in N \quad \tau \in B.$$

Once again following the proof of Theorem 3.1 in [9], $\{f_n\}$ can be shown to be a weakly-null sequence in $L^{q'}(E, w_1^{1-q'})$. Then, by using the compactness of $K$ and Theorem A

$$\|L^* F_n\|_{L^{p'}((0,\infty), W_0^{1-p'})} = \|K^* f_n\|_{L^{p'}(E, w_0^{1-p'})} \to 0.$$

Hence $L$ is also compact.

□

## 3 The precise conditions

In this section we obtain the precise weight conditions for the boundedness and compactness of the operator $K$ for $1 < p \leq q < \infty$ using the weight characterization for the boundedness and compactness of the operator $L$ which have already been obtained by Jain and Gupta [7]. Theorem 2.1 of [7] holds for all locally integrable functions if $U_i, V_i$, $i = 1, 2$ and $W_0, W_1$ are taken to be locally integrable. We state their results below:

**Theorem B** *Let $1 < p \leq q < \infty$, $W_0, W_1$ be locally integrable weight functions defined on $(0, \infty)$ and $U_i, V_i$, $i = 1, 2$ be certain locally integrable functions on $(0, \infty)$. Then the inequality (2.4) holds for all locally integrable functions $F$ if and only if*

$$\mathcal{B} = \max(\mathcal{B}_1, \mathcal{B}_2) < \infty, \tag{3.1}$$

*where*

$$\mathcal{B}_1 := \sup_{\substack{t<s \\ a(s)<b(t)}} \mathcal{B}_1(s,t) = \sup_{\substack{t<s \\ a(s)<b(t)}} \left( \int_t^s W_1(s) \, |U_1(s)|^q \, ds \right)^{1/q} \times \left( \int_{a(s)}^{b(t)} W_0^{1-p'}(s) \, |V_1(s)|^{p'} \, ds \right)^{1/p'} \tag{3.2}$$

*and*

$$\mathcal{B}_2 := \sup_{\substack{t<s \\ c(s)<d(t)}} \mathcal{B}_2(s,t) = \sup_{\substack{t<s \\ c(s)<d(t)}} \left( \int_t^s W_1(s) \, |U_2(s)|^q \, ds \right)^{1/q} \times \left( \int_{c(s)}^{d(t)} W_0^{1-p'}(s) \, |V_2(s)|^{p'} \, ds \right)^{1/p'}. \tag{3.3}$$

**Theorem C** *Let $1 < p \leq q < \infty$, $W_0, W_1$ be weight functions defined on $(0, \infty)$, $U_i, V_i$, $i = 1, 2$ be certain general measurable functions on $(0, \infty)$ and*

*$\mathcal{B}_1, \mathcal{B}_2$ be as defined by (3.2) and (3.3). Further assume that $\int_0^\infty W_0^{1-p'}(s)\,ds$, $\int_0^\infty W_1(s)\,ds$ are finite. Then the operator $L$ is compact if and only if $L$ is bounded i.e. (3.1) holds and*

$$\begin{aligned}
&\lim_{t\to s-} \mathcal{B}_1(s,t) = \lim_{t\to b^{-1}(a(s))+} \mathcal{B}_1(s,t) = 0, && \text{for every } s > 0\\
&\lim_{s\to t+} \mathcal{B}_1(s,t) = \lim_{s\to a^{-1}(b(t))-} \mathcal{B}_1(s,t) = 0, && \text{for every } t > 0\\
&\lim_{t\to s-} \mathcal{B}_2(s,t) = \lim_{t\to d^{-1}(c(s))+} \mathcal{B}_2(s,t) = 0, && \text{for every } s > 0\\
&\lim_{s\to t+} \mathcal{B}_2(s,t) = \lim_{s\to c^{-1}(d(t))-} \mathcal{B}_2(s,t) = 0. && \text{for every } t > 0
\end{aligned}$$

Using Theorem 2.1 and Theorem B we now obtain the precise weight conditions for the inequality (2.3).

**Theorem 3.1.** *Let $E$ be a cone in $\mathbb{R}^N$, $1 < p \le q < \infty$, $w_0, w_1$ be locally integrable weight functions defined on $E$ and $u_i, v_i$, $i = 1,2$ be certain locally integrable functions on $E$ depending only on the radial component. Then the inequality (2.3) holds for all locally integrable functions $f$ if and only if*

$$\mathcal{D} = \max(\mathcal{D}_1, \mathcal{D}_2) < \infty, \tag{3.4}$$

*where*

$$\begin{aligned}
\mathcal{D}_1 := \sup_{\substack{t<s\\ a(s)<b(t)}} \mathcal{D}_1(s,t) = \sup_{\substack{t<s\\ a(s)<b(t)}} &\left(\int_{sS\setminus tS} w_1(y)\,|u_1(y)|^q\,dy\right)^{1/q}\\
&\times\left(\int_{b(t)S\setminus a(s)S} w_0^{1-p'}(y)\,|v_1(y)|^{p'}\,dy\right)^{1/p'}
\end{aligned} \tag{3.5}$$

*and*

$$\begin{aligned}
\mathcal{D}_2 := \sup_{\substack{t<s\\ c(s)<d(t)}} \mathcal{D}_2(s,t) = \sup_{\substack{t<s\\ c(s)<d(t)}} &\left(\int_{sS\setminus tS} w_1(y)\,|u_2(y)|^q\,dy\right)^{1/q}\\
&\times\left(\int_{d(t)S\setminus c(s)S} w_0^{1-p'}(y)\,|v_2(y)|^{p'}\,dy\right)^{1/p'}.
\end{aligned} \tag{3.6}$$

*Proof.* Proof of Theorem 3.1 follows from Theorem 2.1 and Theorem B, once we prove that $\mathcal{B}_1 = \mathcal{D}_1$ and $\mathcal{B}_2 = \mathcal{D}_2$. Note that

$$
\begin{aligned}
&\mathcal{B}_1(s,t) \\
&= \left(\int_t^s W_1(s)\,|U_1(s)|^q\,ds\right)^{1/q}\left(\int_{a(s)}^{b(t)} W_0^{1-p'}(s)\,|V_1(s)|^{p'}\,ds\right)^{1/p'} \\
&= \left(\int_t^s \int_B w_1(s\sigma)\,|u_1(s\sigma)|^q\,s^{N-1}d\sigma ds\right)^{1/q} \\
&\quad\times\left(\int_{a(s)}^{b(t)} \int_B w_0^{1-p'}(s\sigma)\,|v_1(s\sigma)|^{p'}\,s^{N-1}d\sigma ds\right)^{1/p'} \\
&= \left(\int_{sS\setminus tS} w_1(y)\,|u_1(y)|^q\,dy\right)^{1/q}\left(\int_{b(t)S\setminus a(s)S} w_0^{1-p'}(y)\,|u_1(y)|^{p'}\,dy\right)^{1/p'} \\
&= \mathcal{D}_1(s,t),
\end{aligned}
$$

for every $s$ and $t$. Therefore $\mathcal{B}_1 = \mathcal{D}_1$ and similarly $\mathcal{B}_2 = \mathcal{D}_2$.

□

Finally we give precise weight conditions for the compactness of the operator $K$.

**Theorem 3.2.** *Let $E$ be a cone in $\mathbb{R}^N$, $1 < p \le q < \infty$, $w_0, w_1$ be weight functions defined on $E$, $u_i, v_i$, $i = 1,2$ be certain general measurable functions on $E$ depending only on the radial component and $\mathcal{D}_1, \mathcal{D}_2$ be as defined by (3.5) and (3.6). Further assume that $\int_E w_0^{1-p'}(y)\,dy$, $\int_E w_1(y)\,dy$ are finite. Then the operator $K$ is compact if and only if $K$ is bounded i.e. (3.4) holds and*

$$
\begin{aligned}
\lim_{t\to s-} \mathcal{D}_1(s,t) &= \lim_{t\to b^{-1}(a(s))+} \mathcal{D}_1(s,t) = 0, && \text{for every } s > 0 \\
\lim_{s\to t+} \mathcal{D}_1(s,t) &= \lim_{s\to a^{-1}(b(t))-} \mathcal{D}_1(s,t) = 0, && \text{for every } t > 0 \\
\lim_{t\to s-} \mathcal{D}_2(s,t) &= \lim_{t\to d^{-1}(c(s))+} \mathcal{D}_2(s,t) = 0, && \text{for every } s > 0 \\
\lim_{s\to t+} \mathcal{D}_2(s,t) &= \lim_{s\to c^{-1}(d(t))-} \mathcal{D}_2(s,t) = 0. && \text{for every } t > 0
\end{aligned}
$$

*Proof.* Theorem 3.2 follows by using Theorem 2.2, Theorem C and the fact that $\mathcal{B}_1 = \mathcal{D}_1$ and $\mathcal{B}_2 = \mathcal{D}_2$. □

**Remark 3.3** *Note that Theorem 2.1 and 2.2 are proved for $1 < p, q < \infty$, however Theorem 3.1 and 3.2 hold for $1 < p \le q < \infty$. Once the weight characterization for the boundedness of the operator L are obtained for $1 < q < p < \infty$ , Theorem 3.1 can also be obtained for $1 < q < p < \infty$. In view of the general principle of Ando [1], the weight characterization for the compactness* *of $L$ and $K$ for $1 < q < p < \infty$ would be same as that for the boundedness of these operators.*

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