

ON THE APPROXIMATION BY VILENKIN-FEJÉR MEANS IN LEBESGUE SPACES

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ABSTRACT. In this paper, in terms of modulus of continuity, we study the approximation properties of Fejér means with respect to bounded Vilenkin groups in Lebesgue spaces for any $1 \leq p < \infty$.

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1. INTRODUCTION

Let \mathbb{N}_+ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ denote a sequence of the positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k .

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j} 's. The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$. In this paper we discuss bounded Vilenkin groups only, that is $R := \sup_{n \in \mathbb{N}} m_n < \infty$.

The elements of G_m are represented by the sequences

$$x := (x_0, x_1, \dots, x_k, \dots) \quad (x_k \in Z_{m_k}).$$

It is easy to describe a neighborhood base of G_m , namely

$$\begin{aligned} I_0(x) &:= G_m, \\ I_n(x) &:= \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, n \in \mathbb{N}). \end{aligned}$$

The intervals $I_n(x)$ ($n \in \mathbb{N}, x \in G_m$) are called Vilenkin intervals. Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\bar{I}_n := G_m \setminus I_n$. Let

$$e_n := (0, \dots, 0, 1, 0, \dots) \in G_m \quad (n \in \mathbb{N}).$$

where the n th coordinate is 1, the others are 0.

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The norms (or quasi-norms) of Lebesgue spaces $L^p(G_m)$ are defined by

$$\|f\|_p := \left(\int_{G_m} |f|^p d\mu \right)^{1/p}.$$

The modulus of continuity of a function $f \in L^p(G_m)$, $1 \leq p \leq \infty$ is defined by

$$\omega_p(\delta, f) = \sup_{|t| < \delta} \|f(x+t) - f(x)\|, \quad \delta > 0.$$

where

$$|t| := \sum_{i=0}^{\infty} \frac{t_i}{M_{i+1}}.$$

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \quad M_{k+1} := m_k M_k, \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{j=0}^{\infty} n_j M_j, \quad \text{where } n_j \in Z_{m_j} \quad (j \in \mathbb{N})$$

and only a finite number of n_j 's differ from zero.

Let

$$|n| := \max\{j \in \mathbb{N}, n_j \neq 0\}.$$

Next, we introduce on G_m an orthonormal system, which is called the Vilenkin system. First define the complex valued function $r_k(x) : G_m \rightarrow \mathbb{C}$, the generalized Rademacher functions, as

$$r_k(x) := \exp(2\pi i x_k / m_k) \quad (i^2 = -1, x \in G_m, k \in \mathbb{N}).$$

It is clear that

$$(1) \quad |r_k| = 1, \quad \text{for all } x \in G_m.$$

We define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

In particular, this system is called the Walsh-Paley one if $m \equiv 2$ (for details see [12] and [25]). The Vilenkin system is orthonormal and complete in $L^2(G_m)$ (for details see e.g. [4, 25, 31]).

If $f \in L^1(G_m)$, we can define the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Vilenkin system in the usual manner:

$$\begin{aligned}
\widehat{f}(k) & : = \int_{G_m} f \overline{\psi}_k d\mu, \quad (k \in \mathbb{N}), \\
S_n f & : = \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, \quad (n \in \mathbb{N}_+, S_0 f := 0), \\
\sigma_n f & : = \frac{1}{n} \sum_{k=1}^n S_k f, \quad (n \in \mathbb{N}_+), \\
D_n & : = \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+), \\
K_n & : = \frac{1}{n} \sum_{k=1}^n D_k, \quad (n \in \mathbb{N}_+).
\end{aligned}$$

In [29] Toledo (see also [5]) showed that

$$\sup_n \|K_n\|_1 = \frac{17}{15},$$

where K_n is the Fejér kernel with respect to Walsh system. For one-dimensional bounded Vilenkin-Fourier series it was proved in [20] (see also [5, 23]) that

$$\sup_n \|K_n\|_1 \leq C,$$

where C is an absolute constant. But the exact value of the supremum depending on the sequence m is still open problem with respect bounded Vilenkin systems. It follows that (see e.g. [4] and [10]) there exists an absolute constant C_p , depending only on p such that

$$\|\sigma_n f\|_p \leq C_p \|f\|_p, \quad (1 \leq p \leq \infty)$$

If we define the maximal operator σ^* of Fejér means by

$$\sigma^* f := \sup_{n \in \mathbb{N}_+} |\sigma_n f|,$$

then the weak type inequality

$$\mu(\sigma^* f > \lambda) \leq \frac{c}{\lambda} \|f\|_1, \quad (\lambda > 0)$$

holds for any integrable function. For example this result can be found in Zygmund [35] for trigonometric series, in Schipp [24] for Walsh series and in Pál, Simon [20] (see also [22], [32], [33], [34]) for bounded Vilenkin series. It follows that the Fejér means with respect to trigonometric and Vilenkin systems of any integrable function converge a.e to this function.

Almost everywhere convergence and summability of Fejér means and some more general summability methods were studied by several authors. We mention Anakidze, Areshidze and Baramidze [1] (see also [3]), Blahota, Persson

and Tephnadze [8], Memić [13], Tephnadze [27, 28], Fridli, Manchanda, Siddiqi [11], Móricz and Siddiqi [14], Nagy [15], (see also [9], [16], [17], [18], [19] and [21]) and Tutberidze [30].

In [23] it was proved that if $1 \leq p < \infty$, $f \in L^p(G_m)$ and $n \in \mathbb{N}$, then there exists an absolute constant c_p , depending only on p such that the following inequality holds:

$$(2) \quad \|\sigma_n f - f\|_p \leq c_p \omega_p(1/M_N, f) + c_p \sum_{s=0}^{N-1} \frac{M_s}{M_N} \omega_p(1/M_s, f).$$

In this paper we estimate L^1 norms of Vilenkin-Fejér kernels. Moreover, we use a new approach and estimate constants in (2) for any $1 \leq p < \infty$. This new approach is very important to generalize problems investigated in [1] and [2] for the Vilenkin systems.

We note that inequality (2) was proved by Skvortsov [26] for Walsh system. We note also that the rate of the approximation of Vilenkin matrix transform means is discussed in [6] (see also [7]). The Fejér mean is a special matrix transform mean, so in this special case we could reach a more exact estimate with our new approach.

2. AUXILIARY LEMMAS

The next lemmas (see Lemmas 1, 2 and 3) can be found in [23]:

Lemma 1. *Let $n \in \mathbb{N}$, $1 \leq s \leq m_n - 1$. Then*

$$(3) \quad D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n \end{cases}$$

and

$$(4) \quad D_{sM_n} = D_{M_n} \sum_{k=0}^{s-1} r_n^k.$$

Lemma 2. *Let $n > t$, $t, n \in \mathbb{N}$, then*

$$(5) \quad K_{M_n}(x) = \begin{cases} \frac{M_t}{1-r_t(x)}, & x \in I_t \setminus I_{t+1}, \quad x - x_t e_t \in I_n, \\ \frac{M_n}{2}, & x \in I_n, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, if $n, t, s \in \mathbb{N}$ and $1 \leq s \leq m_n - 1$, then

$$(6) \quad sM_n K_{sM_n} = \sum_{l=0}^{s-1} \left(\sum_{i=0}^{l-1} r_n^i \right) (M_n + 1) D_{M_n} + \left(\sum_{l=0}^{s-1} r_n^l \right) M_n K_{M_n}.$$

Lemma 3. *Let $n = \sum_{i=1}^r s_{n_i} M_{n_i}$, where $n_1 > n_2 > \dots > n_r \geq 0$ and $1 \leq s_{n_i} < m_{n_i}$ for all $1 \leq i \leq r$ as well as*

$$(7) \quad n^{(k)} = n - \sum_{i=1}^k s_{n_i} M_{n_i},$$

where $0 < k \leq r$. Then

$$(8) \quad nK_n = \sum_{k=1}^r \left(\prod_{j=1}^{k-1} r^{s_{n_j}} \right) s_{n_k} M_{n_k} K_{s_{n_k} M_{n_k}} + \sum_{k=1}^{r-1} \left(\prod_{j=1}^{k-1} r^{s_{n_j}} \right) n^{(k)} D_{s_{n_k} M_{n_k}}.$$

We next establish several new lemmas that may also be of independent interest:

Lemma 4. *Let $n \in \mathbb{N}$ and $1 \leq s \leq m_n - 1$. Then*

$$(9) \quad |K_{sM_n}| \leq s|K_{M_n}|$$

and

$$(10) \quad n|K_n| \leq 2R^2 \sum_{k=0}^{|n|} M_k |K_{M_k}|.$$

Proof. By combining (3) and (5) we get that

$$(11) \quad D_{M_n} \leq 2|K_{M_n}|.$$

From (6) and (11) we get that

$$(12) \quad \begin{aligned} sM_n|K_{sM_n}| &\leq 2M_n|K_{M_n}| \left(\sum_{l=0}^{s-1} l \right) + sM_n|K_{M_n}| \\ &\leq 2M_n|K_{M_n}| \frac{s(s-1)}{2} + sM_n|K_{M_n}| \\ &= s^2M_n|K_{M_n}|, \end{aligned}$$

which proves inequality (9).

According to (7) we obtain that $n^{(k)} \leq M_{n_k}$. By applying (4) we find that

$$|D_{sM_n}| \leq s|D_{M_n}|.$$

By applying (8) and (9) we find that

$$\begin{aligned} n|K_n| &\leq \sum_{k=1}^r s_{n_k}^2 M_{n_k} |K_{M_{n_k}}| + \sum_{k=1}^{r-1} s_{n_k} M_{n_k} |D_{M_{n_k}}| \\ &\leq R^2 \sum_{k=1}^r M_{n_k} |K_{M_{n_k}}| + 2R \sum_{k=1}^{r-1} M_{n_k} |K_{M_{n_k}}| \\ &\leq (R^2 + 2R) \sum_{k=1}^r M_{n_k} |K_{M_{n_k}}|. \end{aligned}$$

Hence,

$$n|K_n| \leq 2R^2 \sum_{k=0}^{|n|} M_k |K_{M_k}|,$$

which proves (10) and the proof is complete. \square

Lemma 5. *Let $n \in \mathbb{N}$. Then*

$$(13) \quad \int_{G_m} |K_n(x)| d\mu(x) \leq 4R^3.$$

Proof. Since

$$(14) \quad \int_{G_m} |K_n(x)| d\mu(x) \leq \frac{2R^2}{n} \sum_{k=0}^{|n|} M_k \int_{G_m} |K_{M_k}(x)| d\mu(x)$$

we obtain that

$$(15) \quad \begin{aligned} & \int_{G_m} |K_{M_k}(x)| d\mu(x) \\ &= \int_{I_k} |K_{M_k}(x)| d\mu(x) + \sum_{s=0}^{k-1} \sum_{a_s=1}^{m_s-1} \int_{I_k(a_s e_s)} |K_{M_k}(x)| d\mu(x) \\ &= \frac{M_k+1}{2} \frac{1}{M_k} + \sum_{s=0}^{k-1} \sum_{a_s=1}^{m_s-1} \int_{I_k(a_s e_s)} |K_{M_k}(x)| d\mu(x) \end{aligned}$$

on the set $I_k(a_s e_s)$ the modulus of K_{M_k} equals

$$(16) \quad |K_{M_k}| = \left| \frac{M_s}{1 - r_s(x)} \right|$$

Since

$$\left| \frac{1}{r_s(x) - 1} \right| = \left| \frac{1}{2 \sin \frac{\pi x_s}{m_s}} \right| \leq \frac{1}{\left| 2 \sin \frac{\pi}{m_s} \right|},$$

if we apply well-known inequality

$$\frac{1}{2 \sin \frac{t}{2}} \leq \frac{\pi}{2t}, \quad \text{for } 0 < t \leq \pi$$

we get that

$$(17) \quad \left| \frac{1}{r_s(x) - 1} \right| \leq \frac{\pi}{2 \frac{2\pi}{m_s}} = \frac{m_s}{4}$$

and finally we find that

$$|K_{M_k}(x)| \leq \frac{M_{s+1}}{4}, \quad \text{for } x \in I_k(a_s e_s).$$

Therefore, by using (15) we find that

$$\begin{aligned} \int_{G_m} |K_{M_k}(x)| d\mu(x) &\leq \frac{M_k+1}{2} \frac{1}{M_k} + \frac{1}{4M_k} \sum_{s=0}^{k-1} m_s M_{s+1} \\ &\leq \frac{M_k+1}{2} \frac{1}{M_k} + \frac{R}{4M_k} \cdot 2M_k \leq 1 + \frac{R}{2} \leq R. \end{aligned}$$

from (14) we get that

$$(18) \quad \int_{G_m} |K_n(x)| d\mu(x) \leq R \frac{2R^2}{n} \sum_{k=0}^{|n|} M_k \leq 4R^3.$$

The proof is complete. \square

3. APPROXIMATION OF VILENKIN-FEJÉR MEANS

Our main result reads:

Theorem 1. *Let $1 \leq p < \infty$, $f \in L^p(G_m)$, $n \in \mathbb{N}$ and $M_N \leq n < M_{N+1}$. Then*

$$\|\sigma_n f - f\|_p \leq (4R^3 + 2)\omega_p(1/M_N, f) + \frac{R^2}{4} \sum_{s=0}^{N-1} \frac{M_s}{M_N} \omega_p(1/M_s, f).$$

Proof. Let $f \in L^p(G_m)$, $1 \leq p < \infty$ and $M_N \leq n < M_{N+1}$. If we use Young's convolution inequality

$$\|\sigma_n f\|_p = \|f * K_n\|_p \leq \|K_n\|_1 \|f\|_p$$

and the inequality $\|S_{M_n} f - f\|_p \leq \omega_p(1/M_n, f)$, we get that

$$(19) \quad \begin{aligned} & \|\sigma_n f - f\|_p \\ & \leq \|\sigma_n f - \sigma_n S_{M_N} f\|_p + \|\sigma_n S_{M_N} f - S_{M_N} f\|_p + \|S_{M_N} f - f\|_p \\ & = \|\sigma_n (S_{M_N} f - f)\|_p + \|S_{M_N} f - f\|_p + \|\sigma_n S_{M_N} f - S_{M_N} f\|_p \\ & \leq (4R^3 + 1)\omega_p(1/M_N, f) + \|\sigma_n S_{M_N} f - S_{M_N} f\|_p. \end{aligned}$$

By straightforward calculations we obtain that

$$(20) \quad \begin{aligned} & \sigma_n S_{M_N} f - S_{M_N} f \\ & = \frac{1}{n} \sum_{k=1}^{M_N} S_k S_{M_N} f + \frac{1}{n} \sum_{k=M_N+1}^n S_k S_{M_N} f - S_{M_N} f \\ & = \frac{1}{n} \sum_{k=1}^{M_N} S_k f + \frac{1}{n} \sum_{k=M_N+1}^n S_{M_N} f - S_{M_N} f \\ & = \frac{1}{n} \sum_{k=1}^{M_N} S_k f + \frac{n - M_N}{n} S_{M_N} f - S_{M_N} f \\ & = \frac{M_N}{n} \sigma_{M_N} f - \frac{M_N}{n} S_{M_N} f \\ & = \frac{M_N}{n} (S_{M_N} \sigma_{M_N} f - S_{M_N} f) \\ & = \frac{M_N}{n} S_{M_N} (\sigma_{M_N} f - f). \end{aligned}$$

By using (20) and the fact that

$$\|S_{M_N} f\|_p \leq \|f\|_p, \quad f \in L_p(G_m), \quad 1 \leq p < \infty$$

we find that

$$(21) \quad \begin{aligned} \|\sigma_n S_{M_N} f - S_{M_N} f\|_p &= \frac{M_N}{n} \|S_{M_N} (\sigma_{M_N} f - f)\|_p \\ &\leq \|S_{M_N} (\sigma_{M_N} f - f)\|_p \leq \|\sigma_{M_N} f - f\|_p. \end{aligned}$$

Moreover,

$$(22) \quad \begin{aligned} &\sigma_{M_N} f(x) - f(x) \\ &= \int_{G_m} (f(x-t) - f(x)) K_{M_N}(t) d\mu(t) \\ &= \int_{I_N} (f(x-t) - f(x)) K_{M_N}(t) d\mu(t) \\ &+ \sum_{s=0}^{N-1} \sum_{n_s=1}^{m_s-1} \int_{I_N(n_s e_s)} (f(x-t) - f(x)) K_{M_N}(t) d\mu(t) \\ &:= I + II. \end{aligned}$$

If we apply (5) and generalized Minkowski's inequality we get that

$$(23) \quad \begin{aligned} \|I\|_p &\leq \int_{I_N} \|f(x-t) - f(x)\|_p \frac{M_N + 1}{2} d\mu(t) \\ &\leq \omega_p(1/M_N, f) \int_{I_N} \frac{M_N + 1}{2} d\mu(t) \leq \omega_p(1/M_N, f). \end{aligned}$$

and

$$(24) \quad \begin{aligned} \|II\|_p &= \sum_{s=0}^{N-1} \sum_{n_s=1}^{m_s-1} \frac{M_s}{2 \sin((\pi n_s)/m_s)} \int_{I_N(n_s e_s)} \|f(x-t) - f(x)\|_p d\mu(t) \\ &\leq \sum_{s=0}^{N-1} \sum_{n_s=1}^{m_s-1} \frac{M_s}{2 \sin(\pi/m_s)} \int_{I_N(n_s e_s)} \omega_p(1/M_s, f) d\mu(t) \\ &= \frac{1}{2} \sum_{s=0}^{N-1} \sum_{n_s=1}^{m_s-1} \frac{M_s}{\sin(\pi/m_s)} \omega_p(1/M_s, f) \frac{1}{M_N} \\ &\leq \frac{R^2}{4} \sum_{s=0}^{N-1} \frac{M_s}{M_N} \omega_p(1/M_s, f). \end{aligned}$$

The proof is complete by combining (19)-(24). □

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