LYAPUNOV FUNCTIONS AND THEIR COMPUTATION BY LINEAR OPTIMIZATION

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Abstract. Lyapunov functions characterize the stability of attractors in dynamical systems. Their analytical construction for general systems, however, remains a formidable task. Therefore, numerous numerical methods for their computation have been developed. We give a short overview of the theory of Lyapunov functions and discuss how they can be parameterized using linear programming in the so-called CPA method. We discuss a few of the many extensions of the CPA method, in particular to differential inclusions, and we discuss how the linear programming problem generated in the CPA method can often be solved very efficiently. We give numerous concrete examples to illustrate the technique of the CPA method.

1. Lyapunov functions

We consider the ordinary differential equation (ODE)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \ \ \mathbf{f} \colon \mathbb{R}^n \to \mathbb{R}^n. \tag{1.1}$$

Sufficient conditions for (1.1) to define a dynamical system are that $\mathbf{f}$ is locally Lipschitz and bounded, where the second condition is not essential because one can always replace $\mathbf{f}$ with $(1 + \|\mathbf{f}\|)^{-1}\mathbf{f}$ to obtain an ODE with the same trajectories, see e.g. Chapter 3 in [41]. It is well-known that obtaining an analytical solution to (1.1) is in general very difficult or impossible. However, often one is more interested in what happens in the long term, than in exact solutions of the equation. That is, in the qualitative behavior of the dynamical systems rather than the quantitative behaviour. For this kind of analysis the Lyapunov theory has proven itself to be extremely useful [29, 34, 37, 42, 44]. The centrepiece of the Lyapunov theory is the (complete) Lyapunov function, which is a continuous, real-valued function from the state-space $\mathbb{R}^n$ that is non-increasing and decreasing, whenever possible, along solution trajectories of the system. The only case when it is not decreasing along a trajectory is if the trajectory is periodic, or infinitesimally close to being periodic; the technical term is *chain recurrent*.

The usefulness of the Lyapunov theory is that it bypasses the solution of the system. That is, to show that the Lyapunov function is decreasing along solution trajectories, one does not need to know the solution. In more detail, let $t \mapsto \phi(t, \mathbf{x})$ be the solution to (1.1) starting at $\mathbf{x}$ at time $t = 0$, let $V \colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function that has a strict minimum on the compact set $A \subset \mathbb{R}^n$, and let D be a neighbourhood of A. Then

$$\frac{d}{dt}V(\phi(t,\mathbf{x}))\Big|_{t=0} = \nabla V(\phi(t,\mathbf{x})) \cdot \dot{\phi}(t,\mathbf{x})\big|_{t=0} = \nabla V(\phi(t,\mathbf{x})) \cdot \mathbf{f}(\phi(t,\mathbf{x}))|_{t=0} = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$$

so if $\dot{V}(\mathbf{x}) := \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) < 0$ for all $\mathbf{x} \in D \setminus A \subset \mathbb{R}^n$, then V is decreasing along all solution trajectories passing through $D \setminus A$. In particular, if $D \setminus A$ is bounded and forward invariant (solutions cannot leave), all solutions starting in $D \setminus A$ must asymptotically approach the minimum of V at A. In applications, the most common setting is that the system (1.1) has an equilibrium $A = \{\mathbf{x}_0\}$, which can w.l.o.g. be assumed to be at the origin. In this setting a Lyapunov function $V \in C^1(\mathbb{R}^n)$ for the system (1.1) is a function fulfilling

$$\alpha(\|\mathbf{x}\|) \leq V(\mathbf{x}) \ \text{ and } \ \dot{V}(\mathbf{x}) \leq -\beta(\|\mathbf{x}\|) \ \text{ for all } \mathbf{x} \in D,$$

where $D \subset \mathbb{R}^n$ is a neighbourhood of the equilibrium at the origin and $\alpha, \beta \colon [0, \infty) \to [0, \infty)$ are strictly increasing, continuous functions with $\alpha(0) = \beta(0) = 0$ and $\lim_{x \to \infty} \alpha(x) = \infty$; so-called class $\mathcal{K}_\infty$ and class $\mathcal{K}$ comparison functions, respectively [29, 33]. Any compact sublevel set $L_c := \{\mathbf{x} \in \mathbb{R}^n \colon V(\mathbf{x}) \leq c\} \subset D$, $c > 0$, is forward invariant and every solution starting in it is asymptotically attracted to the equilibrium at the origin. Hence, sublevel sets of V act as *traps* for the dynamics; $\mathbf{x} \in L_c$ implies $\phi(t, \mathbf{x}) \in L_c$ for all $t \geq 0$.

The Lyapunov function was introduced by Aleksandr Lyapunov in 1892 to show the stability of equilibrium points [37], but the theory has since be extended to show that the qualitative behaviour of a dynamical system is characterized by a so-called complete Lyapunov function for the system [3, 6, 30]. Indeed, for a dynamical system defined by (1.1) there exists a smooth complete Lyapunov function $V \colon \mathbb{R}^n \to \mathbb{R}$ [27], and, for smooth $\mathbf{f}$, one can even prescribe its orbital derivative on a compact subset of where the flow is gradient-like [21].

Let us give a few simple examples of Lyapunov functions. For a physical system (1.1), in which the total energy E is conserved, one can set $V = E$ as then $\dot{V}(\mathbf{x}) = \dot{E}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^n$. If the system is dissipative, which in practice is most commonly the case as the amount of energy usable to sustain movement or perform work decreases as entropy increases, then $\dot{V}(\mathbf{x}) = \dot{E}(\mathbf{x}) < 0$ unless $\mathbf{x}$ is an equilibrium, and it follows that the state of the system is asymptotically attracted to an equilibrium, because the system cannot lose energy indefinitely. Indeed, a Lyapunov function can be seen as a mathematical extension of the physical concept of (dissipative) energy.

For a gradient system $\dot{\mathbf{x}} = -\nabla U(\mathbf{x})$, $U \colon \mathbb{R}^n \to \mathbb{R}$, $V = U$ is a natural Lyapunov function if U has a strict local minimum at some $\mathbf{x}_0 \in \mathbb{R}^n$, because

$$\frac{d}{dt} U(\phi(t,\mathbf{x})) \Big|_{t=0} = \nabla U(\phi(t,\mathbf{x})) \cdot \mathbf{f}(\phi(t,\mathbf{x}))|_{t=0} = -\nabla U(\phi(t,\mathbf{x})) \cdot \nabla U(\phi(t,\mathbf{x}))|_{t=0} = -\|\nabla U(\mathbf{x})\|^2 < 0$$

for all $\mathbf{x} \in D \setminus \{\mathbf{x}_0\}$, where $D \subset \mathbb{R}^n$ is some neighbourhood of $\mathbf{x}_0$. Hence, all solutions starting close enough to the equilibrium $\mathbf{x}_0$ are asymptotically attracted to it.

For a linear system $\dot{\mathbf{x}} = A\mathbf{x}$, $A \in \mathbb{R}^{n \times n}$, the equilibrium at the origin is asymptotically stable, if and only if for a symmetric and positive definite matrix $Q \in \mathbb{R}^{n \times n}$ there exists a symmetric and positive matrix $P \in \mathbb{R}^{n \times n}$ such that

$$A^{\mathrm{T}} P + PA = -Q. \tag{1.2}$$

Then a global Lyapunov function for the system is given by $V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} P\mathbf{x}$, because V has a strict global minimum at the origin and

$$\frac{d}{dt} V(\phi(t,\mathbf{x})) \Big|_{t=0} = \frac{d}{dt} \phi(t,\mathbf{x})^{\mathrm{T}} P\phi(t,\mathbf{x}) \Big|_{t=0} = \dot{\phi}(t,\mathbf{x})^{\mathrm{T}} P\phi(t,\mathbf{x}) + \phi(t,\mathbf{x})^{\mathrm{T}} P\dot{\phi}(t,\mathbf{x}) \Big|_{t=0}$$

$$= \phi(t,\mathbf{x})^{\mathrm{T}} (A^{\mathrm{T}} P + PA)\phi(t,\mathbf{x}) \Big|_{t=0} = \mathbf{x}^{\mathrm{T}} (A^{\mathrm{T}} P + PA)\mathbf{x} = -\mathbf{x}^{\mathrm{T}} Q\mathbf{x} < 0$$

for $\mathbf{x} \neq 0$. Hence, every solution is asymptotically attracted to the origin, and, moreover, one can show that there are constants $C \geq 1$ and $\alpha > 0$ such that $\|\phi(t,\mathbf{x})\| \leq Ce^{-\alpha t}\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$ and all $t \geq 0$; the equilibrium at the origin is said to be globally exponentially stable. The equation (1.2) is called the *continuous-time Lyapunov equation* and is a special case of the Sylvester equation [43]. It has a positive definite solution $P = P^{\mathrm{T}} \in \mathbb{R}^{n \times n}$, if and only if A is *Hurwitz*, i.e. the real parts of all eigenvalues of A are negative.

This linear analysis is also often used for nonlinear systems to obtain local stability certificates. If $\mathbf{f}(\mathbf{x}_0) = 0$, then the system (1.1) has an equilibrium at $\mathbf{x}_0$ and this equilibrium is locally exponentially stable, i.e. an estimate $\|\phi(t,\mathbf{x}) - \mathbf{x}_0\| \leq Ce^{-\alpha t}\|\mathbf{x} - \mathbf{x}_0\|$ as above holds true for all $\mathbf{x}$ in some neighbourhood of $\mathbf{x}_0$, if and only if the Jacobian $D\mathbf{f}(\mathbf{x}_0) \in \mathbb{R}^{n \times n}$ is Hurwitz. In this case $V(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0)^{\mathrm{T}} P(\mathbf{x} - \mathbf{x}_0)$ is a local Lyapunov function for the nonlinear system (1.1), where P is the solution to (1.2) with $A = D\mathbf{f}(\mathbf{x}_0)$, valid in some neighbourhood of $\mathbf{x}_0$.
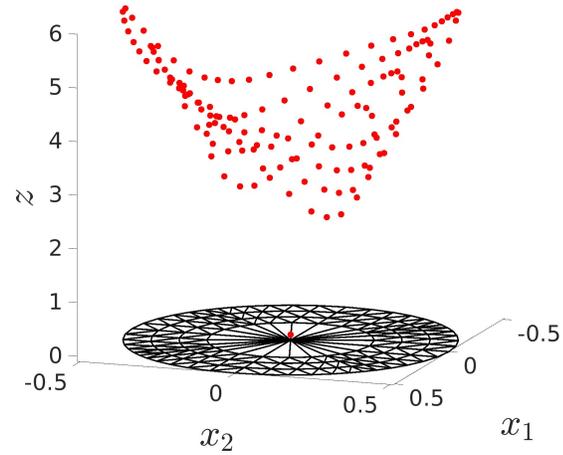
In applications of dynamical systems in engineering and the natural sciences, an equilibrium point of (1.1) is often the intended working point of the system. Local stability is usually the bare minimum for the system to be of any use. Usually, one is also interested in how robust the stability of the equilibrium is and how much disturbances the system can take before stability is lost. That is, one is interested in the existence of a Lyapunov function for the system in a sizeable neighbourhood of the equilibrium, and not only in local stability.

## 2. CPA METHOD: LYAPUNOV FUNCTIONS BY LINEAR OPTIMIZATION

In the CPA method to compute Lyapunov functions, a continuous and piecewise affine (CPA) Lyapunov function is parameterized using linear programming (LP) [11, 31, 32, 38, 39]. It works for general nonlinear systems (1.1) in n-dimension if $\mathbf{f}$ is a C^2 vector field. For pedagogical reasons we illustrate in detail how it
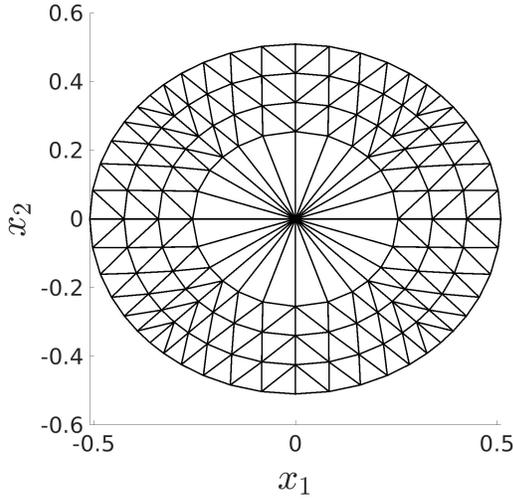
(A) Triangulation we use to compute a CPA Lyapunov function for the system (2.1). This triangulation is used in Figs. 1b and 1c.

(B) On the z-axis we plot the values $V_\mathbf{x}$ of a feasible solution to the LP problem (2.6) for the system (2.1), where the $\mathbf{x} \in \mathcal{V}$ are the vertices of the triangulation plotted in the x_1, x_2-plane.
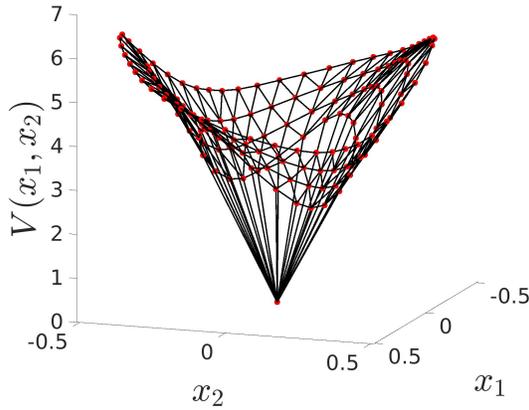
(C) By interpolating the values $V_\mathbf{x}$, $\mathbf{x} \in \mathcal{V}$, from Fig. 1b using formula (2.2), we get a CPA Lyapunov function for the system (2.1).

(D) A CPA Lyapunov function for the system (2.1) computed as in Fig. 1c, but using a denser triangulation of a larger domain.
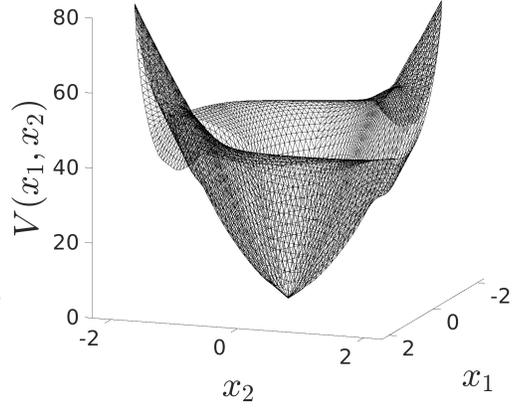
works for the concrete example of the time-reversed van der Pol oscillator

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} -x_2 \\ -(1-x_1^2)x_2 + x_1 \end{pmatrix} \tag{2.1}$$

where $n = 2$, while explaining the method in n-dimensions.

First the domain of interest $D \subset \mathbb{R}^n$, where the CPA method attempts to parameterize a Lyapunov function, is triangulated. That is, subdivided into n-simplices (triangles for $n = 2$), that are pairwise disjoint or intersect in a common face (side or vertex for $n = 2$). In Figure 1a we plot a triangulation, on which we will parameterize a CPA Lyapunov function for the system (2.1).

Denote by $\mathcal{T} = \{\mathfrak{S}_v\}_{v \in I}$, where I is an index set, the triangulation of D. Each $\mathfrak{S}_v$ is the convex hull of its vertices,

$$\mathfrak{S}_v = \mathrm{co}\left(\mathbf{x}_0^v, \mathbf{x}_1^v, \ldots, \mathbf{x}_n^v\right) = \left\{ \sum_{i=0}^n \lambda_i \mathbf{x}_i^v \in \mathbb{R}^n : \sum_{i=0}^n \lambda_i = 1, \ \lambda_i \geq 0 \ \text{ for } i = 0, 1, \ldots, n \right\}.$$

We denote the set of all the vertices of all the simplices in $\mathcal{T}$ by $\mathcal{V}$. The variables of the LP problem constructed in the CPA method are the values $V_{\mathbf{x}}$, $\mathbf{x} \in \mathcal{V}$, of the Lyapunov function $V : D \to [0, \infty)$ we want to parameterize. From the values $V_{\mathbf{x}}$, $\mathbf{x} \in \mathcal{V}$, the function V is defined through

$$V(\mathbf{x}) = \sum_{i=0}^n \lambda_i V_{\mathbf{x}_i^v} \quad \text{for} \quad \mathbf{x} = \sum_{i=0}^n \lambda_i \mathbf{x}_i^v \in \mathfrak{S}_v. \tag{2.2}$$

Note that the representation of $\mathbf{x} \in \mathfrak{S}_v$ as a convex combination of the vertices $\mathbf{x}_i^v$ of $\mathfrak{S}_v$ is unique because $\mathfrak{S}_v$ is an n-simplex. It is not difficult to see that V defined this way is continuous on D and has a constant gradient $\nabla V_v \in \mathbb{R}^n$ in the interior of each simplex $\mathfrak{S}_v$. For V to be a Lyapunov function it should have a minimum at the equilibrium at the origin. This is very easy to enforce using linear constraints, as long as we ensure that the origin is a vertex of simplices in $\mathcal{T}$, i.e. $0 \in \mathcal{V}$. The linear constraints

$$V_0 = 0 \quad \text{and} \quad V_{\mathbf{x}} \geq \|\mathbf{x}\| \ \text{ for all } \ \mathbf{x} \in \mathcal{V} \setminus \{0\},$$

then enforce $V(0) = 0$ and that for every $\mathbf{x} \in D$ we have

$$V(\mathbf{x}) = \sum_{i=0}^n \lambda_i V_{\mathbf{x}_i^v} \geq \sum_{i=0}^n \lambda_i \|\mathbf{x}_i^v\| \geq \left\| \sum_{i=0}^n \lambda_i \mathbf{x}_i^v \right\| = \|\mathbf{x}\|,$$

with V defined as in (2.2).

As the piecewise affine function V is merely Lipschitz continuous, and not differentiable, the condition

$$D^+V(\mathbf{x}) := \limsup_{h \to 0+} \frac{V(\mathbf{x} + h\mathbf{f}(\mathbf{x})) - V(\mathbf{x})}{h} \leq -\|\mathbf{x}\| \ \text{ for all } \ \mathbf{x} \in D^\circ \ \text{ (upper, right Dini-derivative),} \tag{2.3}$$

is used instead of $\dot{V}(\mathbf{x}) \leq -\|\mathbf{x}\|$ to enforce the decrease of V along solution trajectories (D° denotes the interior of D). The condition (2.3) holds true for all $\mathbf{x} \in D^\circ$ if for every $v \in I$ we have

$$\nabla V_v \bullet \mathbf{f}(\mathbf{x}_i^v) + \|\nabla V_v\|_1 E_i^v(\mathbf{f}) \leq -\|\mathbf{x}_i^v\| \quad \text{for } i = 0, 1, \ldots, n, \tag{2.4}$$

where the

$$E_i^v(\mathbf{f}) := \max_{j=0,1,\ldots,n} \sum_{r,s=1}^n \frac{B_{r,s}^v(\mathbf{f})}{2} |[\mathbf{x}_i^v - \mathbf{x}_d^v]_r| (|[\mathbf{x}_j^v - \mathbf{x}_d^v]_s| + |[\mathbf{x}_i^v - \mathbf{x}_d^v]_s|), \tag{2.5}$$

with $[\mathbf{y}]_i$ as the ith component of the vector $\mathbf{y}$, $d \in \{0, 1, \ldots, n\}$ arbitrary, and the $B_{r,s}^v(\mathbf{f})$ are system specific constants fulfilling

$$B_{r,s}^v(\mathbf{f}) \geq \max_{\substack{i=1,2,\ldots,n \\ \mathbf{x} \in \mathfrak{S}_v}} \left| \frac{\partial^2 f_i}{\partial x_r \partial x_s}(\mathbf{x}) \right|.$$

This is proved in [39, Lemma 4.16, Theorem II]. Because the components of the vectors ∇V_v are linear in the variables $V_{\mathbf{x}_i^v}$, $i = 0, 1, \ldots, n$, the condition (2.4) can be enforced using the auxiliary variables C_j^v, $v \in I$ and $j = 1, 2, \ldots, n$. The LP problem that parameterizes a CPA Lyapunov for the system (1.1) is then given by:

$$V_0 = 0 \quad \text{and} \quad V_{\mathbf{x}} \geq \|\mathbf{x}\| \ \text{ for all } \ \mathbf{x} \in \mathcal{V} \setminus \{0\}, \tag{2.6}$$

$$-C_j^v \leq [\nabla V_v]_j \leq C_j^v \ \text{ for all } v \in I \text{ and } j = 1, 2, \ldots, n, \text{ and}$$

$$\nabla V_v \bullet \mathbf{f}(\mathbf{x}_i^v) + E_i^v(\mathbf{f}) \sum_{j=1}^n C_j^v \leq -\|\mathbf{x}_i^v\| \ \text{ for all } v \in I \text{ and } i = 0, 1, \ldots, n.$$

For our concrete example of system (2.1) it is not difficult to see that we can set

$$B_{1,1}^v(\mathbf{f}) = 2 \cdot \max_{\mathbf{x} \in \mathfrak{S}_v} |x_2|, \quad B_{1,2}^v(\mathbf{f}) = B_{2,1}^v(\mathbf{f}) = 2 \cdot \max_{\mathbf{x} \in \mathfrak{S}_v} |x_1|, \quad \text{and} \quad B_{2,2}^v(\mathbf{f}) = 0.$$

In Figure 1b we see a feasible solution $V_{\mathbf{x}}$, $\mathbf{x} \in \mathcal{V}$, to this LP problem for the system (2.1); that is, the conditions in (2.6) all hold true for these values of $V_{\mathbf{x}}$. This solution was obtained using the linear solver Gurobi [22], which

is free for academic use. These values are then interpolated to a CPA Lyapunov function V for the system as in Figure 1c using the formula (2.2). Note that the constants $B_{rs}^v(\mathbf{f})$ are the only inputs to the LP problem (2.6), apart from the triangulation $\mathscr{T}$ and the vector field $\mathbf{f}$ on the right-hand-side of (1.1). Any sublevel set of V, $\{\mathbf{x} \in D\colon V(\mathbf{x}) \leq c\} \subset D^\circ$, $c > 0$, that is a compact subset of the open set D°, is forward invariant for the dynamics; see Figure 2b for an illustrative example.

Note that exactly the same procedure as for the planar system (1.1) can be used in n-dimensions and in [11] it is proved, that if the equilibrium is exponentially stable, then one can always use the LP problem (2.6) to parameterize a CPA Lyapunov function for the system.

3. EXTENSIONS OF THE CPA METHOD

The CPA Method can be extended and adapted to different kinds of systems and different scenarios in various ways. For example, for differential inclusions and switched systems [1, 2, 4, 36], for discrete-time systems [10, 23, 26, 40], for contraction metrics, which are a kind of a Lyapunov function on the tangent space [16–18], and for nonautonomous systems [14, 25].

As an example, let us discuss the extension to differential inclusions. As a concrete example, let us consider the system

$$\dot{\mathbf{x}} \in \mathrm{co}\{\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x})\}, \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} -x_2 \\ -(1 - x_1^2)x_2 + x_1 \end{pmatrix} \text{ and } \mathbf{g}(\mathbf{x}) = \begin{pmatrix} -x_2 \\ x_1 - x_2(1 - x_1^2 + 0.1x_1^4) \end{pmatrix}. \quad (3.1)$$

Note that $\mathbf{f}$ in (3.1) is equal to $\mathbf{f}$ in (2.1). A solution to (3.1) with initial value $\xi \in \mathbb{R}^2$ is a continuous function $\mathbf{x}\colon [0, \infty) \to \mathbb{R}^2$ such that $\mathbf{x}(0) = \xi$ and $\dot{\mathbf{x}}(t) \in \mathrm{co}\{\mathbf{f}(\mathbf{x}(t)), \mathbf{g}(\mathbf{x}(t))\}$ for almost all $t \in [0, \infty)$. The extension of the CPA method to compute a Lyapunov function for the differential inclusion (3.1) is simple, just add to the LP problem (2.6) the linear constraints

$$\nabla V_v \cdot \mathbf{g}(\mathbf{x}_i^v) + E_i^v(\mathbf{g}) \sum_{j=1}^n C_j^v \leq -\|\mathbf{x}_i^v\| \quad \text{for all } v \in I \text{ and } i = 0, 1, \dots, n,$$

with $E_i^v(\mathbf{g})$ defined as in (2.5) with

$$B_{r,s}^v(\mathbf{g}) \geq \max_{\substack{i=1,2,\dots,n \\ \mathbf{x} \in \mathbb{S}_v}} \left| \frac{\partial^2 g_i}{\partial x_r \partial x_s}(\mathbf{x}) \right|.$$

For the system (3.1) we can set

$$B_{1,1}^v(\mathbf{g}) = \max_{\mathbf{x} \in \mathbb{S}_v} |x_2|(2 + 1.2x_1^2), \quad B_{1,2}^v(\mathbf{g}) = B_{2,1}^v(\mathbf{g}) = \max_{\mathbf{x} \in \mathbb{S}_v} |x_1|(2 + 0.4x_1^2), \quad \text{and } B_{2,2}^v(\mathbf{g}) = 0.$$

In Figure 2a we plot a CPA Lyapunov function computed for the difference inclusion (3.1) and in Figure 2b we compare a sublevel set of this function to a sublevel set of the CPA Lyapunov function for the system (2.1) depicted in Figure 1d. Note the simplicity of extending the CPA method to differential inclusions!

4. CPA METHOD COMBINED WITH OTHER METHODS

The main limitation of the CPA method is solving the system dependent LP problem, which can be very large for higher-dimensional system. A way around this is to construct the LP problem in the CPA method, but instead of solving it with a linear solver to use a different method to assign values to the variables $V_{\mathbf{x}}$, $\mathbf{x} \in \mathscr{V}$. Subsequently, the linear constraints of the LP problem are verified to see where a function interpolated from the variables fulfills the conditions of a Lyapunov function. For example, one can use the formulas

$$V_1(\mathbf{x}) = \int_0^T \|\phi(t, \mathbf{x})\| dt \quad \text{or} \quad V_2(\mathbf{x}) = \int_0^T \|\mathbf{f}(\phi(t, \mathbf{x}))\| dt, \quad (4.1)$$

for Lyapunov functions from converse theorems, see e.g. [5,7,9,35]; $V_2(\mathbf{x})$ is the length of the solution trajectory $\phi([0, T], \mathbf{x})$ because $\|\mathbf{f}(\phi(t, \mathbf{x}))\| = \|\dot{\phi}(t, \mathbf{x})\|$. Note that these formulas depend on the solution trajectories of the system, which must be numerically computed for each vertex $\mathbf{x} \in \mathscr{V}$, and $T > 0$ must be sufficiently large. In particular, one can only use the formulas for V_1 and V_2 to numerically approximate values at a finite number of points $\mathbf{x}$ and they cannot be used to compute a Lyapunov function on a whole domain. Hence, the CPA method
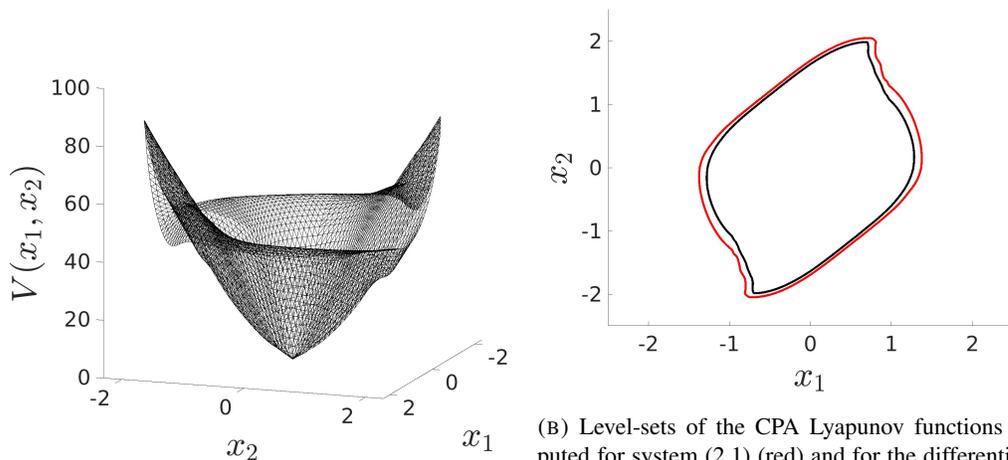
(A) A CPA Lyapunov function for the differential inclusion (3.1). Note that this function is additionally a Lyapunov function for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and for the system $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$, with $\mathbf{f}$ and $\mathbf{g}$ defined in (3.1).

(B) Level-sets of the CPA Lyapunov functions computed for system (2.1) (red) and for the differential inclusion (3.1) (black) for level 40. The corresponding sublevel sets are forward invariant for the systems, i.e., a solution entering the set can never leave. Further, the solution is asymptotically attracted to the equilibrium at the origin.
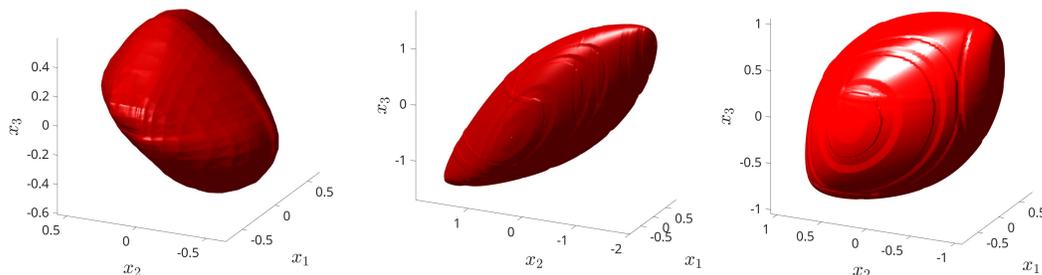

FIGURE 3. Sublevel-set of CPA Lyapunov functions computed for the system (4.2) using a LP solver (left), the formula for V_1 in (4.1) with subsequent certification (middle), and the formula for V_2 in (4.1) with subsequent certification (right).

is used to interpolate this finite number of approximate values of a Lyapunov functions to a true Lyapunov function defined on the domain of interest.

In Figure 3 we plot sublevel sets of CPA Lyapunov functions computed for the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} -x_1 - x_2 - x_3 \\ \sin(x_1) - 2x_2(1+x_1) + x_3 \\ x_1(1+x_1) + x_2 - \sin(x_3) \end{pmatrix} \tag{4.2}$$

using a triangulation with $18^3 \cdot 3! = 34{,}992$ tetrahedra (3-simplices) when solving the LP problem, but a triangulation with $100^2 \cdot 50 \cdot 3! = 3{,}0000{,}000$ tetrahedra when using the formulas for V_1 and V_2. In the LP problems we used the easily derived crude upper bounds $B_{r,s}^v(\mathbf{f}) = 2$ for all $v \in I$ and $r,s = 1,2,3$. Each computation is completed in less than 10 seconds on a standard PC (i285K, 64 GB, Linux Mint). Note that each such sublevel set it a lower bound on the equilibrium's basin of attraction. The formulas V_1 and V_2 usually fail to deliver good values for the Lyapunov function very close to the origin, but this problem is easily fixable by proving local stability of the equilibrium using linearization and quadratic Lyapunov functions as discussed in Section 1.

In [28] different integral formulas than V_1 and V_2 are studied, that give larger lower bounds on the basin of attraction, in [8, 24] the Yoshizawa construction formula [45] is used, and in [12] generalized interpolation in reproducing kernel Hilbert spaces is used to solve the Zubov equation [46] and deliver values to the variables. For computing Lyapunov functions on the tangent space, so-called contraction metrics, this combined approach also works and is even more valuable, as one avoids solving huge semi-definite optimization problem explicitly, see [15] for an overview. Both for usual Lyapunov functions and Lyapunov functions on the tangent space, one can prove that these combined approaches deliver true Lyapunov functions, see e.g. [13, 19, 20] in addition to the references above.

5. CONCLUSIONS

We discussed the Lyapunov stability theory and Lyapunov functions, which are real-valued functions from the state-space of a dynamical system, that are non-increasing along all solution trajectories and decreasing whenever possible. Lyapunov functions are an extension of the physical concept of (dissipative) energy. Then we discussed the CPA method to compute Lyapunov functions for ordinary differential equations (ODEs) using linear optimization and we gave examples of its use to compute continuous and piecewise affine (CPA) Lyapunov functions. Further, we discussed extensions of the CPA method for various kinds of systems and setups and gave an example for differential inclusions. Finally, we discussed how one can avoid solving the LP problem in the CPA method and instead use a different method to *guess* values for the variables of the LP problem, such that these values constitute a feasible solution. Subsequently it is certified where the conditions for a Lyapunov function are fulfilled by verifying the validity of the constraints of the LP problem.

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