ONE-SIDED EXTRAPOLATION IN THE CLASSICAL AND GRAND BALL BANACH FUNCTION SPACES

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Abstract. In this note Rubio de Francía's extrapolation results are presented in one-sided setting, in grand Lebesgue, Lorentz and ball Banach function spaces. These results can be applied, for example, to obtain the boundedness of one-sided operators of Harmonic Analysis in these spaces.

1. Introduction

We present one-sided variant of Rubio de Francía's extrapolation statements generally speaking, in grand ball Banach function spaces (BBFS briefly). In particular, we formulate the results in the classical Lebesgue, Lorentz spaces, and more generally in grand BBFS. BBFS was introduced in [26]. Grand X^p spaces based on a Banach function lattices/spaces X were introduced in [15] (see also [27]). Rubio de Francía's extrapolation properties in various grand function spaces were studied in [16], [17], [19], [18], [20], and in general grand Banach function spaces in [23] (see also [22]). One-sided extrapolation results in various situations were derived in [10], [14]. The same problem in BBFS was investigated in the recent paper [12]. Grand Lebesgue spaces (GLS briefly) $L^{p)}(\Omega)$ were introduced in [13], where the authors studied the integrability problem of the Jacobian. Initially it was introduced on a bounded open set. Later, GLSs on unbounded domains were introduced in [33], [25] by using so-called "grandizers". Recently, in the paper [3] the authors introduced GLSs on unbounded sets without "grandizers". In this note we will study grand function spaces, generally speaking, defined on an interval which might be unbounded. We formulate the results for the diagonal cases, i.e., when the exponents of domain and range spaces are the same, but we investigated the extrapolation problem in off-diagonal cases as well, which will be published later.

The creator of the weighted extrapolation theory is J. L. Rubio de Francía. His method is so much universal that it become as the base of several investigations, new ideas and for further extensions. In this direction it should be singled the monograph [6]. This monograph contains

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several of important topics: weighted norm inequalities, Rubio de Francía extrapolation and its extension, Calderón–Zygmund decomposition, new approach to weak inequalities etc.

In the monograph [6] (Theorem 4.6), and the papers [15], [19, 31] extrapolation problems were studied in general-type Banach function spaces (BFS briefly).

Another A_1 off-diagonal extrapolation theorem in so-called ball quasi-Banach function spaces (BQBFS briefly) was derived in [7].

Various extrapolation results in general weighted Banach function spaces in the on-diagonal case were considered in [2].

It should be mentioned that extrapolation theorems in BBFS and quasi-Banach function spaces were studied in [34] and [31], respectively (see also [26]).

We formulate the main statements only in the right-hand side case and also only for diagonal case. We mention here that the appropriate results in off-diagonal case are also derived.

Let $\mathcal{M}$ and $L^1_{\rm loc}$ denote the space of Lebesgue measurable functions and the space of locally integrable functions on $\mathbb{R}$, respectively. For any $x \in \mathbb{R}$ and $r \in (0, \infty)$, define

$$B(x, r) = \{y \in \mathbb{R} : |y - x| < r\} \quad \text{and} \quad \mathbb{B} = \{B(x, r) : x \in \mathbb{R},\, r \in (0, \infty)\}.$$

We recall the definition of a BQBFS from [26] (Definition 2.2):

Definition 1.1. A quasi-Banach function space $X \subset \mathcal{M}$ is said to be a a BQBFS on $\mathbb{R}$ if it satisfies

(1) $\|f\|_X = 0 \iff f = 0$ a.e.,
(2) $|g| \le |f|$ a.e. $\implies \|g\|_X \le \|f\|_X$,
(3) $0 \le f_n \uparrow f$ a.e. $\implies \|f_n\|_X \uparrow \|f\|_X$,
(4) $B \in \mathbb{B} \implies \chi_B \in X$.

If $\|\cdot\|_X$ is a norm, X is a Banach space that satisfies (1)-(4) and

$$B \in \mathbb{B} \implies \int_B |f(x)|dx < C_B \|f\|_X, \quad \forall f \in X, \tag{1.1}$$

for some $C_B > 0$, then we call X a BBFS.

For any $0 < r < \infty$ and a BQBFS X, the r-convexification of X, X^r is defined as

$$X^r = \{f : |f|^r \in X\}.$$

The vector space X^r is equipped with the quasi-norm $\|f\|_{X^r} = \||f|^r\|_X^{1/r}$. The reader is referred to [29], Volume II, p.53–54. If X is a BQBFS, then X^r is a BQBFS whenever $r \in (0, 1)$.

Let X be a ball Banach function space. The associate space of X, X', consists of all measurable functions f satisfying

$$\|f\|_{X'} = \sup\left\{\int_{\mathbb{R}} |f(x)g(x)|\, dx : g \in X,\ \|g\|_X \le 1\right\} < \infty.$$

Recall that (see [26], Proposition 2.3) if X is a ball Banach function space, then X' is also a ball Banach function space.

1.1. **Grand Lebesgue and Lorentz Spaces.** In 1992 T. Iwaniec and C. Sbordone [13] introduced grand Lebesgue space over the bounded open set Ω in $\mathbb{R}^n$. This space is defined with respect to the norm:

$$\|f\|_{L^{p)}(\Omega)} := \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|\Omega|}\right)^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(\Omega)} < \infty, \quad p > 1.$$

We refer to [11] for further generalizations of this norm.

Later, in [33] (see also [25]) grand Lebesgue spaces were introduced on an unbounded domain Ω by using the "grandizers". Extrapolation results on such grand Lebesgue spaces were received in [28].

Recently, in [3] the authors introduced (see also [24, Definition 4.1]) the norm of grand Lebesgue space for any domain Ω without using "grandizers".

Definition 1.2. [3] Let $1 < r < p < \infty$, $\theta > 0$, and let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set of nonzero measure, possibly infinite. Suppose that w is a weight function on Ω (i.e., w is a.e. positive locally integrable function on Ω). By weighted grand Lebesgue space $L^{(r,p),\theta}(\Omega)$ we mean the space of the (Lebesgue) measurable functions defined over Ω such that

$$\|f\|_{L_w^{(r,p),\theta}(\Omega)} := \sup_{0<\varepsilon<p-r} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L_w^{p-\varepsilon}(\Omega)} := \sup_{0<\varepsilon<p-r} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\int_{\Omega} |f(x)|^{p-\varepsilon} w(x) dx \right)^{\frac{\theta}{p-\varepsilon}} < \infty.$$

If w is constant, then we denote $L_w^{(r,p)}(\Omega)$ by $L^{(r,p)}(\Omega)$. It is obvious that when Ω is bounded, then $L_w^{(1,p)}(\Omega)$ is the classical weighted grand Lebesgue space denoted by $L_w^{p)}(\Omega)$.

By using the Hölder inequality, it is easy to see that for bounded Ω, the spaces $L_w^{(r,p),\theta}(\Omega)$ and $L_w^{p),\theta}(\Omega)$ coincide.

We will also need the following weighted grand Lebesgue space defined with respect to the norm:

$$\|f\|_{\mathcal{L}_w^{(r,p),\theta}(\Omega)} := \sup_{0<\varepsilon<p-r} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|fw\|_{L^{p-\varepsilon}(\Omega)}.$$

Let w be a weight function on Ω. In [19] the authors introduced generalized weighted grand Lorentz space with respect to the norm:

$$\|f\|_{L_w^{p),s,\theta}(\Omega)} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L_w^{p-\varepsilon,s}(\Omega)},$$

where $1 < p < \infty$, $1 \le s \le \infty$, $\theta > 0$, and the symbol $L_w^{q,s}(\Omega)$ denotes the weighted Lorentz space with parameters q, s for which the quantity

$$\|f\|_{L^{q,s}_w} = \begin{cases} \left(s \int\limits_0^\infty \big(w\{x \in \Omega : |f(x)| > \tau\} \big)^{s/q} \tau^{s-1} d\tau \right)^{1/s}, & \text{if } 1 \le s < \infty, \\ \sup_{s>0} s \Big(w(\{x \in \Omega : |f(x)| > s\} \Big)^{1/q}, & \text{if } s = \infty \end{cases}$$

is finite, where $wE := \int\limits_E w(x)dx$.

Similarly to [3] we introduce the space $L^{(\delta,p),s,\theta}_w(\Omega)$ on any domain Ω defined with respect to the norm:

$$\|f\|_{L^{(\delta,p),s,\theta}_w(\Omega)} = \sup_{0<\varepsilon<p-\delta} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon,s}_w(\Omega)}, \ 0 < \delta < p.$$

Let X be a BBFS on Ω, and let Φ_δ be the class of positive non-increasing functions $\varphi(\cdot)$ on $(0,\delta]$, $\delta < p-1$, such that $\lim_{x\to 0+} \varphi(x) = 0$. We introduce grand BBFS similarly to [23] (see also [15], [27]): for a bounded domain Ω and $\varphi(\cdot) \in \Phi_{p-1}$ we define the norm as follows

$$\|f\|_{X^{p),\varphi(\cdot)}(\Omega)} := \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{X^{p-\varepsilon}}, \quad p > 1.$$

If $\varphi(\varepsilon) \equiv \varepsilon^\theta$, where $\theta > 0$, then we denote $X^{p),\varphi(\cdot)}$ by $X^{p),\theta}$.

For $X = L^1$, the space $X^{p),\varphi(\cdot)}$ coincides with the space introduced in [4] for measurable φ (for $\varphi(\varepsilon) \equiv \varepsilon^\theta$, $\theta > 0$, it is Iwaniec-Sbordone space $L^{p),\theta}$).

If Ω is an arbitrary domain, then by the symbol $X^{(r,p),\varphi(\cdot)}(\Omega)$ we denote the space defined with respect to the norm:

$$\|f\|_{X^{(r,p),\varphi(\cdot)}(\Omega)} := \sup_{0<\varepsilon<p-r} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{X^{p-\varepsilon}}.$$

For any $f \in L^1_{\text{loc}}$, the one-sided Hardy-Littlewood maximal operators M^+f and M^-f are defined as follows

$$M^+f(x) = \sup_{t>0} \frac{1}{t} \int_x^{x+t} |f(y)|\, dy, \quad M^-f(x) = \sup_{t>0} \frac{1}{t} \int_{x-t}^{x} |f(y)|\, dy, \quad x \in \mathbb{R},$$

respectively.

We now recall the definition of *one-sided Muckenhoupt classes.*

Let $1 < p < \infty$ and let w be weight function on $\mathbb{R}$. We say that $w \in A_p^+$ if

$$[w]_{A_p^+} := \sup_{a\in\mathbb{R},h>0} \left(\frac{1}{h}\int_{a-h}^{a} w(t)dt\right)\left(\frac{1}{h}\int_{a}^{a+h} w(t)^{-\frac{1}{p-1}}dt\right)^{p-1} < \infty.$$

We say that $w \in A_1^+$ if there exists a constant $C > 0$ such that

$$M^- w(t) \le Cw(t), \quad t \in \mathbb{R}. \tag{1.2}$$

For any $\omega \in A_1^+$, the smallest constant C in (1.2) is denoted by $[\omega]_{A_1^+}$.
Define

$$A_\infty^+ = \bigcup_{p\in[1,\infty)} A_p^+.$$

It is easy to see that the class A_p^+ has the property:

$$[w]_{A^+_{p-\varepsilon_1}} \le [w]_{A^+_{p-\varepsilon_2}}; \;\; 0 < \varepsilon_1 < \varepsilon_2 \le p-1.$$

These classes have also so called openness property: if $w \in A_p^+$, then these exists a constant $\varepsilon_p^+ > 0$ such that $w \in A^+_{p-\varepsilon_p^+}$ (see [32], [30]).

Definition 1.3. Denote by σ_p^+ the best possible constant among those constants ε_p^+ for which the openness property holds.

Main Results

1.2. One-sided Extrapolation in Grand Lebesgue Spaces (GLeSs).

Theorem 1.4 (Bounded Interval.)**.** *Let $p_0 \in [1,\infty)$ and let $\mathcal{F}$ be the class of pairs of non-negative functions defined on $\mathbb{R}$. Suppose that for all $(f,g) \in \mathcal{F}$ and for all $w \in A_1^+$*

$$\left(\int_{\mathbb{R}} g^{p_0} w dx\right)^{\frac{1}{p_0}} \le CN\big([w]_{A_1^+}\big)\left(\int_{\mathbb{R}} f^{p_0} w dx\right)^{\frac{1}{p_0}} \tag{1.3}$$

holds, where the positive constant C is independent of (f,g) and w, and the constant $N\big([w]_{A_1^+}\big)$ is such that the mapping $\cdot \mapsto N(\cdot)$ is non-decreasing. Let I be a bounded interval in $\mathbb{R}$. Then for $1 < p < \infty$, $\theta > 0$, $w \in A_p^+$ and $(f,g) \in \mathcal{F}$ with supports in I, we have

$$\|g\|_{L_w^{p),\theta}} \le \overline{C}\|f\|_{L_w^{p),\theta}}, \tag{1.4}$$

where $\overline{C}$ is the positive constant independent of $(f,g) \in \mathcal{F}$.

Theorem 1.5 (Unbounded Interval)**.** *Suppose that $p_0 \in [1, \infty)$. Let $\mathcal{F}$ be the class of pairs of non-negative functions defined on $\mathbb{R}$. Suppose that for all $(f,g) \in \mathcal{F}$ and for all $w \in A^+_{p_0}$ we have*

$$\left(\int_{\mathbb{R}} g^{p_0} w dx\right)^{\frac{1}{p_0}} \le CN\big([w]_{A_1^+}\big)\left(\int_{\mathbb{R}} f^{p_0} w dx\right)^{\frac{1}{p_0}}, \tag{1.5}$$

where the positive constant C is independent of (f,g) and w, and N is the constant independent of (f,g) and depending on $[w]_{A_1^+(X)}$ such that the mapping $\cdot \mapsto N(\cdot)$ is non-decreasing. Then for $1 < p < \infty$, $\varphi \in \Phi_{\sigma_p^+}$, $w \in A_p^+$ and $(f,g) \in \mathcal{F}$, we have

$$\|g\|_{L_w^{(\sigma_p^+,p),\varphi(\cdot)}(\mathbb{R})} \le \overline{C}\|f\|_{L_w^{(\sigma_p^+,p),\varphi(\cdot)}(\mathbb{R})}, \tag{1.6}$$

where $\overline{C}$ is the positive constant independent of $(f,g) \in \mathcal{F}$, and the constant σ_p^+ is defined in Definition 1.3.

1.3. **Grand BBFSs.** We say that a BBFS denoted by X belongs to $\mathbb{M}^+$ (resp. X belongs to $\mathbb{M}^-$) if the operator M^+ (resp. M^-) is bounded in X.

Theorem 1.6. *[Bounded interval] Let $\mathcal{F}$ be a family of pairs (f,g) of measurable non-negative functions f, g defined on $\mathbb{R}$. Suppose for some $1 \le p_0 < \infty$, for every $w \in A_1^+$ and all $(f,g) \in \mathcal{F}$, the one-weight inequality*

$$\left(\int_{\mathbb{R}} g^{p_0}(x) w(x)\, dx\right)^{\frac{1}{p_0}} \le CN\big([w]_{A_1^+}\big)\left(\int_{\mathbb{R}} f^{p_0}(x) w(x)\, dx\right)^{\frac{1}{p_0}} \tag{1.7}$$

holds, where C and $N\big([w]_{A_1^+}\big)$ are positive constants such that C is independent of (f,g) and w, and $N\big([w]_{A_1^+}\big)$ is independent of (f,g) and depends on $[w]_{A_1^+}$ so that the mapping $\cdot \mapsto N(\cdot)$ is non-decreasing. Let X be a BBFS and let there exist $1 < q_0 < \infty$ such that X^{1/q_0} is again a BBFS.

Let I be a bounded interval in $\mathbb{R}$. Then for any $p > 1$, $\varphi(\cdot) \in \Phi_{p-1}$, there exists a positive constant C such that for all pairs of functions $(f,g) \in \mathcal{F}$ with compact support in I, the inequality

$$\|g\|_{X^{p),\varphi(\cdot)}} \le C\|f\|_{X^{p),\varphi(\cdot)}}, \quad (f,g) \in \mathcal{F},$$

holds provided that $(X^{(p-\varepsilon)/q_0})' \in \mathbb{M}$, $\varepsilon \in (0,\sigma)$, and that $L := \sup_{0<\varepsilon<\sigma} \|M\|_{(X^{(p-\varepsilon)/q_0})'} < \infty$, where σ is some small positive constant.

Theorem 1.7. *[Unbounded interval] Let $\mathcal{F}$ be a family of all pairs (f,g) of measurable non-negative functions f, g defined on $\mathbb{R}$. Suppose that for some $1 \le p_0 < \infty$, for every $w \in A_1^+$ and all $(f,g) \in \mathcal{F}$,* *the one-weight inequality*

$$\left(\int_{\mathbb{R}} g^{p_0}(x)w(x)\,dx\right)^{\frac{1}{p_0}} \le CN\big([w]_{A_1^+}\big)\left(\int_{\mathbb{R}} f^{p_0}(x)w(x)\,dx\right)^{\frac{1}{p_0}}$$

holds, where C and $N\big([w]_{A_1^+}\big)$ are positive constants such that C is independent of (f,g) and w, and $N\big([w]_{A_1^+}\big)$ is independent of (f,g) and depends on $[w]_{A_1}$ so that the mapping $\cdot \mapsto N(\cdot)$ is non-decreasing. Let X be a BFS and assume that there exists $1 < q_0 < \infty$ such that X^{1/q_0} is again a BFS.

Then for any $p > 1$, for all $\varphi(\cdot) \in \Phi_\delta$, where δ is sufficiently small positive constant $\delta \in (0, p-1)$, there exists a positive constant C such that for all pairs of functions $(f,g) \in \mathcal{F}$, the inequality

$$\|g\|_{X^{(\delta,p),\varphi(\cdot)}(\mathbb{R})} \le C\|f\|_{X^{(\delta,p),\varphi(\cdot)}(\mathbb{R})}, \quad (f,g) \in \mathcal{F},$$

holds provided that $(X^{(p-\varepsilon)/q_0})' \in \mathbb{M}$, $\varepsilon \in (0,\delta)$, and that $L := \sup_{0<\varepsilon<\delta} \|M\|_{(X^{(p-\varepsilon)/q_0})'} < \infty$ for some small positive number δ.

Remark 1.8. Taking $g = Tf$ (i.e., $(f,g) = (f,Tf)$) in the statements of this section, as a particular case, we can formulate appropriate extrapolation statements for T, where T is one of the operators of Harmonic Analysis such that it is bounded in weighted Lebesgue spaces under the one-sided Muckenhoupt condition on weights. Such operators are, for example, one-sided Hardy–Littlewood maximal and one-sided Calderón-Zygmund singular integral operators, commutators of one-sided singular integrals, one-sided fractional integral operators etc (see e.g., the papers [1], [30], [32]).

1.4. **One-sided Extrapolation in the Classical and Grand Lorentz Spaces (GLoS).** Now we give one-sided extrapolation results in the classical and grand Lorentz spaces. For simplicity we formulate them only in the right-hand side case.

Theorem 1.9. *Let $\mathcal{F}$ be a family of pairs (f,g) of measurable non-negative functions f,g defined on $\mathbb{R}$. Suppose that for some $1 \le p_0 < \infty$, for every $w \in A_1^+$ and all $(f,g) \in \mathcal{F}$, the one-weight inequality*

$$\left(\int_{\mathbb{R}} g^{p_0}(x)w(x)\,dx\right)^{\frac{1}{p_0}} \le CN\big([w]_{A_1^+}\big)\left(\int_{\mathbb{R}} f^{p_0}(x)w(x)\,dx\right)^{\frac{1}{p_0}} \tag{1.8}$$

holds with the positive constant C independent of (f,g) and w, and the positive constant $N\big([w]_{A_1^+}\big)$ independent of (f,g) and depending on $[w]_{A_1^+}$ so that the mapping $\cdot \to N(\cdot)$ is non-decreasing. Then for any $1 < p, s < \infty$, for all $(f,g) \in \mathcal{F}$ and any $w \in A_p^+$,

$$\|g\|_{L_w^{p,s}(\mathbb{R})} \le C_0\|f\|_{L_w^{p,s}(\mathbb{R})},$$

where the positive constant C_0 is independent of (f,g).

Now we pass to the case of grand Lorentz spaces.

Theorem 1.10. *Let w be a weight function on $\mathbb{R}$ and let $\mathcal{F}$ be a family of pairs (f,g) of measurable non-negative functions f,g defined on $\mathbb{R}$. Suppose that for some $1 \le p_0 < \infty$, for every $w \in A_1^+$ and all $(f,g) \in \mathcal{F}$, the one-weight inequality*

$$\left(\int_{\mathbb{R}} g^{p_0}(x)w(x)\,dx\right)^{\frac{1}{p_0}} \le CN\big([w]_{A_1^+}\big)\left(\int_{\mathbb{R}} f^{p_0}(x)w(x)\,dx\right)^{\frac{1}{p_0}}$$

holds with some positive constant C which does not depend on (f,g) and w, and the positive constant $N\big([w]_{A_1^+}\big)$ such that the mapping $\cdot \mapsto N(\cdot)$ is non-decreasing. Let I be a bounded interval in $\mathbb{R}$. Then for any $1<p<\infty$, $1 \le s < \infty$, $\theta > 0$, $w \in A_p^+$ and for all measurable $(f,g) \in \mathcal{F}$ having support on I,

$$\|g\|_{L_w^{p),s,\theta}} \le C\|f\|_{L_w^{p),s,\theta}},$$

with the positive constant C independent of (f,g).

Further, the next statement holds:

Theorem 1.11. *Let w be a weight function on $\mathbb{R}$ and let $\mathcal{F}$ be a family of pairs (f,g) of measurable non-negative functions f,g defined on $\mathbb{R}$. Suppose that for some $1 \le p_0 < \infty$, for every $w \in A_1^+$ and all $(f,g) \in \mathcal{F}$, the one-weight inequality*

$$\left(\int_{\mathbb{R}} g^{p_0}(x)w(x)\,dx\right)^{\frac{1}{p_0}} \le CN\big([w]_{A_1^+}\big)\left(\int_{\mathbb{R}} f^{p_0}(x)w(x)\,dx\right)^{\frac{1}{p_0}}$$

holds with some positive constant C which does not depend on (f,g) and w, and the positive constant $N\big([w]_{A_1^+}\big)$ such that the mapping $\cdot \mapsto N(\cdot)$ is non-decreasing. Then for any $1<p<\infty$, $1 \le s < \infty$, $\theta > 0$, $w \in A_p^+$ and for all measurable $(f,g) \in \mathcal{F}$, the inequality

$$\|g\|_{L_w^{(\sigma_p^+,p),s,\theta}(\mathbb{R})} \le C\|f\|_{L_w^{(\sigma_p^+,p),s,\theta}(\mathbb{R})},$$

holds with the positive constant C independent of (f,g), where σ_p^+ is the constant from Definition 1.3.

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