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ON CLASSIFICATION OF COMPLEMENTS IN FACTORIZABLE GROUPS

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Abstract. For any factorization $A = XB$ of a group A through its subgroups X and B, we provide a complete classification of the complements of X in A, expressed in terms of certain set-theoretical maps from B into X.

The present note is a continuation of the previous works on the problem of describing factorizations of groups and monoids [1, 2]. It focuses on factorizations of the form $A = XB$ of a group A through its subgroups X and B, with the aim of providing a full classification of complements of X in A.

Preliminaries. Given a group A, we denote its identity element by 1_A. One says that A is *factorizable* by its subgroups X and B (or that A is *factorized* through its subgroups X and B) if

$$A = XB \text{ and } X \cap B = \{1_A\}.$$

Equivalently, every element $a \in A$ admits a unique decomposition

$$a = l_X^B(a) r_X^B(a)$$

with $l_X^B(a) \in X$ and $r_X^B(a) \in B$. This expression is called the (X, B)-*decomposition* of the element $a \in A$.

For a given group A and its fixed subgroup $X \subset A$, let $\mathsf{FAC}(A/X)$ be the set of those subgroups $B \subset A$ for which A is factorizable through X and B. A subgroup X of A is called *complemented* if $\mathsf{FAC}(A/X) \neq \emptyset$; that is, if there exists a subgroup $B \subset A$ such that (X, B) is a factorization of A. In this case, B is called a *complement* of X.

Classification of complements. Suppose that $A = XB$ is a factorization of a group A through its subgroups X and B. Define $\mathscr{M}(B, X)$ to be the set of maps

$$Q : B \to X$$

satisfying the condition

$$Q(b_1 b_2) = l_X^B(b_1 Q(b_2)) Q(r_X^B(b_1 Q(b_2)))$$

for all $b_1, b_2 \in B$.

Proposition 1. *Let $A = XB$ be a factorization of a group A and let $Q \in \mathscr{M}(B, X)$. Then*

$$B_Q = \{Q^{-1}(b)b : b \in B\}$$

is a subgroup of A.

Theorem 1. *For any factorization $A = XB$ of groups, the assignment*

$$Q : B \to X \longmapsto B_Q = \{Q^{-1}(b)b : b \in B\}$$

yields a one-to-one correspondence between the sets $\mathscr{M}(B, X)$ and $\mathsf{FAC}(A/X)$. Its inverse takes $C \in \mathsf{FAC}(A/X)$ to the map $Q_C : B \to X$ defined by $Q_C(b) = l_X^C(b)$.

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Bicrossed products of groups. Let A be a group that factorizes through its subgroups X and B. Then the multiplication map

$$X \times B \to A, (x,b) \to xb$$

is bijective. Transporting the group structure of A across this bijection endows the Cartesian product $X \times B$ with a group law. Explicitly, the multiplication is given by

$$(x_1, b_1) \cdot (x_2, b_2) = (x_1 l_X^B(b_1 x_2), r_X^B(b_1 x_2) b_2).$$

The unit element of this group is $(1_X, 1_B)$ and the inverse $(x,b)^{-1}$of (x,b) is given by the pair $(l_X^B(b^{-1}x^{-1}), r_X^B(b^{-1}x^{-1}))$. The set $X \times B$ with this group structure is called the *bicrossed product* of X and B, denoted by

$$X \bowtie B.$$

Observe that

$$X \times \{1_B\} = \{(x, 1_B) : x \in X\} \text{ and } \{1_X\} \times B = \{(1_X, b) : b \in B\}$$

are subgroups of $X \bowtie B$. Moreover, $X \bowtie B$ factorizes through these two subgroups.

We can now translate Theorem 1 into the setting of bicrossed products. The result is as follows:

Theorem 2. *Let $A = XB$ be a factorization of groups. Then the assignment*

$$Q : B \to X \longmapsto \{(Q^{-1}(b), b) : b \in B\} \subset X \times B$$

establishes a one-to-one correspondence between $\mathscr{M}(B,X)$ and the set of complements of $X \times \{1_B\}$ in $X \bowtie B$.

Remark 1. *Theorems 1 and 2 can be generalized to the setting of groups in cartesian monoidal categories (see* [3]*).*

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