On affine diameters of finite subsets of the plane

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Abstract. It is shown that the total number $af(Z)$ of affine diameters of the set Z of all vertices of a convex n-gon in the plane $\mathbf{R}^2$ does not exceed $3n/2$. Also, it is established that if Z is an arbitrary n-point subset of $\mathbf{R}^2$, then $af(Z) \leq n^2/4 + 2$. Both these estimates are precise.

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Let $\mathbf{R}^m$ denote the m-dimensional Euclidean space and let Z be a subset of $\mathbf{R}^m$ with $\mathrm{card}(Z) \geq 2$.

By the standard definition, an affine diameter of Z is any line segment $[x, y]$ such that $\{x, y\} \subset Z$ and there exist two distinct parallel supporting hyperplanes G_x and G_y of Z passing through the points x and y respectively (see, e.g., [1], [3–5], [9]).

It follows from this definition that every affine diameter of Z is simultaneously an affine diameter of the convex hull of Z, denoted usually by $\mathrm{conv}(Z)$ (the converse assertion fails to be true in general).

Any ordinary diameter of Z (with respect to the standard metric of $\mathbf{R}^m$) is an affine diameter of Z, but not conversely. If Z is a finite subset of the plane $\mathbf{R}^2$ with $\mathrm{card}(Z) = n$, then the total number of ordinary diameters of Z does not exceed n, as a simple inductive argument shows. If $Z \subset \mathbf{R}^3$ and $\mathrm{card}(Z) = n$, where $n \geq 2$, then the total number of ordinary diameters of Z does not exceed $2(n-1)$ (see, for instance, [4], [6]). Various properties of affine diameters of point sets in $\mathbf{R}^m$ are considered in [1–6] and [8]. In this connection, see especially [9].

The goal of this short communication is to prove that if Z is the set of all vertices of a convex n-gon P in $\mathbf{R}^2$, then the total number of affine diameters of Z does not exceed $3n/2$. This estimate is precise, because gives the equality for the set of all vertices of a regular convex $2n$-gon Q in $\mathbf{R}^2$ (and for the set of all vertices of the image of Q under a non-degenerate affine transformation of $\mathbf{R}^2$).

Throughout this note the symbol $Af(Z)$ will stand for the family of all affine diameters of a subset Z of the space $\mathbf{R}^m$, where $m \geq 2$. Also, we stipulate $af(Z) = \mathrm{card}(Af(Z))$.

In the sequel, we need several auxiliary statements.

Lemma 1. *Let Z be a subset of $\mathbf{R}^m$ and let $\{Z_1, Z_2\}$ be a partition of Z. Denote by $\mathcal{A}_1$ the family of all those affine diameters of Z which are incident with at least one point from Z_1.*

Then the inclusion $Af(Z) \subset \mathcal{A}_1 \cup Af(Z_2)$ takes place (in general, this inclusion is not reducible to the equality).

Lemma 2. *Let Z be the set of all vertices of a convex n-gon P in $\mathbf{R}^2$, let $z \in Z$ and let z_1 and z_2 be two points of Z such that the line segments $[z, z_1]$ and $[z, z_2]$ are affine diameters of Z.*

Then any diagonal of P incident to z and lying in the convex angle formed by $[z, z_1]$ and $[z, z_2]$ is also an affine diameter of Z.

According to Lemma 2, if z is any vertex of a convex n-gon P, then there are some successive vertices $z_1, z_2, \ldots, z_k$ of P such that all line segments connecting these vertices with z turn out to be affine diameters of Z. Obviously, these successive vertices determine the corresponding successive sides

$$[z_1, z_2], \quad [z_2, z_3], \quad \ldots, \quad [z_{k-1}, z_k]$$

of P. In the sequel, we shall say that the above-mentioned successive sides constitute in P the affine support of z.

Lemma 3. *Let Z be the set of all vertices of a convex n-gon P in $\mathbf{R}^2$, let z_1, z_2, z_3, z_4 be four distinct points from Z satisfying the following two conditions:*

(a) the enumeration of z_1, z_2, z_3, z_4 is such that these points become successive vertices of the quadrangle $\mathrm{conv}(z_1, z_2, z_3, z_4)$;

(b) both line segments $[z_1, z_4]$ and $[z_2, z_3]$ are affine diameters of Z.

Then $[z_1, z_2]$ and $[z_3, z_4]$ are two parallel sides of P.

Lemma 4. *Let Z be the set of all vertices of a convex n-gon P in $\mathbf{R}^2$ and let a point $z \in Z$ be such that there are at least five distinct affine diameters*

$$[z, z_1], \quad [z, z_2], \quad [z, z_3], \quad [z, z_4], \quad [z, z_5]$$

of Z incident with z, where z_1, z_2, z_3, z_4, z_5 are some successive vertices of P.

Then $[z, z_3]$ is a unique affine diameter of Z incident with z_3.

In connection with Lemma 4 it makes sense to notice that, for any natural number $n \geq 2$, there exists a convexly independent subset Z of $\mathbf{R}^2$ with $\mathrm{card}(Z) = 2n + 1$, having the following properties:

(*) each point of Z is incident with at least two affine diameters of Z;

(**) there is a point of Z incident with exactly four affine diameters of Z.

Lemma 5. *Let Z be the set of all vertices of a convex n-gon P in $\mathbf{R}^2$ and let one of the sides of P be an affine diameter of P.*

Then the disjunction of these two assertions holds true:

(1) P is a parallelogram;

(2) the total number of affine diameters of Z is strictly less than $3n/2$.

Actually, in Lemma 5 the total number of affine diameters of Z does not exceed $n+2$. Obviously, if $n > 4$, then $n + 2 < 3n/2$.

Lemma 6. *Let Z be the set of all vertices of a convex n-gon P in $\mathbf{R}^2$ and let a point $z \in Z$ be such that there are at least four distinct affine diameters $[z, z_1]$, $[z, z_2]$, $[z, z_3]$, $[z, z_4]$ of Z, where z_1, z_2, z_3, z_4 are some successive vertices of P.*

Then, for any vertex $z' \in Z$ distinct from z, z_1, z_2, z_3, z_4, the side $[z_2, z_3]$ does not belong to the affine support of z'.

Lemma 7. *Let Z be the set of all vertices of a convex n-gon P in $\mathbf{R}^2$ and let n_k denote the number of all those vertices of P, each of which is incident with exactly k many affine diameters of Z.*

Then the following relations are fulfilled:

(1) $n_1 + 2n_2 + 3n_3 + ... = 2af(Z)$;

(2) $n_2 + 2n_3 + 3n_4 + ... \leq 2n$.

Theorem 1. *Let Z be the set of all vertices of a convex n-gon P in $\mathbf{R}^2$.*

Then $af(Z) \leq 3n/2$. In addition, the equality $af(Z) = 3n/2$ is valid if and only if there exists a partition of the set of all sides of P into two-element subsets such that the sides of any member of the partition are parallel.

Remark 1. Consider any finite non-collinear convexly independent point set Z in $\mathbf{R}^2$ with $\mathrm{card}(Z) = n$. It is easy to see that $af(Z) \geq n/2$. Therefore, in view of Theorem 1, the number $af(Z)$ is always of linear order with respect to $\mathrm{card}(Z)$.

Remark 2. A substantially different situation can be observed for a non-coplanar convexly independent set $Z \subset \mathbf{R}^3$ with $\mathrm{card}(Z) = n$. In this case, we obviously have

$$n/2 \leq af(Z) \leq n(n-1)/2.$$

Note also that:

(i) if Z coincides with the set of all vertices of an n-gonal prism, then $af(Z)$ is of order n^2;

(ii) if Z coincides with the set of all vertices of a convex n-gonal pyramid, then $af(Z)$ is of order n.

But it may happen that, for a finite convexly independent subset Z of $\mathbf{R}^3$ with $\mathrm{card}(Z) = n$, the value $af(Z)$ is of order strictly between $O(n)$ and $O(n^2)$.

Indeed, consider in $\mathbf{R}^3$ two distinct parallel planes H_1 and H_2. Let M_1 be a convex m-gon in H_1 and let M_2 be a convex n-gon in H_2. Define

$$P = \text{conv}(M_1 \cup M_2).$$

Obviously, P is a convex polyhedron in $\mathbf{R}^3$ and the total number $v(P)$ of vertices of P is equal to $m + n$. Taking into account Theorem 1, we see that

$$m/2 + n/2 + mn \leq af(v(P)) \leq mn + 3m/2 + 3n/2,$$

which implies that the number $af(v(P))$ is of order $mn + m + n$. Suppose now that n is of order $\ln(m)$. Then an easy calculation shows that $af(v(P))$ is of order $m\ln(m)$. So we have the finite convexly independent set $Z = v(P)$ such that $af(Z)$ is of order strictly between $O(\text{card}(Z))$ and $O((\text{card}(Z))^2)$.

Remark 3. Let Z be a finite subset of the space $\mathbf{R}^m$ satisfying the inequality $\text{card}(Z) > 2^m$. Then one necessarily has

$$af(Z) < (1/2)\text{card}(Z)(\text{card}(Z) - 1).$$

This strong inequality follows from one nice result of L. Danzer and Grünbaum [5] (see also [1]). Note that if, for a finite set $Y \subset \mathbf{R}^m$ with $\text{card}(Y) \geq 2$, the equality

$$af(Y) = (1/2)\text{card}(Y)(\text{card}(Y) - 1)$$

holds true, then Y turns out to be the set of all vertices of a k-dimensional parallelepiped, where $k \leq m$. Some other characterizations of k-dimensional parallelepipeds in the space $\mathbf{R}^m$ can be found in [7].

Theorem 2. *Let Z be a finite subset of* $\mathbf{R}^2$ *and let* $\text{card}(Z) = n \geq 4$.

Then the inequality $af(Z) \leq n^2/4 + 2$ holds true.

Moreover, the equality $af(Z) = n^2/4 + 2$ takes place if and only if the following three conditions are simultaneously satisfied:

(a) $n = 2k$ where $k \geq 2$ is a natural number;

(b) $\text{conv}(Z)$ *is a parallelogram in* $\mathbf{R}^2$*;*

(c) there are two opposite sides L_1 and L_2 of $\text{conv}(Z)$ *such that k many points of Z belong to L_1 and the rest k many points of Z belong to L_2.*

Theorem 2 can be proved by using induction on n.

Remark 4. Note that if Z is a finite subset of $\mathbf{R}^3$ such that

$$\text{card}(Z) = n \in \{8, 12, 16, 20, ...\},$$

then it may happen that $af(Z) = n^2/4 + 3n/2$. Therefore, the precise inequality of Theorem 2 does not longer hold for finite subsets of $\mathbf{R}^3$.

Remark 5. Let C be a compact convex set in $\mathbf{R}^2$ containing at least two points. As is well known (see, e.g., [3]), the following two assertions are valid:

(a) if C is not a parallelogram, then C admits a representation in the form $C = \cup\{C_1, C_2, C_3\}$ such that no affine diameter of C_i $(i = 1, 2, 3)$ is an affine diameter of C;

(b) if C is a parallelogram, then C does not admit a representation in the form $C = \cup\{C_1, C_2, C_3\}$ such that no affine diameter of C_i $(i = 1, 2, 3)$ is an affine diameter of C (however, in this case C can be represented in the form $C = \cup\{C_1, C_2, C_3, C_4\}$ such that no affine diameter of C_i $(i = 1, 2, 3, 4)$ is an affine diameter of C).

Note that if Z is a finite subset of $\mathbf{R}^2$ containing at least two points, then the analogs of (a) and (b) for Z can be deduced directly from Theorems 1 and 2 by using induction on $\mathrm{card}(Z)$.

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