Algorithmic problems in tensor completions of 2-nilpotent finitely generated torsion-free groups

M. G. Amaglobeli[1],T. Z. Bokelavadze[2], A. G. Myasnikov[3]

Abstract

In this paper, we study algorithmic problems in the tensor completions $G \otimes_{\mathcal{N}_{2,R}} R$ of finitely generated torsion-free nilpotent groups G of class 2 in the quasivariety $\mathcal{N}_{2,R}$ of R-exponential 2-nilpotent groups over a computable field of characteristic zero with a computable additive basis. We show that the word problem, the conjugacy problem, and the power problem are decidable in $G \otimes_{\mathcal{N}_{2,R}} R$. Note that these results do not hold for arbitrary finitely generated torsion-free 2-nilpotent R-groups. In fact, there are two generated torsion-free 2-nilpotent R-groups for which these problems are undecidable.

1 Introduction

In this paper, we investigate algorithmic problems in the tensor completions $G \otimes R$ of finitely generated torsion-free nilpotent groups G of class 2, within the quasivariety $\mathcal{N}_{2,R}$ of R-exponential 2-nilpotent groups over a computable field R of characteristic zero with a computable additive basis. We prove that the word problem, the conjugacy problem, and the power problem are decidable in $G \otimes R$. Our approach relies on a recent description, given in [1], of the algebraic structure and the R-exponentiation in $G \otimes R$. Specifically, it was shown that $G \otimes R$ is isomorphic to the group $(G \otimes_{\mathcal{H}} R) \times D$, where $G \otimes_{\mathcal{H}} R$ is the classical Hall R-completion of G, D is an Abelian R-group, and the direct product is taken in the category of abstract groups (not of R-groups). The R-exponentiation in $G \otimes R$ is given explicitly by formula (3) (see Theorem 1 in Section 3). We effectively reduce the algorithmic problems for $G \otimes R$ to the corresponding problems for the Hall completions $G \otimes_{\mathcal{H}} R$, which are well understood (see, for example, [8]), and to certain algorithmic problems in the R-vector space D. When the field R is computable with a computable additive basis, these algorithmic problems in $G \otimes_{\mathcal{H}} R$ and in D are decidable.

Recall that a group G is called a *divisible group with unique roots* if, for any $g \in G$ and $0 \neq n \in \mathbb{N}$, the equation $x^n = g$ has a unique solution in G. In this case, there exists a $\mathbb{Q}$-exponentiation on G, that is, a mapping $G \times \mathbb{Q} \to G$ defined by $(g, \frac{m}{n}) \to (g^{\frac{1}{n}})^m$, which satisfies the following properties for any $x, y \in G$ and $\alpha, \beta \in \mathbb{Q}$:

[1]Department of Mathematics, Iv. Javakhishvili Tbilisi State University, Georgia, e-mail: mikheil.amaglobeli@tsu.ge,
[2]Department of Mathematics, Akaki Tsereteli State University, Kutaisi, Georgia, email: tengiz.bokelavadze@atsu.edu.ge,
[3]Stevens Institute of Technology, Hoboken, NJ 07030, USA, e-mail: amiasnikov@gmail.com

1) $x^1 = x$, $x^0 = e$, $e^\alpha = e$;

2) $x^\alpha x^\beta = x^{\alpha+\beta}$, $(x^\alpha)^\beta = x^{\alpha\beta}$;

3) $(y^{-1}xy)^\alpha = y^{-1}x^\alpha y$.

We will call such groups $\mathbb{Q}$-groups. Malcev proved that divisible torsion-free nilpotent groups are $\mathbb{Q}$-groups (see [10, 11]). A thorough study of arbitrary $\mathbb{Q}$-groups was carried out by Baumslag in [4]. Around the same time, while studying equations in free groups, Lyndon [9] introduced the concept of an R-group (and coined this name) for an arbitrary associative ring R with unity. By definition, these are groups equipped with a map $G \times R \to G$, denoted $(g, \alpha) \to g^\alpha$, that exactly satisfy axioms 1)—3) above.

Unfortunately, there are abelian Lyndon R-groups that are not R-modules. Later, Myasnikov and Remeslennikov, in their paper [13], added an additional scheme of axioms (quasi-identities) to the Lyndon axioms. Namely, for any $x, y \in G$ and $\alpha \in R$ (here $[g, h] = g^{-1}h^{-1}gh$):

4) (MR-axiom) $\quad [g, h] = e \longrightarrow (gh)^\alpha = g^\alpha h^\alpha$.

Note that all $\mathbb{Q}$-groups satisfying axioms 1)-3) also satisfy axiom 4), as do many other natural R-groups according to Lyndon. We denote by $\mathcal{M}_R$ the class of all R-groups satisfying axioms 1)-4) and refer to them as R-groups.

The tensor extension of the ring of scalars plays an important role in module theory. In [13], Myasnikov and Remeslennikov defined an exact analog of this construction, the so-called tensor completion, for an arbitrary group, and in [14] they described a concrete way to construct tensor completions for CSA groups, in particular, for free and torsion-free hyperbolic groups (see also [5]). However, the notion of tensor completion in an arbitrary variety of groups requires a special approach since the general construction of tensor completions is not necessarily compatible with the given variety. Amaglobeli, Myasnikov, and Nadiradze introduced the general notion of a variety of R-groups and tensor completions in such varieties in [2]. It turns out that different, although equivalent, definitions of nilpotency in the class of discrete groups can lead to formally different types of nilpotent R-groups. The question of whether these classes are truly distinct remains open. However, it is known [2] that in the class of 2-nilpotent groups, different definitions of nilpotency lead to the same previously mentioned quasivariety of 2-nilpotent R-groups $\mathcal{N}_{2,R}$. It is with respect to this class that we describe the tensor completions $G \otimes_{\mathcal{N}_{2,R}} R$ of finitely generated 2-nilpotent torsion-free groups G. In [7], Hall introduced and studied nilpotent R-groups satisfying the following additional axiom for all $x_1, \ldots, x_n \in G$ and $\alpha \in R$:

5) $x_1^\alpha x_2^\alpha \cdots x_n^\alpha = (x_1 x_2 \cdots x_n)^\alpha \tau_2(\bar{x})^{\binom{\alpha}{2}} \cdots \tau_c(\bar{x})^{\binom{\alpha}{c}}$,

where τ_i are Petresco group words over the alphabet $x_1, \ldots, x_n$, c — the nilpotency class of G, and $\binom{\alpha}{i}$ — the binomial coefficients.

In this case, the ring R must be a binomial domain, i.e., an integral domain, where for each $\alpha \in R$, the binomial coefficient $\binom{\alpha}{i} = \frac{\alpha(\alpha-1)\ldots(\alpha-i+1)}{i!}$, considered as an element of the field of fractions of R, belongs to R. The axiom 5) is a more precise version of axiom 4) in the case of nilpotent groups.

2 Preliminaries

2.1 R-groups

In this section, following the works of [2, 9, 13], we introduce the necessary terminology and discuss some properties of R-groups that are useful for further understanding.

Let R be an associative commutative ring with identity and $G \in \mathcal{M}_R$. The standard group language L_{gr} can be extended by symbols for the unary operations f_α for each $\alpha \in R$, yielding the language of R-groups: $L^R_{gr} = L_{gr} \cup \{f_\alpha \mid \alpha \in R\}$. The operation f_α is interpreted in an R-group G as α-exponentiation $f_\alpha(g) = g^\alpha$ for $g \in G$. As usual in universal algebra, the notions of R-homomorphisms, R-subgroups, and R-generators can be defined in the category $\mathcal{M}_R$ by considering them as groups in the language L^R_{gr}. Thus, a homomorphism $\phi : G \to H$ of two R-groups G and H is an R-homomorphism if $\phi(g^\alpha) = (\phi(g))^\alpha$ for all $g \in G$ and $\alpha \in R$; a subgroup $N \leq G$ is called an *R-subgroup* of G if it is invariant under R-exponentiation; and a subset $X \subseteq G$ *R-generates* an R-subgroup $\langle X \rangle_R$, which is the smallest R-subgroup of G containing X. Note that G is generated (as an R-group) by a set $A \subseteq G$ if every element of G can be represented as $w(a_1, \ldots, a_n)$ for some group R-word $w(z_1, \ldots, z_n)$ and $a_1, \ldots, a_n \in A$. Here, if $z_1, \ldots, z_n$ are variables, then any expression $w(z_1, \ldots, z_n)$ obtained from these variables by multiplication, inversion, and exponentiation by elements of R is called a *group R-word in the variables* $z_1, \ldots, z_n$ (group R-words are precisely terms in the language L^R_{gr}).

Axioms 1)-3) and 5) are identities in the language L^R_{gr}, while axiom 4) is a quasi-identity. Therefore, the class of all Lyndon R-groups (satisfying 1)-3)), $\mathcal{L}_R$, like the class $\mathcal{H}_{c,R}$ of all Hall groups of nilpotency class $\leq c$, is a variety in the language of L^R_{gr}, while the classes $\mathcal{M}_R$ and $\mathcal{N}_{2,R}$ are quasivarieties. It follows from general theorems of universal algebra that these classes contain free R-groups and tensor R-completions with respect to these classes; the standard theory of R-ideals, i.e., normal subgroups whose quotient groups again belong to the same class; we can speak of groups defined by generators and relations in these classes, etc.

2.2 Tensor Completions of Nilpotent R-Groups

For a natural number $c > 1$, we denote by $\mathcal{N}_{c,R}$ the quasivariety of all nilpotent R-groups satisfying the c-nilpotency identity $\forall x_1, \ldots, x_{c+1} [x_1, \ldots, x_{c+1}] = e$, where $[x_1, \ldots, x_{c+1}]$ is the left-normed commutator of weight $c + 1$. The operation of tensor completion plays a decisive role in the study of R-groups. In this section, we discuss tensor completions in the class $\mathcal{N}_{c,R}$.

Let G be a torsion-free nilpotent group of nilpotency class $c \geq 1$, and let R be an associative commutative ring with 1 of characteristic 0.

Recall that an R-group $G \otimes_{\mathcal{N}_{c,R}} R$ is called a *tensor R-completion of G in the class* $\mathcal{N}_{c,R}$ if there exists a homomorphism $\lambda : G \to G \otimes_{\mathcal{N}_{c,R}} R$ such that the following conditions are satisfied:

(1) $\lambda(G)$ R-generates $G \otimes_{\mathcal{N}_{c,R}} R$;

2) For any R-group $H \in \mathcal{N}_{c,R}$ and an arbitrary homomorphism $\varphi : G \to H$, there exists an R-homomorphism $\psi : G \otimes_{\mathcal{N}_{c,R}} R \to H$ that makes the diagram commutative

$$\begin{array}{ccc} G & \xrightarrow{\lambda} & G \otimes_{\mathcal{N}_{c,R}} R \\ {\scriptstyle\varphi}\downarrow & \swarrow{\scriptstyle\psi} & \\ H & & \end{array} \qquad (\varphi = \psi \circ \lambda).$$

It was proved in [2] that for any such group G and any such ring R, the tensor completion $G \otimes_{\mathcal{N}_{c,R}} R$ exists and is unique up to R-isomorphism. Moreover, the homomorphism λ gives a natural embedding of G into $G \otimes_{\mathcal{N}_{c,R}} R$. If G is an abelian R-group (i.e., $c = 1$), then $G \otimes_{\mathcal{N}_{c,R}} R \cong G \otimes R$ is the tensor product of the $\mathbb{Z}$-module G and the ring R.

Tensor R-completions $G \otimes_{\mathcal{H}_{c,R}} R$ of a group G in the class $\mathcal{H}_{c,R}$ are well known and are described by Hall in [7]. Namely, let G be a finitely generated torsion-free nilpotent group. Then G has a tuple of elements $\bar{w} = (w_1, \ldots, w_n)$ (the Malcev base of G) such that every element g of G is uniquely representable as

$$g = w_1^{t_1} \ldots w_n^{t_n} \tag{1}$$

for some $t_i \in \mathbb{Z}$. The set $t(g) = (t_1, \ldots, t_n)$ is called the set of Malcev coordinates g. In this case, the multiplication of elements of the form (1) is performed using certain polynomials $f_1(\bar{x}, \bar{y}), \ldots, f_n(\bar{x}, \bar{y})$, where $\bar{x} = (x_1, \ldots, x_n), \bar{y} = (y_1, \ldots, y_n)$ such that for any $g, h \in G$

$$t(gh) = (f_1(t(g), t(h)), \ldots, f_n(t(g), t(h))).$$

Similarly, raising to the power $\lambda \in \mathbb{Z}$ is introduced through polynomials (see [16]). To construct $G \otimes_{\mathcal{H}} R$, one can take the same Malcev basis $\bar{w} = (w_1, \ldots, w_n)$ of G, consider all formal products

$$w_1^{\alpha_1} \ldots w_n^{\alpha_n},$$

and introduce multiplication and exponentiation $\alpha \in R$ on them using the same polynomials as for G. Since R is binomial, all values of these polynomials are defined in R (see details in [16]).

3 Group Structure of Tensor Completions of Finitely Generated 2-Nilpotent Torsion-Free Groups

In this section, following [1], we describe the algebraic structure and R-exponentiation in tensor completions $G \otimes_{\mathcal{N}_{2,R}} R$ of finitely generated 2-nilpotent torsion-free groups G over a binomial ring R of characteristic 0. For brevity, we denote $G \otimes R = G \otimes_{\mathcal{N}_{2,R}} R$ and $G \otimes_{\mathcal{H}} R = G \otimes_{\mathcal{H}_{2,R}} R$.

In [1] we defined a crucial notion of a *c-commutator* for $g, h \in G \otimes R$ and $\alpha \in R$ as follows:

$$c(g,h)_\alpha = [g,h]^{\binom{\alpha}{2}}(g,h)_\alpha.$$

Then

$$(gh)^\alpha = g^\alpha h^\alpha [g,h]^{-\binom{\alpha}{2}} c(g,h)_\alpha \tag{2}$$

and hence the element $c(g,h)_\alpha$ *measures how much the power* $(gh)^\alpha$ *in* $G \otimes R$ *differs from the corresponding power in* $G \otimes_{\mathcal{H}} R$.

The identity map $G \to G$ canonically extends to an embedding $\eta : G \to G \otimes_{\mathcal{H}} R$. Moreover, since $G \otimes_{\mathcal{H}} R$ is an R-group, the embedding η canonically extends to an R-epimorphism $\mu : G \otimes R \to G \otimes_{\mathcal{H}} R$. This epimorphism plays an essential role in our research.

Let $\bar{u} = (u_1, \ldots, u_m)$ and $\bar{v} = (v_1, \ldots, v_n)$ be bases of the free Abelian groups $G/Z(G)$ and $Z(G)$, respectively. Then $\bar{u} \cdot \bar{v} = (u_1, \ldots, u_m, v_1, \ldots, v_n)$ is a Mal'tsev basis of G. Therefore, each element $g \in G$ can be uniquely represented as a product

$$g = u_1^{\alpha_1} \ldots u_m^{\alpha_m} v_1^{\beta_1} \ldots v_n^{\beta_n} = \bar{u}^{\bar{\alpha}} \cdot \bar{v}^{\bar{\beta}},$$

where $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m), \bar{\beta} = (\beta_1, \ldots, \beta_n)$ are some tuples of integers. By construction, each element of the Hall completion $G \otimes_{\mathcal{H}} R$ has a unique decomposition:

$$g = u_1^{\alpha_1} \ldots u_m^{\alpha_m} v_1^{\beta_1} \ldots v_n^{\beta_n} = \bar{u}^{\bar{\alpha}} \cdot \bar{v}^{\bar{\beta}},$$

where $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m), \bar{\beta} = (\beta_1, \ldots, \beta_n)$ are some tuples of elements from R. Multiplication and R-exponentiation in $G \otimes_{\mathcal{H}} R$ are defined by the same polynomials as in G with respect to the basis $\bar{u} \cdot \bar{v}$.

Lemma 1. *[1] Let G be a finitely generated 2-nilpotent torsion-free group, and R a binomial domain of characteristic zero. Let $\bar{u} \cdot \bar{v}$ be a Mal'tsev basis of G, as defined above. Then, in the group $G \otimes R$, the set of elements*

$$H_{\bar{u} \cdot \bar{v}} = \{u^{\bar{\alpha}} \cdot v^{\bar{\beta}} \mid \bar{\alpha} \in R^m, \bar{\beta} \in R^n\}$$

forms a subgroup (not necessarily an R-subgroup) that is, as an abstract group, isomorphic to the Hall completion $G \otimes_{\mathcal{H}} R$ of the group G. Moreover, $G = H_{\bar{u} \cdot \bar{v}} \times D$ where D is an abelian R-group (R-module).

It was shown in [1] that the R-exponentiation in $G \otimes R = H_{\bar{u}\cdot\bar{v}} \times D$ is determined by the following formula:

$$(u_1^{\alpha_1} \dots u_m^{\alpha_m} v_1^{\beta_1} \dots \dots v_n^{\beta_n} d)^{\alpha} = u_1^{\alpha_1 \alpha} \dots u_m^{\alpha_m \alpha} v_1^{\beta_1 \alpha + \sigma_1} \dots v_n^{\beta_n \alpha + \sigma_n} d^{\alpha} c(u_1^{\alpha_1}, \dots, u_m^{\alpha_m})_{\alpha}, \tag{3}$$

where

$$v_1^{\sigma_1} \dots v_n^{\sigma_n} = \tau_2(u_1^{\alpha_1}, \dots, u_m^{\alpha_m})^{-\binom{\alpha}{2}}, \tag{4}$$

$\tau_2(u_1^{\alpha_1}, \dots, u_m^{\alpha_m})$ is the second Petresco word (see [1]) and

$$c(u_1^{\alpha_1}, \dots, u_m^{\alpha_m})_{\alpha} = \Pi_{i=1}^{n-1} c(u_1^{\alpha_1} \dots u_i^{\alpha_i}, u_{i+1}^{\alpha_{i+1}})_{\alpha}. \tag{5}$$

Theorem 1. *[1] Let G be an arbitrary 2-nilpotent finitely generated torsion-free group and R an arbitrary binomial domain. Then the R-group $G \otimes R$ satisfies the following conditions:*

1) *$G \otimes R$, as an abstract group (not an R-group!), is isomorphic to $(G \otimes_{\mathcal{H}} R) \times D$, where $G \otimes_{\mathcal{H}} R$ is the Hall R-completion of G, considered as an abstract group (not an R-group), and D is the R-module generated by all c-commutators $c(x^{\alpha}, y^{\beta})_{\mu}, \alpha, \beta, \mu \in R$;*

2) *R-exponentiation in the group $G \otimes R$ is defined by formulas (3), (4) and (5);*

3) *D is a free R-module.*

4 Decidability of the classical algorithmic problems

4.1 Computable fields

Recall (see [6, 12, 15]) that a ring R is called computable if it has an isomorphic copy R_0 of the form $R_0 = \langle \mathbb{N}; \oplus, \otimes \rangle$, where $\mathbb{N}$ is the set of natural numbers and $\oplus, \otimes$ are computable functions. In this situation, we will identify R with one of its copies R_0. We say that a computable field R of characteristic zero has a computable (recursive) additive basis (for brevity, R is CFCB) if there exists a computable subset $B \subseteq \mathbb{N}$ that forms a basis for the additive group R^+ of R viewed as a $\mathbb{Q}$-vector space. A purely transcendental extension $R = \mathbb{Q}(t)$ of $\mathbb{Q}$ is CFCB; for instance, the set of all simple fractions

$$B = \left\{ \frac{t^k}{p(t)^m} \,\middle|\, p(t) \text{ is monic irreducible over } \mathbb{Q},\ m \geq 1,\ 0 \leq k < \deg(p) \right\}$$

can be used as an additive basis of $\mathbb{Q}(t)$. Likewise, any finite simple algebraic extension $R = \mathbb{Q}(\alpha)$ of $\mathbb{Q}$ is CFCB. More generally, every finitely generated extension of $\mathbb{Q}$ is CFCB. There are many more CFCBs (see, for example, [6]).

Note that if R is CFCB and B is a computable additive basis of R, then there is an algorithm which, given an element $a \in R$, computes the unique linear combination

$$a = r_1 b_1 + \dots + r_n b_n, \tag{6}$$

where $r_i \in \mathbb{Q}$ and $b_i \in B$. Indeed, if $B = \{b_1, \dots, b_n\}$ is finite, the algorithm simply enumerates all tuples $(r_1, \dots, r_n) \in \mathbb{Q}^n$ and checks, one by one, whether (6) holds. In the infinite case, we effectively enumerate larger and larger finite subsets of B and, for each such subset, exhaustively test all corresponding tuples as above; in this way, the algorithm eventually finds the desired representation.

4.2 The algorithmic problems

We consider here three basic algorithmic problems for a given finitely generated R-group H with a fixed finite generating set $a_1, \dots, a_n$.

The word problem. *Given an R-word $w(a_1, \dots, a_n)$ in the generators $a_1, \dots, a_n$ decide if the word is equal to 1 in H.*

The conjugacy problem. *Given two R-words $u(a_1, \dots, a_n)$ and $v(a_1, \dots, a_n)$ decide if they are conjugate in H.*

The power problem. *Given two R-words $u(a_1, \dots, a_n)$ and $v(a_1, \dots, a_n)$ decide if there is $\alpha \in R$ such that $u(a_1, \dots, a_n)^\alpha = v(a_1, \dots, a_n)$.*

Theorem 2. *Let G be a finitely generated, torsion-free, 2-nilpotent group, and let R be a computable field of characteristic zero with a computable additive basis. Then the word problem, the conjugacy problem, and the power problem are decidable in the tensor completion $G \otimes R$.*

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