On intersections of Mazurkiewicz type sets with plane algebraic curves

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Abstract. In the present note several results are discussed, concerning the richness of intersections of Mazurkiewicz type sets in $\mathbb{R}^2$ with various plane algebraic irreducible curves. Mazurkiewicz type sets are considered here with respect to the family of all straight lines in $\mathbb{R}^2$ and also with respect to the family of all circles in $\mathbb{R}^2$.

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Each Mazurkiewicz type set with respect to the family of all straight lines of the Euclidean plane $\mathbb{R}^2$ and each plane irreducible algebraic curve in the $\mathbb{R}^2$ of degree at least 2 has a finite intersection with any straight line in the plane. Therefore it is natural to investigate how the above-mentioned finiteness property influences for these two classes of sets their close interrelation. In this context several results concerning the intersections of Mazurkiewicz type sets with plane algebraic curves are examined. Some theorems are also presented, regarding similar questions for Mazurkiewicz type sets with respect to the family of all circles in the plane.

Let us recall some definitions that we will use below.

Definition 1. For a natural number $n > 1$, a set $B \subset \mathbb{R}^2$ is called a Mazurkiewicz n-type set (or n-point set) in $\mathbb{R}^2$ if $\text{card}(B \cap l) = n$ for any straight line l lying in $\mathbb{R}^2$.

In connection with Definition 1 note that in 1914, Mazurkiewicz proved that there exists a set $X \subset \mathbb{R}^2$ such that every straight line in $\mathbb{R}^2$ meets X at exactly two points (see [12]). Such a set X is called a Mazurkiewicz set. Later on, Sierpinski proved that for every natural number $n > 1$, there exists a set $Y \subset \mathbb{R}^2$ such that every straight line in $\mathbb{R}^2$ meets Y at exactly n points. The proof of the existence of n-point sets initiated extensive studies of their various aspects, leading to the discovery of many significant properties of these sets. Actually, n-point sets remain extensively studied objects of contemporary mathematics (see [2], [3], [4], [5], [6], [8], [9], [10], [11], [15]).

Definition 2. For a natural number $k > 2$, a set $B \subset \mathbb{R}^2$ is called a Mazurkiewicz k-type set in $\mathbb{R}^2$ with respect to the family of all circles if $\text{card}(B \cap C) = k$ for any circle C lying in $\mathbb{R}^2$.

For instance, some theorems related to the existence and various properties of Mazurkiewicz type sets with respect to the family of all circles of $\mathbb{R}^2$ are presented in [1] and [7].

Recall that, by definition, a plane algebraic curve C is any subset of the plane $\mathbb{R}^2$ formed by all solutions (x, y) of a non-constant polynomial equation f in two variables, i.e.,

$$C = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}.$$

Similarly to any Mazurkiewicz type set with respect to the lines, each irreducible algebraic curve of degree at least two intersects every line in finitely many points. Likewise, if one considers a Mazurkiewicz type set with respect to the family of all circles in $\mathbb{R}^2$, then any such set has finite intersection with every circle in $\mathbb{R}^2$ (in this connection, note that any irreducible algebraic curve of degree at least three also meets every circle of $\mathbb{R}^2$ at finitely many points).

In view of the said above, it is natural to ask how a Mazurkiewicz type set with respect to straight lines (with respect to circles) and an irreducible algebraic curve of degree at least two (at least three) may be disposed relative to one another in the plane.

In this context, Larman's result is known.

Larman's result. No Mazurkiewicz set in $\mathbb{R}^2$ contains a simple arc (i.e. it does not contain a homeomorphic image of the closed unit interval $[0, 1]$) (see [13]). Also, no three-point set contains a simple arc.

On the other hand, it was proved in [14] that for every natural number $n \geq 4$ there exists an n-point set which contains a simple arc.

Theorem 1. *Let $C \subset \mathbb{R}^2$ be any irreducible algebraic curve of degree at least two and n be an arbitrary fixed natural number such that $n \geq 2$. Then there exist Mazurkiewicz n-type sets: $M_1 \subset \mathbb{R}^2$, $M_2 \subset \mathbb{R}^2$, $M_3 \subset \mathbb{R}^2$, $M_4 \subset \mathbb{R}^2$ each of them with respect to the family of all straight lines in $\mathbb{R}^2$, and possessing the following properties:*

1) *the intersection of M_1 with any nondegenerate simple arc contained in C is of cardinality continuum, has Lebesgue linear measure zero and is of Baire first category;*

2) *the intersection of M_2 with any nondegenerate simple arc contained in C is of cardinality continuum, has Lebesgue linear measure zero and is of Baire second category;*

3) *the intersection of M_3 with any nondegenerate simple arc contained in C is of cardinality continuum, is non measurable with respect to the Lebesgue linear measure (moreover, is thick with respect to this measure) and is of Baire first category;*

4) *the intersection of M_4 with any nondegenerate simple arc contained in C is of cardinality continuum, is non measurable with respect to the Lebesgue linear measure (moreover, is thick with respect to this measure) and is of Baire second category.*

Before formulation of a result concerning the intersections of plane irreducible algebraic curves of degree at least three and Mazurkiewicz type sets with respect to the family of all circles in $\mathbb{R}^2$, let us note that no Mazurkiewicz set with respect to the family of all circles in $\mathbb{R}^2$ contains a simple arc. Note also that a very particular case of the proposition that no Mazurkiewicz set with respect to the family of all circles in $\mathbb{R}^2$ contains a non-degenerate line segment can be proved in the frames of the elementary Euclidean geometry.

Theorem 2. *Let $C \subset \mathbb{R}^2$ be any irreducible algebraic curve of degree at least three and k be an arbitrary fixed natural number such that $k \geq 3$. Then there exist Mazurkiewicz k-type sets: $M_1 \subset \mathbb{R}^2$, $M_2 \subset \mathbb{R}^2$, $M_3 \subset \mathbb{R}^2$, $M_4 \subset \mathbb{R}^2$ each of them with respect to the family of all circles in $\mathbb{R}^2$, and possessing the following properties:*

1) *the intersection of M_1 with any nondegenerate simple arc contained in C is of cardinality continuum, has Lebesgue linear measure zero and is of Baire first category;*

2) *the intersection of M_2 with any nondegenerate simple arc contained in C is of cardinality continuum, has Lebesgue linear measure zero and is of Baire second category;*

3) *the intersection of M_3 with any nondegenerate simple arc contained in C is of cardinality continuum, is non measurable with respect to the Lebesgue linear measure (moreover, is thick with respect to this measure) and is of Baire first category;*

4) *the intersection of M_4 with any nondegenerate simple arc contained in C is of cardinality continuum, is non measurable with respect to the Lebesgue linear measure (moreover, is thick with respect to this measure) and is of Baire second category.*

Note that there exist algebraic curves of degree two or less, for which the analog of Theorem 2 remains valid.

As usual, we denote by $\mathfrak{c}$ the cardinality of the continuum.

Comment 1. *For any given plane irreducible algebraic curve C of degree at least 2 and any cardinal number $0 \leq \tau \leq \mathfrak{c}$ and any natural number $n \geq 2$, there exists a Mazurkiewicz n-type set with respect to the family of all straight lines in the Euclidean plane, whose intersection with C has cardinality τ.*

Comment 2. *For any given plane irreducible algebraic curve C of degree at least 3 and a cardinal number $0 \leq \tau \leq \mathfrak{c}$ and any natural number $k \geq 3$, there exists a Mazurkiewicz k-type set with respect to the family of all circles in the Euclidean plane, whose intersection with C has cardinality τ.*

It should be noted that, besides the plane algebraic curves, there are many transcendental plane curves for which the analogs of Theorems 1 and 2 hold true.

Comment 3. *Let us note that Theorems 1 and 2, as well as Comments 1 and 2, remain valid for subsets of the Euclidean plane that are more general than algebraic and transcendental curves. However, these subsets may no longer possess the "nice" properties of algebraic curves and may have a very pathological structure.*

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