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THE CHARACTERISTIC PROBLEM FOR ONE CLASS OF HIGH-ORDER NONLINEAR EQUATION OF COMPOSITE TYPE

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Abstract. The multidimensional characteristic problem in the conical domain for a class of high-order nonlinear equations of composite type is considered. The theorems on existence, uniqueness and nonexistence of solutions of this problem are proved.

In The Euclidean space $\mathbb{R}^{n+1}$ of variables $x = (x_1, \ldots, x_n)$ and t, we consider the following nonlinear equation of composite type:

$$\square^2 \Delta u + f(u) = F, \tag{1}$$

where $\square := \frac{\partial^2}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the wave operator, $\Delta := \frac{\partial^2}{\partial t^2} + \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, f and F are given, and u is an unknown real functions, $n \geq 2$.

By $D_T : |x| < t < T - |x|, \ \ |x| < \frac{1}{2}T$ we denote the conical domain in $\mathbb{R}^{n+1}$, bounded below by the characteristic cone of the future $S_+ : t = |x|, \ \ x \in \mathbb{R}^n$, with the vertex at the point $O(o, ..., o, o)$, and above by the characteristic cone of the past $S_- : t = T - |x|, \ \ x \in \mathbb{R}^n$, with the vertex at the point $O_1(o, ..., o, T)$.

For the equation (1) in domain D_T, we consider the following characteristic problem : find a solution u of the equation (1) in domain D_T according to the boundary conditions

$$u|_{\partial D_T} = 0, \qquad \square u|_{\partial D_T} = 0. \tag{2}$$

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Note that for equation (1), boundary value problems in domains of different geometric structure were investigated in works $[1-4]$, and for composite equations of a different type then (1) were considered in works $[5-10]$.

Let

$$\overset{o}{C}{}^{k}\left(\overline{D}_T\right) := \left\{u \in C^k\left(\overline{D}_T\right) : u|_{\partial D_T} = 0, \quad \Box u|_{\partial D_T} = 0\right\}, \quad k \geq 2. \tag{3}$$

Introduce the Hilbert space $\overset{o}{W}{}^{2}_{\Box}(D_T)$ as a completion of the classical space $\overset{o}{C}{}^{6}\left(\overline{D}_T\right)$ with respect to the norm

$$\|u\|^2_{\overset{o}{W}{}^{2}_{\Box}(D_T)} = \|u\|^2_{W^1_2(D_T)} + \|\Box u\|^2_{W^1_2(D_T)} =$$

$$\int_{D_T}\left[u^2 + \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]dxdt +$$

$$+\int_{D_T}\left[(\Box u)^2 + \sum_{i=1}^{n}\left(\frac{\partial \Box u}{\partial x_i}\right)^2 + \left(\frac{\partial \Box u}{\partial t}\right)^2\right]dxdt. \tag{4}$$

Before introducing a notion of a weak generalized solution of the problem $(1),(2)$ from the space $\overset{o}{W}{}^{2}_{\Box}(D_T)$, let us suppose that $u \in \overset{o}{C}{}^{6}\left(\overline{D}_T\right)$ is a classical solution of this problem. Multiplying both parts of the equation (1) by an arbitrary function $\varphi \in \overset{o}{C}{}^{3}\left(\overline{D}_T\right)$ and integrating the obtained equality by parts over the domain D_T, due to $(2),(3)$ we obtain

$$\int_{D_T}\left[\sum_{i=1}^{n}\frac{\partial \Box u}{\partial x_i}\frac{\partial \Box \varphi}{\partial x_i} + \frac{\partial \Box u}{\partial t}\frac{\partial \Box \varphi}{\partial t}\right]dxdt - \int_{D_T} f(u)\,\varphi dxdt =$$

$$= -\int_{D_T} F\varphi dxdt \quad \forall \varphi \in \overset{o}{C}{}^{6}\left(\overline{D}_T\right). \tag{5}$$

When obtaining (5), it was taken into account that on the characteristic surface ∂D_T the derivative with respect to the conormal $\frac{\partial}{\partial \mathcal{N}} = \nu_t\frac{\partial}{\partial t} - \sum_{i=1}^{n}\nu_{x_i}\frac{\partial}{\partial x_i}$ is an inner differential operator, where $\nu = (\nu_{x_1}, ..., \nu_{x_n}, \nu_t)$ is the unit vector of the outer normal to ∂D_T, and thus $\frac{\partial \varphi}{\partial \mathcal{N}}|_{\partial D_T} = 0$, since $\varphi|_{\partial D_T} = 0$, and it was also taken into account that, by virtue of (2) and the formulas for integration by parts

$$\int_{D_T}\Box^2\Delta u \cdot \varphi dxdt = \int_{\partial D_T}\left[\varphi\frac{\partial}{\partial \mathcal{N}}\Box\Delta u - \Box\Delta u \cdot \frac{\partial}{\partial \mathcal{N}}\varphi\right]ds +$$

$$+\int_{D_T}\Box\Delta u \cdot \Box\varphi dxdt = \int_{D_T}\Delta\Box u \cdot \Box\varphi dxdt = \int_{\partial D_T}\left[\frac{\partial \Box u}{\partial \nu}\cdot\Box\varphi - \Box u \cdot \frac{\partial \Box \varphi}{\partial \nu}\right]ds -$$

$$-\int_{D_T}\left[\sum_{i=1}^{n}\frac{\partial\Box u}{\partial x_i}\frac{\partial\Box\varphi}{\partial x_i}+\frac{\partial\Box u}{\partial t}\frac{\partial\Box\varphi}{\partial t}\right]dxdt=-\int_{D_T}\left[\sum_{i=1}^{n}\frac{\partial\Box u}{\partial x_i}\frac{\partial\Box\varphi}{\partial x_i}+\frac{\partial\Box u}{\partial t}\frac{\partial\Box\varphi}{\partial t}\right]dxdt.$$

In a certain sense, the converse statement is also true, i.e., is $u\in \overset{o}{C}{}^{6}\left(\overline{D}_T\right)$ satisfies the integral equality (5) for any $\varphi\in \overset{o}{C}{}^{3}\left(\overline{D}_T\right)$, then standard reasoning implies that u is a solution of equation (1) in the domain D_T and, by virtue of the definition (3) of the space $\overset{o}{C}{}^{6}\left(\overline{D}_T\right)$, satisfies the boundary conditions (2). Below, we will use equality (5) as the basis of definition of a weak generalized solution of the problem $(1),(2)$ in Hilbert space $\overset{o}{W}{}^{2}_{\Box}(D_T)$ from (4) under certain conditions imposed on the growth rate of the nonlinearity of the function $f(u)$.

Now we present the conditions imposed on the function $f(u)$ from (1):

$$f\in C(\mathbb{R}),\quad |f(u)|\le M_1+M_2|u|^{\alpha},\quad \alpha=const\ge 0,\quad u\in\mathbb{R},\tag{6}$$

where $M_i=const\ge 0,\ \ i=1,2$, and

$$0\le\alpha=const<\frac{n+1}{n-1}.\tag{7}$$

Remark 1. The embedding operator $I: W_2^1(D_T)\to L_q(D_T)$ is a linear compact operator for $1<q<\frac{2(n+1)}{n-1}$ and $n>1$ [11]. At the same time, the Nemitski operator $K: L_q(D_T)\to L_2(D_T)$, acting according to the formula $Ku=f(u)$, where $u\in L_q(D_T)$ and the function f satisfies conditions (6) and (7), is continuous and bounded for $q\ge 2\alpha$ [12]. Therefore, if $\alpha<\frac{n+1}{n-1}$, then there exists a number q such that $1<q<\frac{2(n+1)}{n-1}$ and $q\ge 2\alpha$. In this case, the operator

$$K_0=KI: W_2^1(D_T)\to L_2(D_T)$$

is continuous and compact. It follows in particular that if $u\in \overset{o}{W}{}^{2}_{\Box}(D_T)\subset W^{1}_{2}(D_T)$, then $f(u)\in L_2(D_T)$.

Definition 1. Let conditions $(6),(7)$ and $F\in L_2(D_T)$ be satisfied. The function $u\in \overset{o}{W}{}^{2}_{\Box}(D_T)$ is called a weak generalized solution of the problem $(1),(2)$, if the integral equality (5) is valid for any function $\varphi\in \overset{o}{W}{}^{2}_{\Box}(D_T)$, i.e.

$$\int_{D_T}\left[\sum_{i=1}^{n}\frac{\partial\Box u}{\partial x_i}\frac{\partial\Box\varphi}{\partial x_i}+\frac{\partial\Box u}{\partial t}\frac{\partial\Box\varphi}{\partial t}\right]dxdt-$$

$$-\int_{D_T}f(u)\varphi dxdt=-\int_{D_T}F\varphi dxdt\quad \forall\varphi\in \overset{o}{H}{}^{2}_{\Box}(D_T).\tag{8}$$

Note that, by virtue of Remark 1, the integral $\int_{D_T} f(u)\varphi dxdt$ in the left $-$ hand side of equality (8) is defined correctly, since from $u \in \overset{o}{W}{}^2_\square(D_T)$ follows that $f(u) \in L_2(D_T)$ and, consequently, $f(u)\varphi \in L_1(D_T)$.

Let us consider the following condition, imposed on the function f

$$\lim_{|u|\to\infty} inf \frac{f(u)}{u} \leq 0. \tag{9}$$

Theorem 1. Let the conditions $(6), (7)$ and (9) be fulfilled. Then for any function $F \in L_2(D_T)$ the boundary value problem $(1), (2)$ has at least one weak generalized solution in the space $\overset{o}{W}{}^2_\square(D_T)$ in the sense of Definition 1.

Theorem 2. Let the conditions $(6), (7)$ be fulfilled and the function f is non$-$inereasing, i.e. $(f(u) - f(v))(u - v) \leq 0 \quad \forall u, v \in \mathbb{R}$. Then for any function $F \in L_2(D_T)$ the boundary value problem $(1), (2)$ cannot have more then one weak generalized solution $u \in \overset{o}{W}{}^2_\square(D_T)$ in the sense of Definition 1.

Corollary. Let the conditions $(6), (7), (9)$ be fulfilled and the function f is non-increasing. Then for any function $F \in L_2(D_T)$ the boundary value problem $(1), (2)$ has a unique weak generalized solution in the space $\overset{o}{W}{}^2_\square(D_T)$ in the sense of Definition 1.

Note that if condition (9) is violated, a sufficiently wide class of functions $F \in L_2(D_T)$ can be given when the problem $(1), (2)$ does not have a weak generalized solution from the space $\overset{o}{W}{}^2_\square(D_T)$ in the sense of Definition 1. Indeed, the following theorem holds.

Theorem 3. Let the conditions $(6), (7)$ be fulfilled and

$$f(u) \leq -|u|^\gamma \qquad \forall u \in \mathbb{R}, \qquad \gamma = const > 1. \tag{10}$$

If $F = \lambda F_0$ with $\lambda = const > 0$, $\; F_0 > 0$ and $F_0 \in L_2(D_T)$, then there exists a number $\lambda_0 = \lambda_0(F_0, \gamma)$ such that for $\lambda > \lambda_0$ the problem $(1), (2)$ does not have a weak generalized solution from the space $\overset{o}{W}{}^2_\square(D_T)$ in the sense of Definition 1.

Note than when condition (10) holds, then condition (9) violates.

REFERENCES

1. S. Kharibegashvili, On the solvability of the Cauchy characteristic problem for a nonlinear equation with iterated wave operator in the principal part. J. Math. Anal. Appl. **338** (2008), no. 1, 71-81.

2. S. Kharibegashvili, B. Midodashvili, Solvability of characteristic boundary-value problems for nonlinear equations with iterated wave operator in the principal part. Electron. J. Differential Equations **2008**, no, 72, 1-12.
3. S. Kharibegashvili, B. Midodashvili, On one boundary value problem for a nonlinear equation with the iterated wave operator in the principal part. Georgian Math. J. **15** (2008), no. 3, 541-554.
4. T. Bibilashvili, Darboux type multi-dimensional problem for a class of higher-order nonlinear hyperbolic equations. Transactions of A. Razmadze Mathematical Institute **177** (2023), issue 1, 135-137.
5. A. V. Bitsadze, Some problem of mathematics and mechanics. Novosibirsk, 1991, 47-49 (in Russian).
6. A. V. Bitsadze, M.S. Salakhitdinov, On the theory of equations of mixed-composite type. Sib. Math. Journal **2** (1961), no. 1, 7-19 (in Russian).
7. M. S. Salakhitdinov, Equation of mixed-composite types. Fan, Tashkent, 1974 (in Russian).
8. T. D. Dzhuraev, Boundary value problems for equations of mixed and mixed-composite types. Fan, Tashkent, 1996 (in Russian).
9. A. I. Kozhanov, Composite type equations and inverse problems. Utrecht, The Netherlands, VSP, 1999.
10. V. I. Korzyuk, O. A. Konopelko, E. S. Cheb, Boundary-value problems for fourth-order equations of hyperbolic and composite types. Journal of Mathematical Sciences **177:1** (2010), 89-115.
11. O. A. Ladyzhenskaya, The boundary value problems of mathematical physics. Springer-Verlag, New York, 1988.
12. A. Kufner and S. Fučik, Nonlinear Differential Equations. Amsterdam -New York, 1980.

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