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ON THE ONE NONPARAMETRIC ESTIMATE OF POISSON REGRESSION FUNCTION WITH THE GENERALIZED WEIGHT

PETRE BABILUA

Abstract. The limiting distribution of the integral square deviation of kernel-type nonparametric estimator of Poisson regression function with the generalized weight function is established. The test of the hypothesis testing about Poisson regression function is constructed. The question of consistency of the constructed test is studied. The power asymptotic of the constructed test is also studied for certain types of close alternatives.

Let random variable Y take values $0,1,2,\dots$ with probabilities $\Pi(k,\lambda)=\mathbb{P}\{Y=k\}=\frac{\lambda^k}{k!}\,e^{-\lambda}$, $\lambda>0$, $k=0,1,2,\dots$. Assume that the parameterλ is the function of an independent variable $x\in[0,1]$, i.e. $\mathbb{P}\{Y=k\}=\frac{\lambda^k(x)}{k!}\,e^{-\lambda(x)}$. $\lambda(x)$ is known as Poisson regression function (see [3], [7]). Let x_i, $i=1,2,\dots,n$, be the division points of the interval $[0,1]$:

$$x_i=\frac{2i-1}{2n}\,,\ \ i=1,2,\dots,n.$$

Let further Y_i, $i=1,2,\dots,n$, be independent Poisson random variables with $\mathbb{P}\{Y_i=k\mid\ x_i\}=\Pi(k,\lambda(x_i))$. The problem consists in estimating the function $\lambda(x)$, $x\in[0,1]$, by sample $Y_1,Y_2,\dots,Y_n$ [3]. Problems of this kind arise, for example, in medicine [6], [13], in astrophysics [8] and so on.

As an estimator for $\lambda(x)$ we consider the following statistic (see [10], [15])

$$\widehat{\lambda}_n(x)=\lambda_{1n}(x)\lambda_{2n}^{-1}(x),$$

$$\lambda_{\nu n}(x)=\frac{1}{nb_n}\sum_{i=1}^{n}K\Big(\frac{x-x_i}{b_n}\Big)Y_i^{2-\nu},\ \ \nu=1,2,$$

where $K(x)$ is some distribution density satisfying the requirements which we formulate in what follows, and $b_n\to 0$ is a sequence of positive numbers.

We assume that kernel $K(x)\ge 0$ is chosen so that it is a function with finite variation and satisfies the conditions $K(x)=K(-x)$, $K(x)=0$ as $|x|\ge\tau>0$, $\int K(x)\,dx=1$. The class of such functions is denoted by $H(\tau)$.

Let $C^{(i)}$ denotes the class of functions $\lambda(x)$, $x\in[0,1]$ having bounded derivatives up to order i, $i=1,2$.

We also introduce the notation

$$\overline{T}_n=nb_n\int\limits_{\Omega_n(\tau)}\big[\lambda_{1n}(x)-\mathbb{E}\,\lambda_{1n}(x)\big]^2 r(x)\,dx,\ \ \Omega_n(\tau)=[\tau b_n,1-\tau b_n],$$

$$T_n=nb_n\int\limits_{\Omega_n(\tau)}\big[\widehat{\lambda}_n(x)-\lambda(x)\big]^2\lambda_{2n}^2(x)r(x)\,dx,$$

where $r(x)$, $x\in[0,1]$ is piecewise continuous and bounded function.

The introduction of the generalized weight function $r(x)$ allows us to consider, within a unified framework, a broad class of integral quadratic functionals. Different choices of $r(x)$ lead to different test statistics, including the classical L^2-type criterion obtained as a special case when $r(x)\equiv 1$, as

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well as localized criteria that emphasize specific subregions of the domain. In this sense, the weight function $r(x)$ serves as a flexible tool for unifying various integral deviation measures within a single theoretical approach.

In many practical situations, the Poisson regression function $\lambda(x)$ is not equally important over the entire interval $[0,1]$. For instance, in applications arising in medicine or physics, interest may be focused on a particular range of the covariate, while in other problems observations may be more reliable or more informative in certain regions. Moreover, in areas where the intensity function $\lambda(x)$ is small, it may be desirable to reduce the contribution of the estimation error. The presence of the weight function $r(x)$ makes it possible to regulate the influence of different parts of the domain in the integral squared deviation and to reflect such heterogeneity in importance.

The use of a generalized weight function also has important implications for the power properties of the test. In particular, under local alternatives of Pitman type, the contribution of the alternative enters the limiting power function through integrals involving $r(x)$. This makes it possible, at least in principle, to choose the weight function in a way that increases sensitivity to specific directions of departure from the null hypothesis, and thus to construct adaptive procedures with improved power characteristics.

Finally, it should be emphasized that the generalized formulation includes the unweighted case $r(x) \equiv 1$ as a special instance, and all main results of the paper remain valid for this choice. Integral quadratic statistics with general weight functions have been widely used as global measures of deviation in the works $[2,4,11,14]$.

$$Q_{ij} = \psi_n(x_i, x_j), \quad \psi_n(u,v) = \int\limits_{\Omega_n(\tau)} K\Big(\frac{x-u}{b_n}\Big) K\Big(\frac{x-v}{b_n}\Big) r(x)\, dx,$$

$$\sigma_n^2 = 4(nb_n)^{-2} \sum_{k=2}^{n} \lambda_k \sum_{i=1}^{k-1} \lambda_i Q_{ik}^2, \quad \lambda_i = \lambda(x_i), \quad i = \overline{1,n},$$

$$\eta_{ij}^{(n)} = \frac{2\varepsilon_i \varepsilon_j Q_{ij}}{nb_n \sigma_n}\,, \quad \varepsilon_i = Y_i - \lambda(x_i),$$

$$\xi_k^{(n)} = \sum_{i=1}^{k-1} \eta_{ik}^{(n)}, \quad k = 2, \dots, n, \quad \xi_1^{(n)} = 0, \quad \xi_k^{(n)} = 0, \quad k > n,$$

$$\mathscr{F}_k^{(n)} = \sigma(\omega:\ \varepsilon_1, \dots, \varepsilon_k),$$

where $\mathscr{F}_k^{(n)}$ is the σ-algebra generated by random variable $\varepsilon_1, \dots, \varepsilon_k$, $\mathscr{F}_0^{(n)} = (\Theta, \Omega)$ (in what follows, for the sake of simplicity, instead of $\xi_k^{(n)}$ and $\eta_{ij}^{(n)}$ we will write ξ_k and η_{ij}).

Lemma 1. *The stochastic sequence $(\xi_k, \mathscr{F}_k)_{k\ge 1}$ is a martingale-difference.*

Lemma 2. *Let $K(x) \in H(\tau)$ and $\lambda(x)$, $0 \le x \le 1$, be also a function with bounded variation, $r(x)$ is piecewise continuous and bounded function. If $nb_n \to \infty$, then*

$$\frac{1}{nb_n} \sum_{i=1}^{n} K^{\nu_1}\Big(\frac{x-x_i}{b_n}\Big) K^{\nu_2}\Big(\frac{y-x_i}{b_n}\Big) \lambda^{\nu_3}(x_i) r(x)$$
$$= \frac{1}{b_n} \int\limits_0^1 K^{\nu_1}\Big(\frac{x-u}{b_n}\Big) K^{\nu_2}\Big(\frac{y-u}{b_n}\Big) \lambda^{\nu_3}(u) r(x)\, du + O\Big(\frac{1}{nb_n}\Big), \quad (1)$$

uniformly in $x, y \in [0,1]$, $\nu_i \in \mathbb{N} \cup \{0\}$, $i = 1,2,3$.

The *proof* of (1) is analogous to that of the assertion of Lemma 1 in [12, Section 1, Lemma 1, formula (2), p. 1643].

Lemma 3. *Let $K(x) \in H(\tau)$ and $\lambda(x) \in C^{(1)}$, $r(x)$ is piecewise continuous and bounded function. If $nb_n^2 \to \infty$, then*

$$b_n^{-1}\sigma_n^2 \longrightarrow \sigma^2(\lambda) = 2\int_0^1 \lambda^2(x)r(x)\,dx \int_{|x|\le 2\tau} K_0^2(x)\,dx \tag{2}$$

and

$$\begin{gathered}\Delta_n = \mathbb{E}\,\overline{T}_n = \Delta(\lambda) + O(b_n) + O\Big(\frac{1}{nb_n}\Big),\\ \Delta(\lambda) = \int_0^1 \lambda(x)r(x)\,dx \int_{|x|\le\tau} K^2(x)\,dx, \quad K_0 = K * K.\end{gathered} \tag{3}$$

Proof. We have

$$\sigma_n^2 = 2(nb_n)^{-2}\Big\{\sum_{k,i=1}^n \lambda_k\lambda_i Q_{ik}^2 - \sum_{i=1}^n \lambda_i^2 Q_{ii}^2\Big\} = d_1(n) + d_2(n), \quad \lambda_i = \lambda(x_i), \quad i = 1,\dots,n. \tag{4}$$

It is easy to see that

$$b_n^{-1}|d_2(n)| = 2n^{-2}b_n^{-3}\sum_{i=1}^n \lambda_i^2\Big(\int_{\Omega_n(\tau)} K^2\Big(\frac{x-x_i}{b_n}\Big)r(x)\,dx\Big)^2 \le c_1\,\frac{1}{nb_n}\,. \tag{5}$$

Further, using the definition of Q_{ki} and Lemma 2, we obtain

$$d_1(n) = 2\int_{\Omega_n(\tau)}\int_{\Omega_n(\tau)}\Big(\int_{\frac{x-1}{b_n}}^{\frac{x}{b_n}} \lambda(x-b_nt)K(t)K\Big(\frac{x-y}{b_n}-t\Big)\,dt\Big)^2 r(x)r(y)\,dx\,dy + O\Big(\frac{1}{nb_n}\Big). \tag{6}$$

Since $\lambda(x) \in C^{(1)}$ and $\big[\frac{x-1}{b_n}\,,\frac{x_n}{b_n}\big] \supset (-\tau,\tau)$ for all $x \in \Omega_n(\tau)$, from (6) we find

$$d_1(n) = 2\int_{\tau b_n}^{t-\tau b_n} \lambda^2(x)r(x)\,dx \int_{\tau b_n}^{t-\tau b_n} K_0^2\Big(\frac{x-y}{b_n}\Big)r(y)\,dy + O(b_n^2) + O\Big(\frac{1}{nb_n}\Big).$$

It can be easily established that

$$d_1(n) = 2\int_{\tau b_n}^{t-\tau b_n} \lambda^2(x)r(x)\,dx\cdot b_n \int_{\frac{x-1}{b_n}+\tau}^{\frac{x}{b_n}-\tau} K_0^2(z)r(x-b_nz)\,dz + O(b_n^2) + O\Big(\frac{1}{nb_n}\Big)$$

and

$$b_n^{-1}d_1(n) = 2\int_{\tau}^{t-\tau b_n} \lambda^2(x)r(x)\,dx \int_{\frac{x-1}{b_n}+\tau}^{\frac{x}{b_n}-\tau} K_0^2(z)r(x-b_nz)\,dz + O(b_n) + O\Big(\frac{1}{nb_n^2}\Big).$$

Therefore

$$b_n^{-1}d_1(n) \longrightarrow 2\int_0^1 \lambda^2(x)r^2(x)\,dx \int_{|x|\le 2\tau} K_0^2(x)\,dx. \tag{7}$$

From (5) and (7) assertion (2) follows.

Now let us prove (3). We have

$$\mathbb{D}\lambda_{1n}(x) = \frac{1}{(nb_n)^2}\sum_{i=1}^n K^2\Big(\frac{x-x_i}{b_n}\Big)\lambda(x_i).$$

By virtue of Lemma 2, from this we find

$$\mathbb{D}\lambda_{1n}(x) = \frac{1}{nb_n^2}\int_0^1 K^2\Big(\frac{x-u}{b_n}\Big)\lambda(u)\,du + \frac{1}{(nb_n)^2}\,. \tag{8}$$

Furthermore, since $\big[\frac{x-1}{b_n}\,,\frac{x}{b_n}\big] \supset [-\tau,\tau]$ for all $x \in \Omega_n(\tau)$ from (8) we obtain

$$\mathbb{D}\lambda_{1n}(x) = \frac{1}{nb_n}\,\lambda(x)\int_{|u|\le\tau} K^2(u)\,du + O\Big(\frac{1}{n}\Big) + O\Big(\frac{1}{(nb_n)^2}\Big).$$

Therefore

$$\mathbb{E}\,\overline{T}_n = \int_0^1 \lambda(x)r(x)\,dx \int_{|u|\le\tau} K^2(u)\,du + O(b_n) + O\Big(\frac{1}{nb_n}\Big). \qquad \square$$

The Lemma 3 is proved.

Theorem 1. *Let $K(x) \in H(\tau)$ and $\lambda(x) \in C^{(1)}$, $r(x)$ is piecewise continuous and bounded function. If $nb_n^2 \to \infty$, then*

$$\frac{b_n^{-1/2}(\overline{T}_n - \Delta(\lambda))}{\sigma(\lambda)} \xrightarrow{d} N(0,1),$$

where $\Delta(\lambda)$ and $\sigma(\lambda)$ are defined as in Lemma 3 and $\xrightarrow{d}$ denotes convergence in distribution, and $N(0,1)$ is a random variable having standard normal distribution $\Phi(x)$.

Proof. We have

$$\frac{\overline{T}_n - \Delta_n}{\sigma_n} = H_n^{(1)} + H_n^{(2)},$$

where

$$H_n^{(1)} = \sum_{k=1}^n \xi_k, \quad H_n^{(2)} = \frac{\sum\limits_{i=1}^n(\varepsilon_i^2 - \mathbb{E}\,\varepsilon_i^2)Q_{ii}}{nb_n\sigma_n}\,.$$

We will show that, $H_n^{(2)}$ converges to zero in probability. Indeed,

$$\mathbb{D}H_n^{(2)} = \frac{1}{(nb_n\sigma_n)^2}\Big[\sum_{i=1}^n(\lambda_i + 2\lambda_i^2)\Big]Q_{ii}^2,$$

where $\lambda_i = \lambda(x_i)$.

Since $Q_{ij} \le c_2 b_n$ and $b_n^{-1}\sigma_n^2 \to \sigma^2(\lambda) > 0$ as $n \to \infty$, we establish that

$$\mathbb{D}H_n^{(2)} \le c_3\,\frac{1}{nb_n} \longrightarrow 0.$$

Therefore $H_n^{(2)} \xrightarrow{\mathbb{P}} 0$ (here and in what follows, $\mathbb{P}$ denotes convergence in probability).

Now let us show that $H_n^{(1)} \xrightarrow{d} N(0,1)$. To this end, we will verify the applicability of Corollaries 2 and 6 of Theorem 2 in [11]. We have to prove the fulfillment of the conditions given in these assertions and guaranteing asymptotic normality of a square-integrable martingale-difference and, by Lemma 1, such is our sequence $\{\xi_k, \mathscr{F}_k\}_{k\ge1}$.

Direct calculation shows that $\sum\limits_{k=1}^n \mathbb{E}\xi_k^2 = 1$. Asymptotic normality will take place if as $n \to \infty$

$$\sum_{k=1}^n \mathbb{E}\Big[\xi_k^2 I\big(|\xi_k| \ge \varepsilon\big) \mid \mathscr{F}_{k-1}\Big] \xrightarrow{\mathbb{P}} 0 \tag{9}$$

and

$$\sum_{k=1}^n \xi_k^2 \xrightarrow{\mathbb{P}} 1. \tag{10}$$

In [9] it is proved that then (10) and the condition $\sup\limits_{1\le k\le n}|\xi_k|\xrightarrow{\mathbb{P}}0$ are fulfilled, the condition (9) takes place too.

Since for $\varepsilon>0$

$$\mathbb{P}\Big\{\sup_{1\le k\le n}|\xi_k|\ge\varepsilon\Big\}\le\varepsilon^{-4}\sum_{k=1}^n\mathbb{E}\,\xi_k^4,$$

by virtue of relation (11) given in what follows, to prove

$$H_n^{(1)}\xrightarrow{d}N(0,1)$$

it remains only to verify (10). For this it suffices to ascertain that

$$\mathbb{E}\Big(\sum_{k=1}^n\xi_k^2-1\Big)^2\longrightarrow 0\ \text{ as }\ n\to\infty,$$

i.e. since $\sum\limits_{k=1}^n\mathbb{E}\,\xi_k^2=1$,

$$\mathbb{E}\Big(\sum_{k=1}^n\xi_k^2\Big)^2=\sum_{k=1}^n\mathbb{E}\,\xi_k^4+2\sum_{1\le k_1<k_2\le n}\mathbb{E}\,\xi_{k_1}^2\xi_{k_2}^2\longrightarrow 1.$$

First we establish that

$$\sum_{k=1}^n\mathbb{E}\,\xi_k^4\longrightarrow 0\ \text{ as }\ n\to\infty.$$

Using the definition of ξ_k and η_{ij}, we write

$$\sum_{k=1}^n\mathbb{E}\,\xi_k^4=I_n^{(1)}+I_n^{(2)},$$

where

$$I_n^{(1)}=\frac{16}{(nb_n)^4\sigma_n^4}\sum_{k=2}^n\mathbb{E}\,\varepsilon_k^4\sum_{j=1}^{k-1}\mathbb{E}\,\varepsilon_j^4Q_{jk}^4,$$

$$I_n^{(2)}=\frac{48}{(nb_n)^4\sigma_n^4}\sum_{k=2}^n\sum_{i\ne j}\mathbb{E}\,\varepsilon_j^2\mathbb{E}\,\varepsilon_i^2Q_{jk}^2Q_{ik}^2.$$

Since

$$Q_{ij}\le c_3b_n,\ \ \mathbb{E}\,\varepsilon_j^4=(\lambda_i+3\lambda_i^2)\le c_4,\ \ b_n^{-1}\sigma_n^2\to\sigma^2(\lambda),$$

we have

$$I_n^{(1)}=O\Big(\frac{1}{(nb_n)^2}\Big),\quad I_n^{(2)}=O\Big(\frac{1}{nb_n^2}\Big).$$

Therefore

$$\sum_{k=1}^n\mathbb{E}\,\xi_k^4\xrightarrow{P}0\ \text{ as }\ n\to\infty. \tag{11}$$

Let us now show that

$$2\sum_{1\le k_1<k_2\le n}\mathbb{E}\,\xi_{k_1}^2\xi_{k_2}^2\longrightarrow 1\ \text{ as }\ n\to\infty.$$

From the definition of ξ_i it follows that

$$\xi_{k_1}^2\xi_{k_2}^2=B_{k_1k_2}^{(1)}+B_{k_1k_2}^{(2)}+B_{k_1k_2}^{(3)}+B_{k_1k_2}^{(4)},$$

where

$$B_{k_1k_2}^{(1)}=\sigma_2(k_1)\sigma_2(k_2),\quad B_{k_1k_2}^{(2)}=\sigma_2(k_1)\sigma_1(k_2),$$

$$B_{k_1k_2}^{(3)}=\sigma_1(k_1)\sigma_2(k_2),\quad B_{k_1k_2}^{(4)}=\sigma_1(k_1)\sigma_1(k_2),$$

$$\sigma_1(k)=\sum_{1\le i\ne j\le k-1}\eta_{ik}\eta_{jk},\quad \sigma_2(k)=\sum_{i=1}^{k-1}\eta_{ik}^2.$$

Therefore

$$2\sum_{1\le k_1<k_2\le n}\mathbb{E}\,\xi_{k_1}^2\xi_{k_2}^2=\sum_{i=1}^{4}A_n^{(i)},$$

where

$$A_n^{(i)}=2\sum_{1\le k_1<k_2\le n}\mathbb{E}\,B_{k_1k_2}^{(i)},\quad i=1,2,3,4.$$

Let us consider $A_n^{(3)}$. Using the definition of η_{ij}, it is easy to show that

$$\mathbb{E}\,B_{k_1k_2}^{(3)}=0$$

and therefore

$$A_n^{(3)}=0. \tag{12}$$

Let us estimate $A_n^{(2)}$. We have

$$|\mathbb{E}\,B_{k_1k_2}^{(2)}|=\frac{16}{n^4b_n^4\sigma_n^4}\Big|\sum_{i=1}^{k_1-1}\mathbb{E}\,\varepsilon_i^3\mathbb{E}\,\varepsilon_{k_1}^3\,\mathbb{E}\,\varepsilon_{k_2}^2Q_{ik_1}^2Q_{ik_2}Q_{k_1k_2}\Big|$$

$$=\frac{16}{(n^4b_n\sigma_n)^4}\Big|\sum_{i=1}^{k_1-1}\lambda_i\lambda_{k_1}\lambda_{k_2}Q_{ik_1}^2Q_{ik_2}Q_{k_1k_2}\Big|\le\frac{c_4(k_1-1)}{(n\sigma_n)^4}\,.$$

Since $\sum\limits_{1\le k_1<k_2\le n}(k_1-1)=O(n^3)$ and $b_n^{-1}\sigma_n^2\to\sigma^2(\lambda)>0$, we obtain

$$|A_n^{(2)}|\le\frac{c_5n^3}{n^4\sigma_n^4}=c_5\,\frac{1}{nb_n^2(b_n^{-1}\sigma_n^2)^2}=O\Big(\frac{1}{nb_n^2}\Big). \tag{13}$$

We will establish that $A_n^{(1)}\to1$ as $n\to\infty$. It is obvious that

$$A_n^{(1)}=2\sum_{1\le k_1<k_2\le n}\mathbb{E}\,B_{k_1k_2}^{(1)}=D_n^{(1)}+D_n^{(2)},$$

where

$$D_n^{(1)}=2\sum_{1\le k_1<k_2\le n}\Big(\sum_{i=1}^{k_1-1}\mathbb{E}\,\eta_{ik_1}^2\Big)\Big(\sum_{j=1}^{k_2-1}\mathbb{E}\,\eta_{jk_2}^2\Big),$$

$$D_n^{(2)}=2\bigg(\sum_{k_1<k_2}\mathbb{E}\,B_{k_1k_2}^{(1)}-\sum_{k_1<k_2}\Big(\sum_{i=1}^{k_1-1}\mathbb{E}\,\eta_{ik_1}^2\Big)\Big(\sum_{j=1}^{k_2-1}\mathbb{E}\,\eta_{jk_2}^2\Big)\bigg).$$

From the definition of σ_n^2 it follows that

$$D_n^{(1)}=1-\sum_{k=2}^{n}\Big(\sum_{i=1}^{k-1}\mathbb{E}\,\eta_{ik}^2\Big)^2.$$

But

$$\sum_{k=2}^{n}\Big(\sum_{i=1}^{k-1}\mathbb{E}\,\eta_{ik}^2\Big)^2\le c_6\,\frac{b_n^4n^3}{(nb_n)^4\sigma_n^4}=O\Big(\frac{1}{nb_n^2}\Big).$$

Thus

$$D_n^{(1)}\to1\ \text{ as }\ n\to\infty. \tag{14}$$

Further, we will show that $D_n^{(2)}\to0$. It is easy to see that

$$D_n^{(2)}=2\sum_{k_1<k_2}\Big[\sum_{i=1}^{k_1-1}\operatorname{cov}(\eta_{ik_1}^2,\eta_{ik_2}^2)+\sum_{i=1}^{k_1-1}\operatorname{cov}(\eta_{ik_1}^2,\eta_{k_1k_2}^2)\Big].$$

But

$$\mathbb{E}\,\eta_{ik_1}^2\eta_{ik_2}^2 \le c_7\,\frac{Q_{ik_1}^2 Q_{ik_2}^2}{(nb_n)^4\sigma_n^4} \le c_8\,\frac{1}{n^4\sigma_n^4}$$

and

$$\mathbb{E}\,\eta_{ij}^2 = O\Big(\frac{1}{n^2\sigma_n^2}\Big).$$

Therefore

$$\operatorname{cov}(\eta_{ik_1}^2, \eta_{ik_2}^2) = O\Big(\frac{1}{n^4\sigma_n^4}\Big). \tag{15}$$

Further, since $\sum\limits_{1\le k_1<k_2\le n}(k_1-1) = O(n^3)$ from (15) we have

$$D_n^{(2)} = O\Big(\frac{1}{n\sigma_n^4}\Big) = O\Big(\frac{1}{nb_n^2}\Big). \tag{16}$$

So, according (14) and (16),

$$A_n^{(1)} = 1 + O\Big(\frac{1}{nb_n^2}\Big). \tag{17}$$

Finally, let us prove that $A_n^{(4)} \to 0$ as $n\to\infty$. Using the definition of η_{ij} and relations $Q_{ij}\ge 0$ and $\mathbb{E}\,(Y_i-\lambda(x_i))^2 = \lambda(x_i)$, we obtain

$$|\mathbb{E}\,B_{k_1,k_2}^{(4)}| = 4\Big|\sum_{1\le t<s\le k_1-1}\mathbb{E}\,\eta_{sk_1}\eta_{tk_1}\eta_{sk_2}\eta_{tk_2}\Big| \le \frac{c_8}{n^4 b_n^4\sigma_n^4}\sum_{1\le t<s\le k_1-1} Q_{sk_1}Q_{tk_1}Q_{sk_2}Q_{tk_2}.$$

Therefore,

$$|A_n^{(4)}| \le \frac{c_9}{n^2 b_n^4\sigma_n^4}\sum_{k_1<k_2} A_{k_1k_2},$$

where

$$A_{k_1k_2} = \frac{1}{n^2}\sum_{1\le t<s\le k_1-1} Q_{sk_1}Q_{tk_1}Q_{sk_2}Q_{tk_2}.$$

However

$$\sum_{k_1<k_2} A_{k_1k_2} \le \sum_{k_1,k_2=1}^{n}\Big(\frac{1}{n}\sum_{t=1}^{n} Q_{tk_1}Q_{tk_2}\Big)^2.$$

Thus

$$\begin{aligned}
|A_n^{(4)}| &\le c_9\,\frac{1}{n^2 b_n^4\sigma_n^4}\sum_{k_1,k_2=1}^{n}\Big(\frac{1}{n}\sum_{i=1}^{n} Q_{ik_1}Q_{ik_2}\Big)^2\\
&\le c_9\,\frac{1}{n^2 b_n^4\sigma_n^4}\sum_{k_1,k_2=1}^{n}\Bigg[\int\limits_{\Omega_n(\tau)}\int\limits_{\Omega_n(\tau)} K\Big(\frac{x-x_{k_1}}{b_n}\Big)K\Big(\frac{y-x_{k_2}}{b_n}\Big)r(x)r(y)\,dx\,dy\\
&\qquad\times\frac{1}{n}\sum_{i=1}^{n} K\Big(\frac{x-x_i}{b_n}\Big)K\Big(\frac{y-x_i}{b_n}\Big)r(x)r(y)\,dx\,dy\Big)\Bigg]^2.
\end{aligned} \tag{18}$$

Further, using Lemma 2 from (18), it can be established that

$$\begin{aligned}
|A_n^{(4)}| \le c_9\,\frac{1}{b_n^4\sigma_n^4}\sum_{k_1,k_2=1}^{n}\Bigg\{\frac{1}{n}\int\limits_0^1\int\limits_{\Omega_n(\tau)}\int\limits_{\Omega_n(\tau)} K\Big(\frac{x-x_{k_1}}{b_n}\Big)K\Big(\frac{y-x_{k_2}}{b_n}\Big)\\
\times K\Big(\frac{x-u}{b_n}\Big)K\Big(\frac{y-u}{b_n}\Big)r(u)r(x)r(y)\,du\,dx\,dy\Bigg\}^2 + O\Big(\frac{1}{nb_n^2}\Big).
\end{aligned} \tag{19}$$

Analogously, again applying Lemma 2 in (19), it can be shown that

$$|A_n^{(4)}| \le \frac{c_{10}}{b_n^4\sigma_n^4} \int_0^1\int_0^1\int_0^1\int_0^1 \psi_n(u_1,v_2)\psi_n(u_1,v_1)\psi_n(u_2,v_1)\psi_n(u_2,v_2)\,du_1\,du_2\,dv_1\,dv_2 + O\Big(\frac{1}{nb_n^2}\Big), \quad (20)$$

where

$$\psi_n(x,y) = \int_{\Omega_n(\tau)} K\Big(\frac{u-x}{b_n}\Big)K\Big(\frac{u-y}{b_n}\Big)r(u)\,du.$$

Now let us estimate the integral I_n contained in (20). We have

$$I_n = \int_0^1\int_0^1\int_0^1 \psi_n(u_2,v_1)\psi_n(u_2,v_2)\,dv_1\,dv_2\,du_2 \int_0^1 \psi_n(u_1,v_2)\psi_n(u_1,v_1)\,du_1.$$

But since $\big[\frac{x-1}{b_n}, \frac{x}{b_n}\big] \supseteq [-\tau,\tau]$ for all $x\in\Omega_n(\tau)$, we obtain

$$\int_0^1 \psi_n(u_1,v_2)\psi_n(u_1,v_1)\,du_1$$

$$= b_n \int_{\Omega_n(\tau)}\int_{\Omega_n(\tau)} K\Big(\frac{t-v_2}{b_n}\Big)K\Big(\frac{z-v_1}{b_n}\Big)K_0\Big(\frac{z-t}{b_n}\Big)r(t)r(z)\,dt\,dz \le c_{11}b_n^3, \quad K_0 = K*K.$$

Therefore

$$|A_n^{(4)}| \le c_{12}\,\frac{1}{b_n\sigma_n^4}\int_0^1\int_0^1\int_0^1 \psi_n(u_2,v_1)\psi_n(u_2,v_2)\,du_2\,dv_1\,dv_2 + O\Big(\frac{1}{nb_n^2}\Big). \quad (21)$$

Repeating the same argumentation for (21), we finally obtain

$$|A_n^{(4)}| \le c_{13}\,\frac{b_n^4}{b_n\sigma_n^4} + O\Big(\frac{1}{nb_n^2}\Big) = O\Big(\frac{b_n^4}{b_n^3(b_n^{-1}\sigma_n^2)^2}\Big) + O\Big(\frac{1}{nb_n^2}\Big) = O(b_n) + O\Big(\frac{1}{nb_n^2}\Big). \quad (22)$$

Combining relations (12), (13), (17) and (22), we conclude that

$$2\sum_{1\le k_1<k_2\le n} \mathbb{E}\,\xi_{k_1}^2\xi_{k_2}^2 \longrightarrow 1.$$

This and (11) imply that

$$\mathbb{E}\Big(\sum_{k=1}^n \xi_k^2 - 1\Big)^2 \longrightarrow 0 \ \text{ as } \ n\to\infty.$$

Therefore

$$\frac{\overline{T}_n - \Delta_n}{\sigma_n} \xrightarrow{d} N(0,1). \quad (23)$$

Further, using Lemma 3 from (23) we find

$$b_n^{-1/2}\,\frac{\overline{T}_n - \Delta(\lambda)}{\sigma(\lambda)} \xrightarrow{d} N(0,1). \qquad \square$$

Theorem 2. *Let $K(x)\in H(\tau)$, $\lambda(x)\in C^{(2)}$ and $r(x)$ is piecewise continuous and bounded function. Moreover, if $nb_n^2\to\infty$ and $nb_n^4\to 0$, then*

$$b_n^{-1/2}\,\frac{T_n - \Delta(\lambda)}{\sigma(\lambda)} \xrightarrow{d} N(0,1).$$

Proof. We have

$$T_n = \overline{T}_n + R_n^{(1)} + R_n^{(2)},$$

where

$$R_n^{(1)} = 2nb_n \int\limits_{\Omega_n(\tau)} \big[\lambda_{1n}(x) - \mathbb{E}\,\lambda_{1n}(x)\big]\,\big[\mathbb{E}\,\lambda_{1n}(x) - \lambda_{2n}(x)\lambda(x)\big]\,r(x)\,dx,$$

$$R_n^{(2)} = nb_n \int\limits_{\Omega_n(\tau)} \big[\mathbb{E}\,\lambda_{1n}(x) - \lambda_{2n}(x)\lambda(x)\big]^2 r(x)\,dx.$$

It is easy to see that

$$\mathbb{E}\,\lambda_{1n}(x) = \frac{1}{nb_n}\sum_{i=1}^{n} K\Big(\frac{x-x_i}{b_n}\Big)\lambda(x_i).$$

Hence by virtue of Lemma 2 it follows

$$\mathbb{E}\,\lambda_{1n}(x) = \frac{1}{b_n}\int\limits_0^1 K\Big(\frac{x-u}{b_n}\Big)\lambda(u)\,du + O\Big(\frac{1}{nb_n}\Big) = \int\limits_{\frac{x-1}{b_n}}^{\frac{x}{b_n}} K(t)\lambda(x-b_n t)\,dt + O\Big(\frac{1}{nb_n}\Big). \tag{24}$$

Since $\lambda(x) \in C^{(2)}$ and $\big[\frac{x-1}{b_n}\,, \frac{x}{b_n}\big] \supset [-\tau,\tau]$, for all $x \in \Omega_n(\tau) = [\tau b_n, 1-\tau b_n]$ from (24) we find

$$\mathbb{E}\,\lambda_{1n}(x) = \lambda(x) + O(b_n^2) + O\Big(\frac{1}{nb_n}\Big). \tag{25}$$

Furthermore, by Lemma 2

$$\lambda_{2n}(x) = \frac{1}{b_n}\int\limits_0^1 K\Big(\frac{x-u}{b_n}\Big)\,du + O\Big(\frac{1}{nb_n}\Big) = \int\limits_{-\tau}^{\tau} K(u)\,du + O\Big(\frac{1}{nb_n}\Big) = 1 + O\Big(\frac{1}{nb_n}\Big). \tag{26}$$

From (24), (25) and (26) it follows that

$$b_n^{-1/2} R_n^{(2)} \le c_{14}\Big(nb_n^{9/2} + b_n^{3/2} + \frac{1}{nb_n^{3/2}}\Big) \longrightarrow 0. \tag{27}$$

Let us now estimate $b_n^{-1/2}\mathbb{E}\,|R_n^{(1)}|$. From (25) and (26) we obtain the inequality

$$b_n^{-1/2}\mathbb{E}\,|R_n^{(1)}| \le c_{15} n b_n^{1/2}\Big[b_n^2 + \frac{1}{nb_n}\Big] \int\limits_{\Omega_n(\tau)} \Big(\mathbb{E}\,\big[\lambda_{1n}(x) - \mathbb{E}\,\lambda_{1n}(x)\big]^2\Big)^{1/2} r(x)\,dx. \tag{28}$$

However, according to Lemma 2, we have

$$\mathbb{E}\,\big[\lambda_{1n}(x) - \mathbb{E}\,\lambda_{1n}(x)\big]^2 = \frac{1}{nb_n}\,\lambda(x) \int\limits_{|u|\le\tau} K^2(u)\,du + O\Big(\frac{1}{(nb_n)^2}\Big) + O\Big(\frac{1}{n}\Big).$$

From this and (28) we find that

$$b_n^{-1/2}\mathbb{E}\,|R_n^{(1)}| \le c_{16}\Big(\sqrt{n}\,b_n^2 + \frac{1}{\sqrt{n}\,b_n}\Big) \longrightarrow 0. \tag{29}$$

Finally, the assertion of Theorem 2 directly follows from Theorem 1 and relations (27) and (29). □

The assertion of Theorem 2 allows us to construct the test of asymptotic level α, $0 < \alpha < 1$, for testing hypothesis H_0, according to which $\lambda(x) = \lambda_0(x)$, $x \in \Omega_n(\tau)$. The critical region is defined by the inequality

$$T_n \ge q_n(\alpha), \tag{30}$$

where

$$q_n(\alpha) = \Delta(\lambda_0) + \lambda_\alpha \sqrt{b_n}\,\sigma(\lambda_0),$$

$$\Delta(\lambda_0)=\int_0^1\lambda_0(x)r(x)\,dx\int_{|u|\le\tau}K^2(u)\,du,\quad \sigma^2(\lambda_0)=2\int_0^1\lambda_0^2(x)r^2(x)\,dx\int_{|u|\le 2\tau}K_0^2(u)\,du,$$

and z_α is defined by the equality $\Phi(z_\alpha)=1-\alpha$.

Now let us investigate the asymptotic property of the test (30) (i.e. the behavior of the power function as $n\to\infty$). In first place, we consider the question of whether the test is consistent. The following assertion is true.

Theorem 3. *Let all the conditions of Theorem 2 be fulfilled. Then as* $n\to\infty$

$$\Pi_n(\lambda)=\mathbb{P}_{H_1}\big\{T_n\ge q_n(\alpha)\big\}\longrightarrow 1,$$

i.e. the test defined in (30) *is consistent against any alternative* $H_1:\lambda(x)\ne\lambda_0(x)$, $0\le x\le 1$.

Proof. Denote

$$m_n(\lambda)=b_n^{-1/2}\bigg(nb_n\int_{\Omega_n(\tau)}\big[\widehat{\lambda}_n(x)-\lambda(x)\big]^2\lambda_{2n}^2(x)r(x)\,dx-\Delta(\lambda)\bigg)\sigma^{-1}(\lambda).$$

It is not difficult to show that

$$\Pi_n(\lambda)=\mathbb{P}_{H_1}\bigg\{m_n(\lambda)\ge -nb_n^{1/2}\bigg(\int_0^1(\lambda(x)-\lambda_0(x))^2r(x)\,dx+o_{\mathbb{P}}(1)\bigg)\bigg\}.$$

Since $m_n(\lambda)$ has asymptotically normal distribution with parameters $(0,1)$ for the hypothesis H_1 and $nb_n^{1/2}\to\infty$, then $\Pi_n(\lambda)\to 1$ as $n\to\infty$. □

Thus for any fixed alternative the power of the test based on T_n tends to 1. However, if with a change of n the alternative changes converging to the basic Hypothesis H_0, then the power of the test will no longer necessarily converge to 1. Let us consider, for example, the sequence of Pitment-type alternatives that are close to hypothesis H_0:

$$H_1:\ \ \lambda_1^{(n)}(x)=\lambda_0(x)+\gamma_n\varphi(x)+o(\gamma_n),\ \ \gamma_n\to 0. \tag{31}$$

Theorem 4. *Let* $\lambda_0(x),\varphi(x)\in C^{(2)}$, $K(x)\in H(\tau)$ *and* $r(x)$ *is piecewise continuous and bounded function. If* $b_n=n^{-\delta}$, $\gamma_n=n^{-1/2+\delta/4}$, $1/4<\delta<1/2$, *then statistic* $b_n^{-1/2}(T_n-\Delta(\lambda_0))\sigma^{-1}(\lambda_0)$ *for alternative* H_1 *distributed in limit normally with parameters*

$$\bigg(\frac{1}{\sigma(\lambda_0)}\int_0^1\varphi^2(x)r(x)\,dx,1\bigg),$$

i.e. the limiting power of the test is equal to

$$1-\Phi\bigg(\lambda_\alpha-\frac{1}{\sigma(\lambda_0)}\int_0^1\varphi^2(u)r(u)\,du\bigg).$$

Proof. Let us write T_n in the form

$$\begin{aligned}T_n&=nb_n\int_{\Omega_n(\tau)}\big(\widehat{\lambda}_n(x)-\lambda_1^{(n)}(x)\big)^2\lambda_{2n}^2(x)r(x)\,dx\\&\quad+nb_n\int_{\Omega_n(\tau)}\big(\lambda_1^{(n)}(x)-\lambda_0(x)\big)^2\lambda_{2n}^2(x)r(x)\,dx\\&\quad+2nb_n\int_{\Omega_n(\tau)}\big(\widehat{\lambda}_n(x)-\lambda_1^{(n)}(x)\big)\big(\lambda_1^{(n)}(x)-\lambda_0(x)\big)^2\lambda_{2n}^2(x)r(x)\,dx\\&=T_n^*+A_1(n)+A_2(n).\end{aligned}$$

Since $\lambda_{2n}(x) = 1 + O\big(\frac{1}{nb_n}\big)$ uniformly with respect $x \in \Omega_n(\tau)$, we have

$$b_n^{-1/2} A_1(n) = \int_0^1 \varphi^2(u)\,du + o(1). \tag{32}$$

Denote

$$D_n = \int\limits_{\Omega_n(\tau)} \big(\lambda_{1n}(x) - \mathbb{E}_1 \lambda_{1n}(x)\big)\varphi(x)\lambda_{2n}(x) r(x)\,dx,$$

where $\mathbb{E}_1(\,\cdot\,)$ is the mathematical expectation for hypothesis H_1. Then

$$b_n^{-1/2} A_2(n) = nb_n^{1/2}\gamma_n D_n + nb_n^{1/2}\gamma_n\Big(b_n^2 + O\Big(\frac{1}{nb_n}\Big)\Big) = nb_n^{1/2}\gamma_n D_n + O(nb_n^{5/2}\gamma_n) + O\Big(\frac{\gamma_n}{\sqrt{b_n}}\Big). \tag{33}$$

It is easy to show

$$nb_n^{1/2}\gamma_n \mathbb{E}_1|D_n| \le c_{17} n^{-\delta/4}.$$

Indeed,

$$nb_n^{1/2}\gamma_n D_n \xrightarrow{\mathbb{P}} 0. \tag{34}$$

Furthermore, a random variable

$$b_n^{-1/2}\big(T_n^* - \Delta(\lambda_1^{(n)})\big)\sigma^{-1}(\lambda_1^{(n)})$$

is asymptotically normal with mean 0 and variation 1. From this and (32), (33) and (34) the proof of Theorem 4 follows. □

Remark 1. It should be emphasized that the estimator $\widehat{\lambda}_n(x)$ behaves worse near the boundary of the interval $[0,1]$ than in the interior interval $[\tau b_n, 1-\tau b_n]$ (see [5]). We therefore consider the integral square deviation on $\Omega_n(\tau)$ in order to avoid difficulties connected with this boundary effect; however it can be shown that in the conditions of Theorems 1 and 2 the results obtained above are also valid for the modified estimator (see [5], [11]) of the function $\lambda(x)$.

Remark 2. The idea of proof of Theorem 1 is analogous of proof Theorem 1 from paper [1].

Remark 3. Let x_i be the division points of the interval $[0,1]$ chosen so that relation $H(x_j) = \frac{2j-1}{2n}$, $j = \overline{1,n}$, where $H(x) = \int_0^x h(u)\,du$, $h(u)$ is some known continuous distribution density on $[0,1]$. Then, arguing analogously to the above, one can obtain a generalization of the results of this paper.

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Department of Mathematics, Faculty of Exact and Natural Sciences, Ivane Javakhishvili Tbilisi State University, 13 University Str., Tbilisi 0186, Georgia

Email address: petre.babilua@tsu.ge