EXTENSIONS OF CLASSICAL INEQUALITIES VIA E-CONVEXITY AND FRACTIONAL INTEGRALS

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Abstract. In this paper, we derive novel integral inequalities similar to Hadamard's inequality for e-convex functions using fundamental analytical methods. Our findings contribute to the growing body of work on convex analysis and provide a foundation for further generalizations in functional inequalities.

1. Introduction

Hadamard's inequality [2, 12] is a fundamental result in mathematical analysis that establishes bounds on the integral of a convex function over a closed interval. Originally formulated by Jacques Hadamard in 1893, this inequality has significant applications in optimization [17], economics [18], and functional analysis [5]. For a convex function $f : [l, m] \to \mathbb{R}$, Hadamard's inequality is given by

$$\phi\left(\frac{l+m}{2}\right) \leq \frac{1}{m-l}\int_l^m \phi(x)\,dx \leq \frac{\phi(l)+\phi(m)}{2}. \tag{1.1}$$

This result provides a crucial relationship between the function's value at the midpoint, its integral, and the arithmetic mean of its values at the endpoints. It is widely utilized in estimating convex functions and has led to numerous extensions and generalizations.

Hadamard's inequality has been extensively analyzed and expanded within different mathematical frameworks. In 2003, Dragomir and Pearce [9] discussed various generalizations of Hadamard's inequality for convex functions, particularly under higher-order convexity conditions. Later, in 2006, Niculescu and Persson [20] examined integral inequalities within convex analysis, providing refinements to the classical Hadamard inequality.

Recent studies have explored connections between Jensen's inequality and Hadamard's theorem [7], reinforcing the deep relationship between convex analysis and integral inequalities.

Convex functions [1, 5, 22] play a pivotal role in mathematical analysis, especially in optimization and economics. A function f is convex on an interval I if for any $u, v \in I$ and $\alpha \in [0, 1]$, the following inequality holds:

$$\phi(\alpha u + (1-\alpha)v) \leq \alpha\phi(u) + (1-\alpha)\phi(v). \tag{1.2}$$

This convexity property ensures that the secant line lies above the function's graph, leading to fundamental inequalities such as Jensen's inequality, the Hermite-Hadamard inequality, and convex combination inequalities.

Extensions of Hadamard's inequality have been formulated for various types of convexity, including strongly convex [23], quasi-convex [8], and harmonically convex functions [13]. These

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developments highlight the continued significance and flexibility of the Hermite-Hadamard inequality [6, 10, 14, 15, 21, 24, 26] in modern mathematical research, showcasing its extensive influence across multiple fields. The incorporation of fractional and quantum calculus, along with its applications in stochastic processes and optimization, underscores its effectiveness in solving intricate mathematical and practical challenges. Additionally, recent investigations into weighted, integral, and multidimensional extensions have expanded its theoretical scope, reinforcing its importance in disciplines such as financial modeling, machine learning, and mathematical physics. As novel convexity concepts emerge and computational methods advance, the Hermite-Hadamard inequality remains a vital link between classical mathematical analysis and contemporary applied mathematics.

2. e-convex functions and some integral inequlities

We begin this section by recalling the concept of an e-convex function, which generalizes classical convexity using an exponential weighting scheme.

Definition 2.1 ([16])**.** Let $\phi : \mathcal{I} \to \mathbb{R}$ be a non-negative function. The function ϕ is said to be e-convex if, for any $u, v \in \mathcal{I}$ and for all $\alpha \in [0, 1]$, the following inequality holds:

$$\phi(\alpha u + (1-\alpha)v) \leq e^{\alpha}\phi(u) + e^{1-\alpha}\phi(v).$$

This definition extends the classical notion of convexity by incorporating an exponential factor, which influences the way the function interpolates between two points. The exponential weights e^{α} and $e^{1-\alpha}$ provide a distinctive scaling property that may lead to unique applications in optimization, functional analysis, and mathematical economics. It is also worth noting that the concept of e-convexity explored in our paper is more general than those considered in [3,25].

Example 2.2. ([16]) A real-valued function $\phi : [0, 1] \to \mathbb{R}$, defined by $\phi(x) = x^{1/p}$, $p \in \mathbb{N} \setminus \{1\}$ is e-convex, but not convex.

In this work, we extend and generalize the integral inequalities established in [21, 26] to the broader class of e-convex functions. We derive new inequalities that serve as analogous to Hadamard's inequality within this generalized framework. We begin by giving the following lemma:

Lemma 2.3. *The following statements are equivalent for a mapping $\phi : [l, m] \to \mathbb{R}$:*

(1) *ϕ is e-convex on $[l, m]$.*
(2) *For all $x, y \in [l, m]$, the mapping $\psi : [0, 1] \to \mathbb{R}$ defined by $\psi(t) = \phi(tx + (1-t)y)$ is e-convex on $[0, 1]$.*

Proof. (1) $\Rightarrow$ (2)

Assume that ϕ is e-convex on $[l, m]$ and let $x, y \in [l, m]$. We shall show that the mapping $\psi : [0, 1] \to \mathbb{R}$ is e-convex on $[0, 1]$. Then, for $\alpha, p, q \in [0, 1]$, we have

$$\begin{aligned}
\psi(\alpha p + (1-\alpha)q) &= \phi((\alpha p + (1-\alpha)q)x + (1 - \alpha p - (1-\alpha)q)y) \\
&= \phi(\alpha p x + (1-\alpha)q x + y - \alpha p y - q y + \alpha q y + \alpha y - \alpha y) \\
&= \phi\left(\alpha(px + (1-p)y) + (1-\alpha)(qx + (1-q)y)\right) \\
&\leq e^{\alpha}\phi(px + (1-p)y) + e^{1-\alpha}\phi(qx + (1-q)y) \\
&= e^{\alpha}\psi(p) + e^{1-\alpha}\psi(q).
\end{aligned}$$

Thus, we conclude that

$$\psi(\alpha p + (1-\alpha)q) \leq e^{\alpha}\psi(p) + e^{1-\alpha}\psi(q),$$

which implies that ψ is e-convex on $[0,1]$.

(2) $\Rightarrow$ (1)
We show that $\phi : [l,m] \to \mathbb{R}$ is e-convex.

Let $x, y \in [l,m]$. Then there exist $p, q \in [0,1]$ such that

$$x = pl + (1-p)m,$$
$$y = ql + (1-q)m.$$

Define the function $\psi : [0,1] \to \mathbb{R}$ by

$$\psi(\lambda) = \phi(\lambda l + (1-\lambda)m).$$

Then, we have

$$\psi(p) = \phi(x), \quad \psi(q) = \phi(y).$$

Since ψ is e-convex, it follows that

$$\begin{aligned} \psi(\alpha p + (1-\alpha)q) &\leq e^{\alpha}\psi(p) + e^{1-\alpha}\psi(q) \\ &= e^{\alpha}\phi(x) + e^{1-\alpha}\phi(y). \end{aligned} \tag{2.1}$$

Also, we compute

$$\begin{aligned} \psi(\alpha p + (1-\alpha)q) &= \phi\left((\alpha p + (1-\alpha)q)l + (1 - \alpha p - (1-\alpha)q)m\right) \\ &= \phi\left(\alpha pl + ql - \alpha ql + m - \alpha pm - qm + \alpha qm + \alpha m - \alpha m\right) \\ &= \phi\left(\alpha(pl + (1-p)m) + (1-\alpha)(lq + (1-q)m)\right) \\ &= \phi\left(\alpha x + (1-\alpha)y\right). \end{aligned} \tag{2.2}$$

Using (2.1) and (2.2), we have

$$\phi(\alpha x + (1-\alpha)y) \leq e^{\alpha}\phi(x) + e^{1-\alpha}\phi(y), \quad \forall\, x, y \in [l,m], \quad \alpha \in [0,1].$$

This shows that ϕ is e-convex on $[l,m]$. □

Theorem 2.4. *Let ϕ and ψ be real-valued, integrable, and e-convex functions on $[l,m]$. Then*

$$\frac{1}{m-l}\int_l^m \phi(x)\,\psi(x)dx \leq \left(\frac{e^2-1}{2}\right) U(l,m) + e\,V(l,m),$$

where

$$U(l,m) = \phi(l)\,\psi(l) + \phi(m)\,\psi(m), \quad V(l,m) = \phi(l)\,\psi(m) + \phi(m)\,\psi(l). \tag{2.3}$$

Proof. Since ϕ and ψ are e-convex on $[l,m]$, for $t \in [0,1]$, we have

$$\phi(tl + (1-t)m) \leq e^t\phi(l) + e^{1-t}\phi(m),$$
$$\psi(tl + (1-t)m) \leq e^t\psi(l) + e^{1-t}\psi(m).$$

Thus,

$$\begin{aligned} &\phi(tl + (1-t)m)\,\psi(tl + (1-t)m) \\ &\leq (e^t)^2\phi(l)\,\psi(l) + (e^{1-t})^2\phi(m)\,\psi(m) + e^t e^{1-t}\left[\phi(l)\,\psi(m) + \psi(l)\,\phi(m)\right] \\ &= (e^t)^2\phi(l)\,\psi(l) + (e^{1-t})^2\phi(m)\,\psi(m) + e\,V(l,m). \end{aligned}$$

Now, integrating both sides w.r.t. t, we get

$$\begin{aligned}
&\int_0^1 \phi(tl + (1-t)m)\, \psi(tl + (1-t)m)dt \\
&\leq \int_0^1 (e^t)^2 \phi(l)\, \psi(l)dt + \int_0^1 (e^{1-t})^2 \phi(m)\, \psi(m)dt + e\, V(l,m) \\
&\leq \phi(l)\, \psi(l) \int_0^1 (e^t)^2 dt + \phi(m), \psi(m) \int_0^1 (e^{1-t})^2 dt + e\, V(l,m) \\
&= \phi(l)\, \psi(l) \frac{(e^2-1)}{2} + \phi(m)\, \psi(m) \frac{(e^2-1)}{2} + e\, V(l,m) \\
&= \left(\frac{e^2-1}{2}\right) (\phi(l)\, \psi(l) + \phi(m)\, \psi(m)) + e\, V(l,m) \\
&= \left(\frac{e^2-1}{2}\right) U(l,m) + e\, V(l,m).
\end{aligned}$$

Putting $tl + (1-t)m = x$, then $(l-m)dt = dx$, we get

$$\frac{1}{m-l} \int_l^m \phi(x)\, \psi(x)dx \leq \left(\frac{e^2-1}{2}\right) U(l,m) + e\, V(l,m).$$

□

Theorem 2.5. *Let ϕ and ψ be real-valued, integrable, and e-convex functions on $[l,m]$. Then*

$$\phi\left(\frac{l+m}{2}\right) \psi\left(\frac{l+m}{2}\right) \leq \frac{2e}{m-l} \int_l^m \phi(x)\, \psi(x)dx + 2e^2\, U(l,m) + (e^2-1)\, V(l,m),$$

where $U(l,m)$ and $V(l,m)$ are as defined in (2.3).

Proof. For $\vartheta \in [0,1]$, we have

$$\begin{aligned}
&\phi\left(\frac{l+m}{2}\right) \psi\left(\frac{l+m}{2}\right) \\
&= \phi\left(\frac{\vartheta l + (1-\vartheta)l + \vartheta m + (1-\vartheta)m}{2}\right) \psi\left(\frac{\vartheta l + (1-\vartheta)l + \vartheta m + (1-\vartheta)m}{2}\right) \\
&\leq \left[e^{\frac{1}{2}} \phi\left(\vartheta l + (1-\vartheta)m\right) + e^{\frac{1}{2}}\, \phi\left((1-\vartheta)l + \vartheta m\right)\right] \\
&\quad \times \left[e^{\frac{1}{2}}\, \psi\left(\vartheta l + (1-\vartheta)m\right) + e^{\frac{1}{2}}\, \psi\left((1-\vartheta)l + \vartheta m\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq e\,[\phi(\vartheta l+(1-\vartheta)m)\,\psi(\vartheta l+(1-\vartheta)m)+\phi((1-\vartheta)l+\vartheta m)\,\psi((1-\vartheta)l+\vartheta m)]\\
&\quad+e\,\Big[\left(e^{\vartheta}\,\phi(l)+e^{1-\vartheta}\,\phi(m)\right)\,\left(e^{1-\vartheta}\,\psi(l)+e^{\vartheta}\,\psi(m)\right)\\
&\quad+\left(e^{1-\vartheta}\,\phi(l)+e^{\vartheta}\,\phi(m)\right)\,\left(e^{\vartheta}\,\psi(l)+e^{1-\vartheta}\,\psi(m)\right)\Big]\\
&=e\,[\phi(\vartheta l+(1-\vartheta)m)\,\psi(\vartheta l+(1-\vartheta)m)+\phi((1-\vartheta)l+\vartheta m)\,\psi((1-\vartheta)l+\vartheta m)]\\
&\quad+e\,\left[2e^{\vartheta}e^{1-\vartheta}\phi(l)\,\psi(l)+2e^{\vartheta}\,e^{1-\vartheta}\,\phi(m)\,\psi(m)\right]\\
&\quad+\left((e^{1-\vartheta})^2+(e^{\vartheta})^2\right)^2\,\left(\phi(l)\,\psi(m)+\phi(m)\,\psi(l)\right)\\
&=e\,[\phi(\vartheta l+(1-\vartheta)m)\,\psi(\vartheta l+(1-\vartheta)m)+\phi((1-\vartheta)l+\vartheta m)\,\psi((1-\vartheta)l+\vartheta m)]\\
&\quad+e\,\big[2e\,(\phi(l)\,\psi(l)+\phi(m)\,\psi(m))\\
&\quad+(e^{\vartheta})^2+(e^{1-\vartheta})^2\,\left(\phi(l)\,\psi(m)+\phi(m)\,\psi(l)\right)\big]\,.
\end{aligned}
$$

This gives

$$
\begin{aligned}
&\phi\left(\frac{l+m}{2}\right)\,\psi\left(\frac{l+m}{2}\right)\\
&\leq e\,[\phi(\vartheta l+(1-\vartheta)m)\,\psi(\vartheta l+(1-\vartheta)m)+\phi((1-\vartheta)l+\vartheta m)\,\psi((1-\vartheta)l+\vartheta m)]\\
&\quad+2e^2U(l,m)+\left(e^{2\vartheta}+e^{2(1-\vartheta)}\right)V(l,m).
\end{aligned}
$$

Now, integrating both sides w.r.t. ϑ, we get

$$
\begin{aligned}
&\phi\left(\frac{l+m}{2}\right)\,\psi\left(\frac{l+m}{2}\right)\\
&\leq e\int_0^1[\phi(\vartheta l+(1-\vartheta)m)\,\psi(\vartheta l+(1-\vartheta)m)+\phi((1-\vartheta)l+\vartheta m)\,\psi((1-\vartheta)l+\vartheta m)]\;d\vartheta\\
&\quad+2e^2U(l,m)+V(l,m)\int_0^1\left(e^{2\vartheta}+e^{2(1-\vartheta)}\right)\,d\vartheta\\
&=e\int_0^1[\phi(\vartheta l+(1-\vartheta)m)\,\psi(\vartheta l+(1-\vartheta)m)+\phi((1-\vartheta)l+\vartheta m)\,\psi((1-\vartheta)l+\vartheta m)]\;d\vartheta\\
&\quad+2e^2\,U(l,m)+(e^2-1)\,V(l,m)\\
&=2e\int_0^1\phi(\vartheta l+(1-\vartheta)m)\,\psi(\vartheta l+(1-\vartheta)m)d\vartheta+2e^2\,U(l,m)+(e^2-1)\,V(l,m).
\end{aligned}
$$

Now putting $\vartheta l+(1-\vartheta)m=x$, then we have

$$
\begin{aligned}
&\phi\left(\frac{l+m}{2}\right)\,\psi\left(\frac{l+m}{2}\right)\\
&\leq\frac{2e}{m-l}\int_l^m\phi(x)\,\psi(x)\,dx+2e^2\,U(l,m)+(e^2-1)\,V(l,m).
\end{aligned}
$$

□

Theorem 2.6. *Let ϕ and ψ be real-valued, integrable and e-convex functions on [l, m]. Then*

$$\frac{1}{(m-l)^2}\int_l^m\int_l^m\int_0^1 \phi(tx+(1-t)y)\,\psi(tx+(1-t)y)\,dt\,dy\,dx$$
$$\leq \frac{e^2-1}{(m-l)}\int_l^m \phi(x)\,\psi(x)\,dx + 2e\,(e-1)^2\,[U(l,m)+V(l,m)]\,,$$

where $U(l,m)$ and $V(l,m)$ are as defined in (2.3).

Proof. Since ϕ and ψ are e-convex on $[l,m]$, then for $x, y \in [l,m]$ and $t \in [0,1]$, we have

$$\phi(tx+(1-t)y) \leq e^t\,\phi(x) + e^{1-t}\,\phi(y)$$

and

$$\psi(tx+(1-t)y) \leq e^t\,\psi(x) + e^{1-t}\,\psi(y).$$

We have

$$\phi(tx+(1-t)y)\,\psi(tx+(1-t)y)$$
$$\leq e^{2t}\,\phi(x)\,\psi(x) + e^{2(1-t)}\,\phi(y)\,\psi(y) + e\,[\phi(x)\,\psi(y)+\phi(y)\,\psi(x)]\,.$$

This gives

$$\int_0^1 \phi(tx+(1-t)y)\,\psi(tx+(1-t)y)\,dt$$
$$\leq \left(\frac{e^2-1}{2}\right)[\phi(x)\,\psi(x)+\phi(y)\,\psi(y)] + e\,[\phi(x)\,\psi(y)+\phi(y)\,\psi(x)]\,.$$

Integrating both sides over $[l,m]\times[l,m]$ and using Hermite-Hadamard type integral inequality for e-convex functions (Theorem 3.7, [16]), we get

$$\int_l^m\int_l^m\int_0^1 \phi(tx+(1-t)y)\,\psi(tx+(1-t)y)\,dt\,dy\,dx$$
$$\leq \frac{e^2-1}{2}(m-l)\,\left[\int_l^m \phi(x)\,\psi(x)\,dx + \int_l^m \phi(y)\,\psi(y)\,dy\right]$$
$$+\,e\left[\int_l^m \phi(x)\,dx\int_l^m \psi(y)\,dy + \int_l^m \phi(y)\,dy\int_l^m \psi(x)\,dx\right]$$
$$\leq (e^2-1)\,(m-l)\int_l^m \phi(x)\,\psi(x)\,dx$$
$$+\,2e\,(m-l)^2\,(e-1)^2\,\left[(\phi(l)+\phi(m))\,(\psi(l)+\psi(m))\right]$$
$$= (e^2-1)\,(m-l)\,\int_l^m \phi(x)\,\psi(x)\,dx + 2e\,(m-l)^2\,(e-1)^2\,\left[U(l,m)+V(l,m)\right].$$

Thus

$$\frac{1}{(m-l)^2}\int_l^m\int_l^m\int_0^1 \phi(tx+(1-t)y)\,\psi(tx+(1-t)y)\,dt\,dy\,dx$$
$$\leq \frac{e^2-1}{(m-l)}\int_l^m \phi(x)\,\psi(x)\,dx + 2e\,(e-1)^2\,\left[U(l,m)+V(l,m)\right].$$

□

Theorem 2.7. *Let ϕ and ψ be real-valued, integrable and e-convex functions on [l, m]. Then*

$$\frac{1}{m-l}\int_l^m \int_0^1 \phi\left(tx+(1-t)\frac{l+m}{2}\right)\psi\left(tx+(1-t)\frac{l+m}{2}\right)dt\,dx$$
$$\leq \frac{e^2-1}{2(m-l)}\int_l^m \phi(x)\,\psi(x)\,dx + e(e-1)\left(\frac{e+1+4\sqrt{e}}{2}\right)\left(U(l,m)+V(l,m)\right),$$

where $U(l,m)$ and $V(l,m)$ are as defined in (2.3)*.*

Proof. Since ϕ and ψ are e-convex on $[l,m]$, it follows that for $t \in [0,1]$, we have

$$\phi\left(tx+(1-t)\frac{l+m}{2}\right) \leq e^t\phi(x) + e^{1-t}\phi\left(\frac{l+m}{2}\right),$$
$$\psi\left(tx+(1-t)\frac{l+m}{2}\right) \leq e^t\psi(x) + e^{1-t}\psi\left(\frac{l+m}{2}\right).$$

Note that

$$\phi\left(tx+(1-t)\frac{l+m}{2}\right)\psi\left(tx+(1-t)\frac{l+m}{2}\right)$$
$$\leq e^{2t}\,\phi(x)\,\psi(x) + e^{2(1-t)}\,\phi\left(\frac{l+m}{2}\right)\psi\left(\frac{l+m}{2}\right)$$
$$+\,e\left[\phi(x)\,\psi\left(\frac{l+m}{2}\right)+\phi\left(\frac{l+m}{2}\right)\psi(x)\right].$$

This gives

$$\int_0^1 \phi\left(tx+(1-t)\frac{l+m}{2}\right)\psi\left(tx+(1-t)\frac{l+m}{2}\right)dt$$
$$\leq \left(\frac{e^2-1}{2}\right)\left[\phi(x)\,\psi(x)+\phi\left(\frac{l+m}{2}\right)\psi\left(\frac{l+m}{2}\right)\right]$$
$$+\,e\left[\phi(x)\,\psi\left(\frac{l+m}{2}\right)+\phi\left(\frac{l+m}{2}\right)\psi(x)\right].$$

This further gives

$$
\begin{aligned}
&\int_l^m \int_0^1 \phi\left(tx+(1-t)\frac{l+m}{2}\right)\,\psi\left(tx+(1-t)\frac{l+m}{2}\right)dt\,dx \\
&\le \left(\frac{e^2-1}{2}\right)\int_l^m \phi(x)\,\psi(x)\,dx + \left(\frac{e^2-1}{2}\right)(m-l)\,\phi\left(\frac{l+m}{2}\right)\,\psi\left(\frac{l+m}{2}\right) \\
&\quad + e\left[\int_l^m \phi(x)\,\psi\left(\frac{l+m}{2}\right)\,dx + \int_l^m \phi\left(\frac{l+m}{2}\right)\,\psi(x)\,dx\right] \\
&\le \left(\frac{e^2-1}{2}\right)\int_l^m \phi(x)\,\psi(x)\,dx + \left(\frac{e^2-1}{2}\right)(m-l)\,e\,\left[(\phi(l)+\phi(m))\,(\psi(l)+\psi(m))\right] \\
&\quad + e\,\left[\sqrt{e}\,(\psi(l)+\psi(m))\,(m-l)\,(e-1)\,(\phi(l)+\phi(m))\right. \\
&\quad \left.+\sqrt{e}\,(\phi(l)+\phi(m))\,(m-l)\,(e-1)\,(\psi(l)+\psi(m))\right] \\
&\le \frac{(e^2-1)}{2}\int_l^m \phi(x)\,\psi(x)\,dx + \frac{(e^2-1)e}{2}(m-l)\,(U(l,m)+V(l,m)) \\
&\quad + 2e^{3/2}\,(m-l)\,(e-1)\,\left(U(l,m)+V(l,m)\right) \\
&= \frac{(e^2-1)}{2}\int_l^m \phi(x)\,\psi(x)\,dx + (m-l)e(e-1)\,\left(U(l,m)+V(l,m)\right)\,\left(\frac{e+1}{2}+2\sqrt{e}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{1}{m-l}\int_l^m \int_0^1 \phi\left(tx+(1-t)\frac{l+m}{2}\right)\,\psi\left(tx+(1-t)\frac{l+m}{2}\right)dt\,dx \\
&\le \frac{e^2-1}{2(m-l)}\int_l^m \phi(x)\,\psi(x)\,dx + e(e-1)\,\left(\frac{e+1+4\sqrt{e}}{2}\right)(U(l,m)+V(l,m))\,.
\end{aligned}
$$

□

Theorem 2.8. *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be an e-convex function on $[l, m]$. Then*

$$
\frac{1}{m-l}\int_l^m \phi(x)\,\phi(l+m-x)\,dx \le Q(\phi(l),\phi(m)) + (e^2-1) + 2e\,\left(M(\phi(l),\phi(m))\right)^2,
$$

where $Q(p,q) = \frac{p^2+q^2}{2}$, $\quad M(p,q) = \sqrt{pq}$.

Proof. Since ϕ is e-convex on $[l,m]$, it follows that for $t\in[0,1]$, we have

$$
\begin{aligned}
&\frac{1}{m-l}\int_l^m \phi(x)\,\phi(l+m-x)\,dx\\
&=\int_0^1 \phi(tl+(1-t)m)\,\phi((1-t)l+tm)\,dt\\
&\le \frac{1}{2}\int_0^1 \left[(\phi(tl+(1-t)m))^2+(\phi((1-t)l+tm))^2\right]\,dt\\
&\le \frac{1}{2}\int_0^1 \left[\left(e^t\,\phi(l)+e^{1-t}\,\phi(m)\right)^2+\left(e^{1-t}\,\phi(l)+e^t\,\phi(m)\right)^2\right]dt\\
&=\frac{1}{2}\int_0^1 \left[e^{2t}\,((\phi(l))^2+(\phi(m))^2)+e^{2(1-t)}\,((\phi(l))^2+(\phi(m))^2)\right.\\
&\qquad\left.+\,4e\,\phi(l)\,\phi(m)\right]dt\\
&=\frac{1}{2}\int_0^1 \left(e^{2t}+e^{2(1-t)}\right)\left((\phi(l))^2+(\phi(m))^2\right)\,dt+2e\,\phi(l)\,\phi(m)\\
&=\frac{1}{2}\left((\phi(l))^2+(\phi(m))^2\right)\int_0^1 \left(e^{2t}+e^{2(1-t)}\right)\,dt+2e\,\phi(l)\,\phi(m)\\
&=\left((\phi(l))^2+(\phi(m))^2\right)\int_0^1 e^{2t}\,dt+2e\,\phi(l)\,\phi(m)\\
&=\left(\phi(l)^2+\phi(m)^2\right)\cdot\frac{(e^2-1)}{2}+2e\,\phi(l)\,\phi(m)\\
&=\left(\frac{\phi(l)^2+\phi(m)^2}{2}\right)(e^2-1)+2e\,\phi(l)\,\phi(m)\\
&=Q(\phi(l),\phi(m))\,(e^2-1)+2e\,M(\phi(l),\phi(m))^2.
\end{aligned}
$$

□

Recall from [4, 11, 19] that for an integrable function ϕ on the interval $[l,m]$, the Riemann-Liouville fractional integrals $F^{\alpha}_{l+}\phi$ and $F^{\alpha}_{m-}\phi$ of order $\alpha>0$, with $l\ge 0$, are defined as follows:

$$
\begin{aligned}
F^{\alpha}_{l+}\phi(x)&=\frac{1}{\Gamma(\alpha)}\int_l^x (x-t)^{\alpha-1}\,\phi(t)\,dt,\quad x>l,\\
F^{\alpha}_{m-}\phi(x)&=\frac{1}{\Gamma(\alpha)}\int_l^x (t-x)^{\alpha-1}\,\phi(t)\,dt,\quad m>x,
\end{aligned}
$$

where $\Gamma(\alpha)=\int_0^\infty e^{-s}s^{\alpha-1}\,ds$ denotes the Gamma function. It is also understood that $F^0_{l+}\phi(x)=F^0_{m-}\phi(x)=\phi(x)$.

Next, we give Hermite-Hadamard type fractional integral inequalities for e-convex functions.

Theorem 2.9. *Let $\phi:[l,m]\to\mathbb{R}$ be an e-convex, integrable function on $[l,m]$. Then the following inequalities for fractional integrals hold:*

$$
\frac{1}{2\sqrt{e}}\phi\left(\frac{l+m}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(m-l)^{\alpha}}\left(F^{\alpha}_{l^+}f(m)+F^{\alpha}_{m^-}f(l)\right)\le\frac{e^{\alpha}}{2}(\phi(l)+\phi(m))\int_0^1 t^{\alpha}(e^t+e^{1-t})\,dt
$$

for $\alpha>0$.

Proof. Since ϕ is e-convex on $[l,m]$, we have

$$\phi\left(\frac{x+y}{2}\right) \leq \sqrt{e}\,(\phi(x)+\phi(y)) \quad \text{for } x,y \in [l,m].$$

Let $x = tl + (1-t)m$ and $y = (1-t)l + tm$, where $t \in (0,1)$. Then

$$\frac{1}{\sqrt{e}}\phi\left(\frac{l+m}{2}\right) \leq \phi(tl+(1-t)m) + \phi((1-t)l+tm), \quad \forall\, t \in (0,1). \tag{2.4}$$

On multiplying both sides of (2.4) by $t^{\alpha-1}$ and integrating with respect to t, we get

$$\begin{aligned}
\frac{1}{\alpha\sqrt{e}}\phi\left(\frac{l+m}{2}\right) &\leq \int_0^1 t^{\alpha-1}\,\phi(tl+(1-t)m)\,dt + \int_0^1 t^{\alpha-1}\,\phi((1-t)l+tm)\,dt \\
&= \frac{1}{(m-l)^\alpha}\int_l^m (m-l)^{\alpha-1}\,\phi(u)\,du + \frac{1}{(m-l)^\alpha}\int_l^m (v-l)^{\alpha-1}\,\phi(v)\,dv \\
&= \frac{\Gamma(\alpha)}{(m-l)^\alpha}\left(F^\alpha_{l^+}\phi(m) + F^\alpha_{m^-}\phi(l)\right)
\end{aligned}$$

$$\Rightarrow \frac{1}{\sqrt{e}}\phi\left(\frac{l+m}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(m-l)^\alpha}\left(F^\alpha_{l^+}\phi(m) + F^\alpha_{m^-}\phi(l)\right). \tag{2.5}$$

Using e-convexity of ϕ, we have

$$\phi(tl+(1-t)m) \leq e^t\,\phi(l) + e^{1-t}\,\phi(m), \tag{2.6}$$

$$\phi((1-t)l+tm) \leq e^{1-t}\,\phi(l) + e^t\,\phi(m). \tag{2.7}$$

Adding (2.6) and (2.7), we have

$$\phi(tl+(1-t)m) + \phi((1-t)l+tm) \leq (\phi(l)+\phi(m))\left(e^t + e^{1-t}\right). \tag{2.8}$$

Now multiplying both sides of (2.8) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t, we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1}\phi(tl+(1-t)m)\,dt + \int_0^1 t^{\alpha-1}\phi((1-t)l+tm)\,dt \\
&\leq (\phi(l)+\phi(m))\int_0^1 t^{\alpha-1}\left(e^t+e^{1-t}\right)dt \\
\Rightarrow\quad &\frac{1}{(m-l)^\alpha}\int_l^m (m-u)^{\alpha-1}\phi(u)\,du + \frac{1}{(m-l)^\alpha}\int_l^m (v-l)^{\alpha-1}\phi(v)\,dv \\
&\leq (\phi(l)+\phi(m))\int_0^1 t^{\alpha-1}\left(e^t+e^{1-t}\right)dt \\
\Rightarrow\quad &\frac{\Gamma(\alpha)}{(m-l)^\alpha}\left(F^\alpha_{l^+}\,\phi(m) + F^\alpha_{m^-}\,\phi(l)\right) \leq (\phi(l)+\phi(m))\int_0^1 t^{\alpha-1}\left(e^t+e^{1-t}\right)dt \\
\Rightarrow\quad &\frac{\Gamma(\alpha+1)}{2(m-l)^\alpha}\left(F^\alpha_{l^+}\,\phi(m) + F^\alpha_{m^-}\,\phi(l)]\right) \leq \frac{\alpha}{2}(\phi(l)+\phi(m))\int_0^1 t^{\alpha-1}\left(e^t+e^{1-t}\right)dt. \qquad (2.9)
\end{aligned}$$

Therefore, by (2.5) and (2.9), we obtain

$$\frac{1}{2\sqrt{e}}\phi\left(\frac{l+m}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(m-l)^\alpha}\left(F^\alpha_{l^+}\,\phi(m) + F^\alpha_{m^-}\,\phi(l)\right) \leq \frac{\alpha}{2}(\phi(l)+\phi(m))\int_0^1 t^{\alpha-1}\left(e^t+e^{1-t}\right)dt.$$

□

Remark 2.10. If we take $\alpha = 1$, then the above inequality reduces to the Hermite-Hadamard-type inequality for e-convex functions (Theorem 3.7, [16]).

Theorem 2.11. *Let ϕ and ψ be real-valued e-convex functions on $[l, m]$. Then for all $l, m, \alpha > 0$, we have*

$$\frac{F^{\alpha}_{l^+}[\phi(m)\,\psi(m)]}{(m-l)^{\alpha}} \le \frac{e}{\Gamma(\alpha+1)}V(l,m) + \frac{\phi(l)\,\psi(l)}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\,e^{2t}\,dt + \frac{\phi(m)\,\psi(m)}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\,e^{2(1-t)}\,dt.$$

Proof. Since ϕ and ψ are e-convex functions on $[l, m]$, it follows that for $t \in [0, 1]$, we have

$$\phi(tl + (1-t)m) \le e^t\phi(l) + e^{1-t}\phi(m),$$
$$\psi(tl + (1-t)m) \le e^t\psi(l) + e^{1-t}\psi(m).$$

Then

$$\begin{aligned}
&\phi(tl + (1-t)m)\cdot\psi(tl + (1-t)m)\\
&\le \left(e^{2t}\phi(l)\,\psi(l) + e^{2(1-t)}\,\phi(m)\,\psi(m)\right) + e\left(\phi(l)\,\psi(m) + \psi(l)\,\phi(m)\right)\\
&= e^{2t}\,\phi(l)\,\psi(l) + e^{2(1-t)}\,\phi(m)\,\psi(m) + e[V(l,m)].
\end{aligned}$$

Now multiplying both sides of the above inequality by $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and integrating with respect to t, we get

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\,\phi(tl + (1-t)m)\,\psi(tl + (1-t)m)dt\\
&\le \frac{e}{\Gamma(\alpha+1)}V(l,m) + \frac{\phi(l)\,\psi(m)}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\,e^{2t}\,dt + \frac{\phi(m)\,\psi(m)}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\,e^{2(1-t)}\,dt.
\end{aligned}$$

Substituting $u = tl + (1-t)m \Rightarrow t = \frac{m-u}{m-l}$, we obtain

$$\frac{F^{\alpha}_{l^+}[\phi(m)\,\psi(m)]}{(m-l)^{\alpha}} \le \frac{e}{\Gamma(\alpha+1)}V(l,m) + \frac{\phi(l)\,\psi(l)}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\,e^{2t}\,dt + \frac{\phi(m)\,\psi(m)}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\,e^{2(1-t)}\,dt.$$

□

Remark 2.12. If $\alpha = 1$, the above inequality reduces to Theorem 2.4.

Theorem 2.13. *Let ϕ and ψ be real-valued, integrable e-convex functions on $[l, m]$. Then for all $\alpha, l, m > 0$, we have*

$$\phi\left(\frac{l+m}{2}\right)\psi\left(\frac{l+m}{2}\right) \le \frac{e\,\Gamma(\alpha+1)}{(m-l)^{\alpha}}\left(F^{\alpha}_{l^+}(\phi(m)\,\psi(m) + \phi(m)\,\psi(l)) + F^{\alpha}_{m^-}(\phi(l)\,\psi(l) + \phi(l)\,\psi(m))\right).$$

Proof. We have

$$\begin{aligned}
&\phi\left(\frac{l+m}{2}\right)\psi\left(\frac{l+m}{2}\right)\\
&= \phi\left(\frac{tl + (1-t)m}{2} + \frac{(1-t)l + tm}{2}\right) g\left(\frac{tl + (1-t)m}{2} + \frac{(1-t)l + tm}{2}\right)\\
&\le e\,\left(\phi(tl + (1-t)m) + \phi((1-t)l + tm)\right)\cdot\left(\psi(tl + (1-t)m) + \psi((1-t)l + tm)\right)\\
&= e\,\big(\phi(tl + (1-t)m)\,\psi(tl + (1-t)m) + \phi((1-t)l + tm)\,\psi((1-t)l + tm)\\
&\quad + \phi((1-t)l + tm)\,\psi(tl + (1-t)m) + \phi(tl + (1-t)m)\,\psi((1-t)l + tm)\big).
\end{aligned}$$

Now, multiplying both sides of the above inequality by $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and integrating the resulting inequality with respect to t, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\phi\left(\frac{l+m}{2}\right)\psi\left(\frac{l+m}{2}\right)\\
&\leq \frac{e}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\,\phi(tl+(1-t)m)\,\psi(tl+(1-t)m)\,dt\\
&\quad+\frac{e}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\,\phi((1-t)l+tm)\,\psi((1-t)l+tm)\,dt\\
&\quad+\frac{e}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\,\phi((1-t)l+tm)\,\psi(tl+(1-t)m)\,dt\\
&\quad+\frac{e}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\,\phi(tl+(1-t)m)\,\psi((1-t)l+tm)\,dt\\
&=\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_l^m (m-l)^{\alpha-1}\phi(u)\,\psi(u)\,du\\
&\quad+\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_l^m (m-u)^{\alpha-1}\phi(l+m-u)\,\psi(l+m-u)\,du\\
&\quad+\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_0^1 (m-u)^{\alpha-1}\phi(l+m-u)\,\psi(u)\,du\\
&\quad+\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_0^1 (m-u)^{\alpha-1}\phi(u)\,\psi(l+m-u)\,du\\
&=\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_l^m (m-u)^{\alpha}\phi(u)\,\psi(u)\,du\\
&\quad+\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_l^m (v-l)^{\alpha-1}\phi(v)\,\psi(v)\,dv\\
&\quad+\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_l^m (v-l)^{\alpha-1}\phi(v)\,\psi(l+m-v)\,dv\\
&\quad+\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_l^m (m-u)^{\alpha-1}\phi(u)\,\psi(l+m-u)\,du\\
&=\frac{e}{(m-l)^{\alpha}}\Big(F^{\alpha}_{l^+}\phi(m)\,\psi(m)+F^{\alpha}_{m^-}\phi(l)\,\psi(l)+F^{\alpha}_{m^-}\phi(l)\,\psi(m)+F^{\alpha}_{l^+}\phi(m)\,\psi(l)\Big).
\end{aligned}$$

Hence,

$$\phi\left(\frac{l+m}{2}\right)\psi\left(\frac{l+m}{2}\right)\leq\frac{e\,\Gamma(\alpha+1)}{(m-l)^{\alpha}}\left(F^{\alpha}_{l^+}(\phi(m)\,\psi(m)+\phi(m)\,\psi(l))+F^{\alpha}_{m^-}(\phi(l)\,\psi(l)+\phi(l)\,\psi(m))\right).$$

□

Remark 2.14. For $\alpha = 1$, the above inequality reduces to Theorem 2.5.

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