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EXPLICIT SOLUTION OF BOUNDARY VALUE PROBLEMS FOR ELASTIC CIRCLE WITH TRIPLE POROSITY

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Dedicated to professor Merab Svanadze on the occasion of his 70^{th} birthday anniversary

Abstract. Along with theoretical investigations of poroelasticity problems, the development of methods for their solution is of great interest. From the point of view of applications, the actual construction of solutions to problems in explicit form makes it possible to perform a numerical analysis of the problem under study. The main purpose of this work is to present explicit solutions to boundary value problems in the theory of elasticity for solids with triple porosity. Special representations of the general solution of the system of differential equations of linear equilibrium theory of elastic materials are constructed through harmonic, biharmonic, and metaharmonic functions, which allow the original system of equations to be reduced to equations of simple structure and facilitate the solution of the original problems. Using these representations, the boundary value problems are solved for a specific body for an elastic circle with triple porosity. Explicit solutions to the problems are obtained in the form of series. The conditions ensuring absolute and uniform convergence of these series are established.

1. Introduction

Recently, many authors have studied elasticity problems for materials with triple porosity. Such materials include, for example, oil and gas reservoirs, rocks and soils, industrial porous materials such as ceramics and compacted powders, and biomaterials such as a bone.

Mathematical formulations of flow in triple-porosity media were first proposed by Liu [16]. Abdassah and Ershaghi [1] developed a triple-porosity model to describe a borehole. Al Ahmadi and Wattenbarger [2], Liu [16], Liu et al. [17], and Wu et al. [31] presented triple-porosity models for single-phase flow in a fracture-matrix system that includes voids within the rock matrix. Based on the extended Darcy's law, several new triple-porosity models for solids have been presented by Bai and Roegiers [3], Straughan [18].

In [21] Svanadze develops a system of equations for an isotropic linear elastic body that admits a triple-porosity structure. In this case, the body has a normal macroporosity, and the fluid pressure in the pores is associated with it. There is also a smaller scale pore structure to which porosity, known as mesoporosity, is attached. Associated with mesoporosity, it attaches fluid pressure. However, there is another, even smaller porosity structure that leads to microporosity. Straughan [19] generalizes the approach of Svanadze [21] and describes a system of linear partial differential equations for an anisotropic elastic body with an internal triple-porosity structure.

The fundamental solutions of the linear theory of elasticity for materials with triple porosity are constructed and the basic BVPs of this theory are investigated by using the potential method; some basic results of the classical theory of elasticity are generalized (see, for example, [18, 21–25]).

The mathematical problems of the theory of bone poroelasticity are studied in [26]. An extensive review of the results of this theory, as well as references to various relevant works, can be found in [9, 10]. The main results and historical information on the theory of porous media are contained in the books by de Boer [11], Straughan [20], and Svanadze [22].

Along with the generalization and development in various directions of the linear theory of elasticity for materials with triple porosity, much attention is paid to the mathematical study and construction of solutions of boundary value problems for specific areas. The construction of solutions to problems in

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explicit form, which makes it possible to effectively perform a numerical analysis of the problem under study, is relevant. Such issues include, for example, works [4–6, 8, 12–15, 27, 28], in which, using various methods, the explicit solutions to the static boundary value problems of the theory of poroelasticity have been constructed for specific media.

Section 2 presents the fundamental equations of elasticity theory and formulates the fundamental boundary value problems of statics for materials with triple porosity.

In Section 3, a general representation of the solution to the system of equations of elasticity theory is constructed by using metaharmonic, harmonic and biharmonic functions.

In this paper, special representations of the general solution of the system of differential equations of the theory of elastic materials with triple porosity are constructed through harmonic, biharmonic and metaharmonic functions, which allow the original system of equations to be reduced to equations of simple structure and facilitate the solution of the original problems. Using these representations, boundary value problems are solved for a specific body, i.e. for an elastic circle with triple porosity. The solutions to the problems are presented explicitly, in the form of absolutely and uniformly converging series.

2. Basic Equations and Boundary Value Problems

We use Svanadze's model for elastic materials with triple porosity [21]. Let K be an elastic circle of radius R with boundary S and center $O(0,0)$. We formulate the following boundary value problems.

Find in the domain K a regular solution $\mathbf{U}(\mathbf{x}) = (\mathbf{u}(\mathbf{x}), p_1(\mathbf{x}), p_2(\mathbf{x}), p_3(\mathbf{x}))$, where $\mathbf{U}(\mathbf{x}) \in C^1(\overline{K}) \cap C^2(K)$, $\overline{K} = K \cup S$, satisfying the system of equations of the linear theory of equilibrium of elastic materials with triple pores:

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} = \mathrm{grad}(\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3)), \tag{2.1}$$

$$\begin{aligned} (a_1\Delta - b_1)p_1 + a_{12}p_2 + a_{13}p_3 &= 0,\\ a_{21}p_1 + (a_2\Delta - b_2)p_2 + a_{23}p_3 &= 0,\\ a_{31}p_1 + a_{32}p_2 + (a_3\Delta - b_3)p_3 &= 0, \quad \mathbf{x} \in K, \end{aligned} \tag{2.2}$$

and on the boundary S, one of the conditions:

$$\begin{aligned} &(\mathrm{I}) : \mathbf{u}(\mathbf{z}) = \mathbf{f}(\mathbf{z}), \quad p_1(\mathbf{z}) = f_3(\mathbf{z}), \quad p_2(\mathbf{z}) = f_4(\mathbf{z}), \quad p_3(\mathbf{z}) = f_5(\mathbf{z}),\\ &(\mathrm{II}) : \mathbf{R}(\partial_z, n)\mathbf{U}(\mathbf{z}) = \mathbf{f}(\mathbf{z}), \quad \partial_n p_1(\mathbf{z}) = f_3(\mathbf{z}), \quad \partial_n p_2(\mathbf{z}) = f_4(\mathbf{z}), \quad \partial_n p_3(\mathbf{z}) = f_5(\mathbf{z}), \end{aligned} \tag{2.3}$$

where $\mathbf{u} = (u_1, u_2)$ is the displacement vector in a solid, p_j is the average value of liquid pressure in phase j; a_j, b_j, a_{ij} are the known elastic and physical constants, $i, j = 1, 2, 3$; $b_1 = a_{12} + a_{13}$, $b_2 = a_{21} + a_{23}$, $b_3 = a_{31} + a_{32}$; λ, μ are the constants; Δ is the Laplacian operator; β_j are the constants characterizing the body porosity; $\mathbf{z} = (z_1, z_2) \in S$, $\mathbf{n(z)} = (n_1, n_2)$ is the outer normal to S at the point $\mathbf{z}$; $\mathbf{f} = (f_1, f_2)$, f_1, f_2, f_3, f_4 and f_5 are the given functions on S;

$$\mathbf{R}(\partial_{\mathbf{x}}, \mathbf{n})\,\mathbf{U}(\mathbf{x}) = \mathbf{T}(\partial_{\mathbf{x}}, \mathbf{n})\,\mathbf{u}(\mathbf{x}) - \mathbf{n}(\mathbf{x})[\beta_1 p_1(\mathbf{x}) + \beta_2 p_2(\mathbf{x}) + \beta_3 p_3(\mathbf{x})]$$

is the stress vector in the linear theory of elasticity for triple porosity solids,

$$\mathbf{T}\left(\frac{\partial}{\partial\mathbf{x}}, \mathbf{n}\right)\mathbf{u}(\mathbf{x}) = \mu\frac{\partial}{\partial\mathbf{n}}\mathbf{u}(\mathbf{x}) + \lambda\mathbf{n}\,\mathrm{div}\,\mathbf{u}(\mathbf{x}) + \mu\sum_{i=1}^{2} n_i(\mathbf{x})\,\mathrm{grad}\,u_i(\mathbf{x})$$

is the stress vector in the classical theory of elasticity; $\partial_l = \dfrac{\partial}{\partial_l}$, l is an arbitrary variable.

Separately we study the following problems.

1. Find in a circle K a solution of equation (2.1), if on the circumference S the following values are given:

a) the vector $\mathbf{u(x)}$–Problem A_1;

b) the vector $\mathbf{R}(\partial_z, n)\mathbf{U}(\mathbf{z})$–Problem A_2.

2. Find in a circle K the solutions p_1, p_2 and p_3 of the system of equations (2.2), if on the circumference S the following values are given:

a) the functions p_1, p_2 and p_3 –Problem B_1;

b) the derivatives $\partial_n p_1$, $\partial_n p_2$ and $\partial_n p_3$ –Problem B_2.

Thus the above-formulated problems of poroelastostatics can be considered as a union of two Problems: I–(A_1; B_1) and II–(A_2; B_2).

3. Explicit Solution of the Boundary Value Problems

First, we solve Problems B_1 and B_2. The determinant of system (2.2) has the form

$$\det = d(\triangle^3 + e_1\triangle^2 + e_2\triangle + e_3), \tag{3.1}$$

where

$$d = a_1a_2a_3, \quad e_1 = -\frac{1}{d}(b_1a_1a_2a_3 + a_1b_2a_3 + a_1a_2b_3),$$
$$e_2 = \frac{1}{d}(b_1b_2a_3 + b_1b_3a_2 - a_{12}a_{21}a_3 - a_2a_{13}a_3),$$
$$e_3 = \frac{1}{d}(-b_1b_2b_3 + a_{12}a_{21}b_3 + a_{12}a_{31}a_{32} + a_{13}a_{31}b_2).$$

Determinant (3.1) with respect to $\triangle$ is a third-degree polynomial. Let us represent it as a product

$$\det = d(\triangle + \omega_1^2)(\triangle + \omega_2^2)(\triangle + \omega_3^2), \tag{3.2}$$

where $-\omega_1^2, -\omega_2^2, -\omega_3^2$ are the roots of the polynomial. System (2.2) is homogeneous, therefore, according to Cramer's formulas, we write $\det \cdot p_j = 0$, $j = 1, 2, 3$. Taking into account (3.2), for each p_j, we obtain

$$(\triangle + \omega_1^2)(\triangle + \omega_2^2)(\triangle + \omega_3^2)\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = 0. \tag{3.3}$$

Assuming that the parameters ω_1^2, ω_2^2 and ω_3^2 differ from each other, the solutions of equations (3.3) can be represented in the form [29]

$$\begin{gathered} p_1(\mathbf{x}) = c_1\varphi_1(\mathbf{x}) + c_2\varphi_2(\mathbf{x}) + c_3\varphi_3(\mathbf{x}), \\ p_2(\mathbf{x}) = c_4\varphi_1(\mathbf{x}) + c_5\varphi_2(\mathbf{x}) + c_6\varphi_3(\mathbf{x}), \\ p_3(\mathbf{x}) = \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}) + \varphi_3(\mathbf{x}), \end{gathered} \tag{3.4}$$

where φ_1, φ_2 and φ_3 are the desired metaharmonic functions: $(\triangle + \omega_l^2)\varphi_l = 0$, $l = 1, 2, 3$; as well as unknown constants $c_1, c_2, \dots, c_6$ and the parameters of ω_l^2. Using the $\triangle\varphi_l = -\omega_l^2\varphi_l$ formula and representations (2.2) and (3.4), we obtain

$$\begin{gathered} c_l = \frac{a_{12}(a_3\omega_l^2 + b_3) - a_{13}a_{32}}{a_{32}(a_1\omega_l^2 + b_1) + a_{12}a_{31}}, \\ c_{l+3} = \frac{1}{a_2\omega_l^2 + b_2}\left[a_{23} + a_{21}\frac{a_{12}(a_3\omega_l^2 + b_3) - a_{13}a_{32}}{a_{32}(a_1\omega_l^2 + b_1) + a_{12}a_{31}}\right], \quad l = 1, 2, 3. \end{gathered} \tag{3.5}$$

Now, let us find the values of the parameters of ω_l^2. From (3.1), we see that to find the values of parameters ω_l^2, we need to solve a third-degree equation with real coefficients

$$\triangle^3 + e_1\triangle^2 + e_2\triangle + e_3 = 0. \tag{3.6}$$

We use Cardan's method and rewrite (3.6) as

$$y^3 + py + q = 0, \tag{3.7}$$

where

$$\triangle = y - \frac{e_1}{3}, \quad p = -\frac{e_1^2}{3} + e_2, \quad q = \frac{2e_1^2}{27} - \frac{e_1e_2}{3} + e_3. \tag{3.8}$$

We consider the case for $\frac{q^2}{4} + \frac{p^3}{27} > 0$. In this case, equation (3.7) has two conjugate complex roots and one real root:

$$y_1 = -\frac{1}{2}(w_1 + v_1) + i\frac{\sqrt{3}}{2}(w_1 - v_1), \;\; y_2 = -\frac{1}{2}(w_1 + v_1) - i\frac{\sqrt{3}}{2}(w_1 - v_1), \;\; y_3 = w_1 + v_1, \tag{3.9}$$

where

$$w_1 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^2}{27}}, \quad v_1 = \frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^2}{27}}, \tag{3.10}$$

$$\Delta_1 = \frac{3q - 2e_1}{6} + i\sqrt{\frac{3p^2}{4} + \frac{q_2}{9}}, \quad \Delta_2 = \frac{3q - 2e_1}{6} - i\sqrt{\frac{3p^2}{4} + \frac{q_2}{9}}, \quad \Delta_3 = -\Big(q + \frac{e_1}{3}\Big). \tag{3.11}$$

So, $\Delta_k \equiv -\omega_k^2$, $k = 1, 2, 3$.

To find the metaharmonic functions φ_1, φ_2 and φ_3, we use the boundary conditions (2.3) for Problems B_1 and B_2.

Problem $\mathbf{B_1}$. Substituting formulas (3.4) into (2.3)–(I), we obtain the following system of equations with respect to the functions $\varphi_l(\mathbf{z})$:

$$\begin{gathered} c_1\varphi_1(\mathbf{z}) + c_2\varphi_2(\mathbf{z}) + c_3\varphi_3(\mathbf{z}) = f_3(\mathbf{z}), \\ c_4\varphi_1(\mathbf{z}) + c_5\varphi_2(\mathbf{z}) + c_6\varphi_3(\mathbf{z}) = f_4(\mathbf{z}), \\ \varphi_1(\mathbf{z}) + \varphi_2(\mathbf{z}) + \varphi_3(\mathbf{z}) = f_5(\mathbf{z}), \quad \mathbf{z} \in S, \quad l = 1, 2, 3, \end{gathered} \tag{3.12}$$

where c_l, c_{l+3} and $f_l(\mathbf{z})$ are the known quantities. According to Cramer's formulas, we write $d_l(\mathbf{z}) = \dfrac{Q_l}{Q}$, where Q is the determinant of system (3.12), and Q_l is the determinant that differs from Q in that its column l coincides with the right-hand side of system (3.12). The solution of system (3.12) has the form

$$\varphi_l(\mathbf{z}) = d_l(\mathbf{z}), \quad \mathbf{z} \in S, \quad l = 1, 2, 3. \tag{3.13}$$

The metaharmonic function $\varphi_l(\mathbf{x})$ in a circle K can be written as follows [30]:

$$\varphi_l(\mathbf{x}) = J_0(\omega_l r)E_{0l} + \sum_{m=1}^{\infty} J_m(\omega_l r)(E_{ml} \cdot \nu_m(\psi)), \quad \mathbf{x} \in K, \tag{3.14}$$

where $J_m(\omega_l r)$ is the Bessel function, $r^2 = x_1^2 + x_2^2$, $\mathbf{x} = (x_1, x_2) = (r, \psi)$, $\boldsymbol{\nu}_m(\psi) = (\cos m\psi, \sin m\psi)$, $E_{ml} = (L_{ml}, K_{ml})$, $E_{0l} \equiv L_{0l}$. We solve the Dirichlet problem

$$\begin{gathered} (\triangle + \omega_1^2)\varphi_l(\mathbf{x}) = 0, \quad \mathbf{x} \in K, \\ \varphi_l(\mathbf{z}) = d_l(\mathbf{z}), \quad \mathbf{z} \in S, \quad l = 1, 2, 3. \end{gathered} \tag{3.15}$$

Let us expand the function $d_l(\mathbf{z})$ as a Fourier series

$$d_l(\mathbf{z}) = \frac{F_{0l}}{2} + \sum_{m=1}^{\infty} (F_{ml} \cdot \nu_m(\psi)), \tag{3.16}$$

where $F_{ml} = (\alpha_{ml}, \beta_{ml})$ is the known vector,

$$\alpha_{m1} = \frac{1}{\pi}\int_0^{2\pi} d_l(\theta)\cos m\theta d\theta, \quad \beta_{ml} = \frac{1}{\pi}\int_0^{2\pi} d_l(\theta)\sin m\theta d\theta, \quad m = 0, 1, \dots .$$

In (3.14), we pass to the limit when $r \to R$. The result, taking into account (3.16), we substitute into (3.15). For each value of m and l, we get

$$\begin{gathered} L_{0l} = \frac{1}{2\pi J_0(\omega_l R)}\int_0^{2\pi} d_l(\theta)d\theta, \qquad L_{ml} = \frac{1}{\pi J_m(\omega_l R)}\int_0^{2\pi} d_l(\theta)\cos m\theta d\theta, \\ K_{ml} = \frac{1}{\pi J_m(\omega_l R)}\int_0^{2\pi} d_l(\theta)\sin m\theta d\theta, \quad m = 0, 1, \dots, \quad l = 1, 2, 3. \end{gathered} \tag{3.17}$$

The Bessel function has no complex roots. Therefore, for ω_1 and ω_2, from (3.11), $J_m(\omega_l R) \neq 0$, $k = 1,2$. The Bessel function has many real roots. From the asymptotic expansion of the Bessel function

$$J_m(\zeta) \sim \sqrt{\frac{2}{\pi\zeta}} \cos\left(\zeta - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

it is clear that if we assume $\zeta \neq \dfrac{3}{2}(m+1)\pi$, then the function cos differs from zero. Let us assume that $\omega_3 \neq \dfrac{3}{2}(m+1)\pi$, then $J_m(\omega_3 R) \neq 0$.

Using now formulas (3.12), with regard for (3.14), we can find values of functions $p_1(\mathbf{x})$, $p_2(\mathbf{x})$ and $p_3(\mathbf{x})$ for $\mathbf{x} \in$K.

Problem $\mathbf{B_2}$. Substituting formulas (3.4) into (2.3)–(II), we obtain the following system of equations with respect to the functions $\partial_R \varphi_l(\mathbf{z})$:

$$\begin{gathered} c_1\partial_R\varphi_1(\mathbf{z}) + c_2\partial_R\varphi_2(\mathbf{z}) + c_3\partial_R\varphi_3(\mathbf{z}) = f_3(\mathbf{z}), \\ c_4\partial_R\varphi_1(\mathbf{z}) + c_5\partial_R\varphi_2(\mathbf{z}) + c_6\partial_R\varphi_3(\mathbf{z}) = f_4(\mathbf{z}), \\ \partial_R\varphi_1(\mathbf{z}) + \partial_R\varphi_2(\mathbf{z}) + \partial_R\varphi_3(\mathbf{z}) = f_5(\mathbf{z}), \quad \mathbf{z} \in S, \quad l = 1,2,3, \end{gathered} \tag{3.18}$$

where c_l, c_{l+3} and $f_l(\mathbf{z})$ are the known quantities. The solution to system (3.18) has the form

$$\partial_R\varphi_l(\mathbf{z}) = d_l(\mathbf{z}), \tag{3.19}$$

where the value of d_l is defined in formula (3.13).

Now, we define the functions $p_l(\mathbf{x})$ inside the circle K. We solve the Neumann problem

$$\begin{gathered} (\triangle + \omega_1^2)\varphi_l(\mathbf{x}) = 0, \quad \mathbf{x} \in K, \\ \partial_R\varphi_l(\mathbf{z}) = d_l, \quad \mathbf{z} \in S, \quad l = 1,2,3. \end{gathered} \tag{3.20}$$

Considering that $J'_m(\zeta) = \partial_\zeta J_m(\zeta)$, $\partial_\zeta J_m(k\zeta) = kJ'_m(k\zeta)$ and (3.14), we substitute into (3.20). Let us expand the function $d_l(\mathbf{z})$ as a Fourier series (3.16). In (3.14), we pass to the limit as $r \to R$. The result, taking into account (3.16), we substitute into (3.20). For each value of m and l, we get

$$L_{ml} = \frac{\alpha_{ml}}{\omega_l J'_{ml}(\omega_l R)}, \quad K_{ml} = \frac{\beta_{ml}}{\omega_l J'_{ml}(\omega_l R)}, \quad l = 1,2,3, \quad m = 0,1,\dots. \tag{3.21}$$

Substituting (3.21) into (3.14), we obtain the values of the functions $\varphi_l(\mathbf{z})$.

Now, using formulas (3.4), at each point of $\mathbf{x}$, we can determine the values of the functions $p_1(\mathbf{x})$, $p_2(\mathbf{x})$ and $p_3(\mathbf{x})$.

Problem $\mathbf{A_1}$. Taking into account (3.4), we obtain

$$\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 = a\varphi_1 + b\varphi_2 + c\varphi_3, \tag{3.22}$$

where

$$a = \beta_1 c_1 + \beta_2 c_4 + \beta_3, \quad b = \beta_1 c_2 + \beta_2 c_3 + \beta_3, \quad c = \beta_1 c_3 + \beta_2 c_6 + \beta_3. \tag{3.23}$$

Substituting (3.22) into equation (2.1), we get

$$\mu\Delta\mathbf{u} + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \mathrm{grad}(a\varphi_1 + b\varphi_2 + c\varphi_3) = 0. \tag{3.24}$$

A solution of equation (3.24) is sought in the form of a sum

$$\mathbf{u(x)} = \mathbf{v}_0(\mathbf{x}) + \mathbf{v}(\mathbf{x}), \tag{3.25}$$

where $\mathbf{v}_0(\mathbf{x})$ is a particular solution of equation (3.24), and $\mathbf{v}(\mathbf{x})$ is a general solution of the corresponding homogeneous equation (3.24). We look for a solution $\mathbf{v}_0(\mathbf{x})$ in the form

$$\mathbf{v}_0(\mathbf{x}) = \mathrm{grad}\, F(\mathbf{x}).$$

To find the function $F(\mathbf{x})$, we substitute $\mathbf{v}_0(\mathbf{x})$ into (3.24) and take into account the fact that $\varphi_l(\mathbf{z})$ is a metaharmonic function. Finally, for the function $\mathbf{v}_0(\mathbf{x})$, we obtain the following expression:

$$\mathbf{v}_0(\mathbf{x}) = -\frac{1}{\lambda+2\mu}\,\mathrm{grad}\Big(\frac{a}{\omega_1^2}\varphi_1(\mathbf{x}) + \frac{b}{\omega_2^2}\varphi_2(\mathbf{x}) + \frac{c}{\omega_3^2}\varphi_3(\mathbf{x})\Big). \tag{3.26}$$

A solution $\mathbf{v}(\mathbf{x})$ of the homogeneous equation corresponding to (3.24) is sought in the form

$$\mathbf{v}(\mathbf{x}) = \mathrm{grad}[\Phi_1(\mathbf{x}) + \Phi_2(\mathbf{x})] + \mathrm{rot}\,\Phi_3(\mathbf{x}), \tag{3.27}$$

where

$$(\lambda+2\mu)\,\mathrm{grad}\,\Delta\Phi_2(\mathbf{x}) + \mu\,\mathrm{rot}\,\Delta\Phi_3(\mathbf{x}) = 0; \tag{3.28}$$
$$\Delta\Phi_1(\mathbf{x}) = 0, \quad \Delta\Delta\Phi_2(\mathbf{x}) = 0, \quad \Delta\Delta\Phi_3(\mathbf{x}) = 0, \quad \mathrm{rot} = \Big(-\frac{\partial}{\partial_2}, \frac{\partial}{\partial_1}\Big).$$

In view of (3.28), we can repressent the harmonic function Φ_1, biharmonic functions Φ_2 and Φ_3 in the form

$$\Phi_1(\mathbf{x}) = \sum_{m=0}^{\infty}\Big(\frac{r}{R_2}\Big)^m (\mathbf{X}_{m1}\cdot\boldsymbol{\nu}_m(\psi)), \quad \Phi_2(\mathbf{x}) = \sum_{m=0}^{\infty}\Big(\frac{r}{R_2}\Big)^{m+2} (\mathbf{X}_{m2}\cdot\boldsymbol{\nu}_m(\psi)),$$
$$\Phi_3(\mathbf{x}) = -\frac{\lambda+2\mu}{\mu}\sum_{m=0}^{\infty}\Big(\frac{r}{R_2}\Big)^{m+2} (\mathbf{X}_{m2}\cdot s_m(\psi)), \tag{3.29}$$

where $\mathbf{X}_{mk} = (X'_{mk}, X''_{mk})$ are the unknown two-component vectors, $k=1,2$; $\boldsymbol{\nu}_m = (\cos m\psi, \sin m\psi)$, $s_m(\psi) = (-\sin m\psi, \cos m\psi)$, $\mathbf{x} = (r,\psi)$.

Using (3.25), the boundary condition (2.3) for the $\mathbf{u(x)}$ will take the form $\mathbf{v}_0(\mathbf{z}) + \mathbf{v}(\mathbf{z}) = \mathbf{f(z)}$. We solve the Dirichlet problem

$$\mu\Delta\mathbf{v}(\mathbf{x}) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{v}(\mathbf{x}) = 0, \tag{3.30}$$
$$\mathbf{v}(\mathbf{z}) = \boldsymbol{\Psi}(\mathbf{z}), \tag{3.31}$$

where $\boldsymbol{\Psi}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) - \mathbf{v_0}(\mathbf{z})$. The known function $\mathbf{v}_0(\mathbf{z})$ is determined by formula (3.26). Let us rewrite equality (3.31) in normal and tangential components

$$v_n(\mathbf{z}) = \Psi_n(\mathbf{z}), \quad v_s(\mathbf{z}) = \Psi_s(\mathbf{z}), \tag{3.32}$$

where, taking into account (3.27), (3.28) and (3.29), we have

$$v_n(\mathbf{x}) = \sum_{m=1}^{\infty}\frac{m}{R}\Big(\frac{r}{R}\Big)^{m-1}(\mathbf{X}_{m1}\cdot\boldsymbol{\nu}_m(\psi)) + \frac{(\lambda+3\mu)m+2\mu}{\mu R}\Big(\frac{r}{R}\Big)^{m+1}(\mathbf{X}_{m2}\cdot\boldsymbol{\nu}_m(\psi)), \tag{3.33}$$
$$v_s(\mathbf{x}) = \sum_{m=1}^{\infty}\frac{m}{R}\Big(\frac{r}{R}\Big)^{m-1}(\mathbf{X}_{m1}\cdot s_m(\psi)) + \frac{(\lambda+3\mu)m+2\mu}{\mu R}\Big(\frac{r}{R}\Big)^{m+1}(\mathbf{X}_{m2}\cdot s_m(\psi)). \tag{3.34}$$

Let us expand the functions $\Psi_n(\mathbf{z})$ and $\Psi_s(\mathbf{z})$ into the Fourier series

$$\Psi_n(\mathbf{z}) = \frac{\boldsymbol{\alpha}_0}{2} + \sum_{m=1}^{\infty}(\alpha_{ml}\cdot\nu_m(\psi)), \quad \Psi_s(\mathbf{z}) = \frac{\boldsymbol{\beta}_0}{2} + \sum_{m=1}^{\infty}(\beta_{ml}\cdot s_m(\psi)), \tag{3.35}$$

where $\boldsymbol{\alpha}_m = (\alpha_{m1}, \alpha_{m2})$, $\boldsymbol{\beta}_m = (\beta_{m1}, \beta_{m2})$, $\boldsymbol{\alpha}_0 = (\alpha_{01}, 0)$, $\boldsymbol{\beta}_0 = (\beta_{01}, 0)$ are the Fourier coefficients. We obtain a system of linear algebraic equations:

$$\begin{gathered}\frac{2}{R}\mathbf{X}_{01} = \frac{\boldsymbol{\alpha}_0}{2}, \quad \frac{2(\lambda+2\mu)}{\mu R}\mathbf{X}_{02} = \frac{\boldsymbol{\beta}_0}{2},\\ \frac{m}{R}\mathbf{X}_{m1} + \frac{(\lambda+3\mu)+2\mu}{\mu R}\mathbf{X}_{m2} = \boldsymbol{\alpha}_m,\\ \frac{m}{R}\mathbf{X}_{m1} + \frac{(\lambda+3\mu)+2(\lambda+2\mu)}{\mu R}\mathbf{X}_{m2} = \boldsymbol{\beta}_m, \quad m=1,2,\dots.\end{gathered} \tag{3.36}$$

We substitute the solution of system (3.36) into (3.29) and then the obtained values of the function $\Phi(\mathbf{z})$ into (3.27). We obtain the value of $\mathbf{v}(\mathbf{x})$. The value of the function $\mathbf{v}_0(\mathbf{x})$ is also known. Using formula (3.25), we determine $\mathbf{u(x)}$. So, Problem A_1 is solved.

Problem $\mathbf{A_2}$. We solve the Neumann problem. We are looking for a solution to equation (3.24) that satisfies condition (II) in (2.3),

$$\mathbf{R}(\partial_z, n)\mathbf{U}(\mathbf{z}) = \mathbf{f}(\mathbf{z}), \quad \mathbf{z} = (z_1, z_2) \in S, \tag{3.37}$$

where

$$\mathbf{R}\left(\partial_{\mathbf{x}}, \mathbf{n}\right)\mathbf{U}(\mathbf{x}) = \mathbf{T}\left(\partial_{\mathbf{x}}, \mathbf{n}\right)\mathbf{u}(\mathbf{x}) - \mathbf{n}(\mathbf{x})[a\varphi_1(\mathbf{x}) + b\varphi_2(\mathbf{x}) + c\varphi_3(\mathbf{x})]. \tag{3.38}$$

We seek $\mathbf{u}(\mathbf{x})$ in the form of the sum (3.25), in which the particular solution $\mathbf{v}_0(\mathbf{x})$ is known. It is expressed by formula (3.26). The general solution $\mathbf{v}(\mathbf{x})$ of the homogeneous equation corresponding to (3.24) will be sought in the form (3.27). Our goal is to find the values of the harmonic function Φ_1 and the biharmonic functions Φ_2 and Φ_3.

From (3.38), taking into account (3.25) and condition (II) from (2.3), we obtain

$$\mathbf{T}(\partial_z, \mathbf{n})\mathbf{v}(\mathbf{z}) = \mathbf{\Psi}(\mathbf{z}), \tag{3.39}$$

where

$$\mathbf{\Psi}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) + \mathbf{n}(\mathbf{z})[\mathbf{a}\varphi_{\mathbf{1}}(\mathbf{z}) + \mathbf{b}\varphi_{\mathbf{2}}(\mathbf{z}) + \mathbf{c}\varphi_{\mathbf{3}}(\mathbf{z})] - \mathbf{T}\left(\partial_{\mathbf{z}}, \mathbf{n}\right)\mathbf{v_0}(\mathbf{z})$$

is a known vector, $\mathbf{\Psi} = (\mathbf{\Psi_1}, \mathbf{\Psi_1})$.

We rewrite condition (3.39) in the form of normal and tangential components. We obtain:

$$\begin{gathered}
(\mathbf{T}(\partial_z, \mathbf{n})\mathbf{v}(\mathbf{z}))_n = (\lambda + 2\mu)\Big(\frac{\partial v_n(\mathbf{z})}{\partial r}\Big)_{r=R} + \frac{\lambda}{R}\frac{\partial v_s(\mathbf{z})}{\partial \psi}, \\
(\mathbf{T}(\partial_{\mathbf{z}}, \mathbf{n})\mathbf{v}(\mathbf{z}))_s = (\lambda + 2\mu)\Big(\frac{\partial v_s(\mathbf{z})}{\partial r}\Big)_{r=R} + \frac{1}{R}\frac{\partial v_n(\mathbf{z})}{\partial \psi}, \\
\Psi_n(\mathbf{z}) = f_n(\mathbf{z}) + a\varphi_1(\mathbf{z}) + b\varphi_2(\mathbf{z}) + c\varphi_3(\mathbf{z})] - \left(\mathbf{T}\left(\partial_{\mathbf{z}}, \mathbf{n}\right)\mathbf{v}_0(\mathbf{z})\right)_n, \\
\Psi_s(\mathbf{z}) = f_s(\mathbf{z}) - \left(\mathbf{T}\left(\partial_{\mathbf{z}}, \mathbf{n}\right)\mathbf{v}_0(\mathbf{z})\right)_s,
\end{gathered} \tag{3.40}$$

where $v_n(\mathbf{z})$ and $v_s(\mathbf{z})$ are defined by formulas (3.33) and (3.34). Using the expansions of the functions $\Psi_n(\mathbf{z})$ and $\Psi_s(\mathbf{z})$ in Fourier series (3.35), as well as formulas (3.29), from formulas (3.40) for each m we obtain a system of equations with respect to the unknowns $\mathbf{X}_{m1}$ and $\mathbf{X}_{m2}$:

$$\begin{gathered}
\frac{2(\lambda + 2\mu)}{R^2}\mathbf{X}_{01} = \frac{\alpha_0}{2}, \quad \frac{2(\lambda + 2\mu)}{R^2}\mathbf{X}_{02} = \frac{\beta_0}{2}, \\
g_1 = m\mu[2(\lambda + \mu)m - \lambda - 2\mu], \quad g_2 = 2(\lambda + \mu)(\lambda + 3\mu)m^2 + (\lambda + 2\mu)[(3\lambda + 5\mu)m + 2\mu], \\
g_3 = m\mu[2(m - 1), \quad g_4 = (\lambda + 3\mu)m(2m + 3) + 2(\lambda + 2\mu), \quad m = 1, 2, \dots .
\end{gathered} \tag{3.41}$$

Substitute formulas (3.41) into (3.29) and substitute the obtained results into (3.27). This defines the vector $\mathbf{v}(\mathbf{x})$. Formula (3.25) gives the solution to Problem A_2.

Having solved Problems A_1, A_2, B_1 and B_2, we can write solutions of the initial Problems (I) and (II).

The conditions $f_j \in C^3(S)$ in the Problem (I) and the conditions $f_j \in C^2(S)$ in the Problem (II) $(j = 1, 2, 3, 4, 5)$, respectively, ensure absolutely and uniformly convergence of the series obtained for $\mathbf{v}(\mathbf{x})$ and $\mathbf{v}_0(\mathbf{x})$.

4. Concluding Remarks

In this paper, we consider the linear theory of elasticity for isotropic three-porous bodies. The system of general constitutive equations is expressed through the vector field of displacements and fluid pressure in the pore network. The following results are presented: a) A general representation of the solution of the system of equations of elasticity theory using elementary functions is constructed, b) Boundary value problems of linear elasticity theory in the two-dimensional case for isotropic three-porous bodies are presented, c) The posed problems are solved for an elastic three-porous disk. The solutions to the problems are obtained explicitly in the form of absolutely and uniformly convergent series, d) The application of the method under consideration allows us to study a wide class of problems for systems of equations of elasticity or thermoelasticity theory for materials with triple porosity; to construct explicit solutions to the main boundary value problems not only for a circle, but also for a ring, a plane with a circular hole, etc.

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References

1. D. Abdassah, I. Ershaghi, Triple-porosity systems for representing naturally fractured Reservoirs. *SPE Form Eval.* **1** (1986), no. 02, 113–127.
2. H. A. Al-Ahmadi, R. A. Wattenbarger, Triple-porosity models: one further step towards capturing fractured reservoirs heterogeneity. In: *SPE Kingdom of Saudi Arabia Annual Technical Symposium and Exhibition*, SPE–149054–MS, 2011.
3. M. Bai, J. -C. Roegiers, Triple-porosity analysis of solute transport. *J. Contam. Hydrol.* **28** (1997), no. 3, 247–266.
4. M. Basheleishvili, L. Bitsadze, Explicit solutions of the boundary value problems of the theory of consolidation with double porosity for the half-plane. *Georgian Math. J.* **19** (2012), no. 1, 41–48.
5. J. G. Berryman, H. F. Wang, The elastic coefficients of double-porosity models for fluid transport in jointed rock. *J. Geo. Res.* **100** (1995), no. B12, 24611–24627.
6. D. E. Beskos, E. C. Aifantis, On the theory of consolidation with double porosity. II. *Internat. J. Engrg. Sci.* **24** (1986), no. 11, 1697–1716.
7. L. Bitsadze, On some solutions in the plane equilibrium theory for solids with triple-porosity. *Bull. TICMI* **21** (2017), no. 1, 9–20.
8. L. Bitsadze, I. Tsagareli, The solution of the Dirichlet BVP in the fully coupled theory of elasticity for spherical layer with double porosity. *Meccanica* **51** (2016), no. 6, 1457–1463.
9. S. C. Cowin, Bone poroelasticity. *J. Biomech.* **32** (1999), no. 3, 217–238.
10. S. C. Cowin, G. Gailani, M. Benalla, Hierarchical poroelasticity: movement of interstitial fluid between porosity levels in bones. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **367** (2009), no. 1902, 3401–3444.
11. R. de Boer, *Theory of Porous Media*. Highlights in historical development and current state. Springer-Verlag, Berlin, 2000.
12. B. Gulua, R. Janjgava, The Dirichlet boundary value problem of porous Cosserat media with triple-porosity for the concentric circular ring. *Appl. Math. Inform. Mech.* **22** (2017), no. 1, 42–49.
13. R. Janjgava, Elastic equilibrium of porous Cosserat media with double porosity. *Adv. Math. Phys.* **2016**, Art. ID 4792148, 9 pp.
14. R. Janjgava, Approximate solution of plane problems about stress concentrations in elastic bodies with voids. *J. Engrg. Math.* **144** (2024), Paper no. 17, 32 pp.
15. R. Kumar, R. Vohra, State space approach to plane deformation in elastic material with double porosity. *Materials Physics and Mechanics* **24** (2015), no. 1, 9–17.
16. CQ. Liu, Exact solution for the compressible flow equations through a medium with triple-porosity. *Appl. Math. Mech.* **2** (1981), no. 4, 457–462.
17. JC. Liu, GS. Bodvarsson, YS. Wu, Analysis of pressure behaviour in fractured lithophysical reservoirs. *J. Contam Hydrol.* **62-63** (2003), 189–211.
18. B. Straughan, Modelling questions in multi-porosity elasticity. *Meccanica* **51** (2016), no. 12, 2957–2966.
19. B. Straughan, Waves and uniqueness in multi-porosity elasticity. *J. Therm. Stresses* **39** (2016), no. 6, 704–721.
20. B. Straughan, *Mathematical Aspects of Multi-Porosity Continua*. Advances in Mechanics and Mathematics, 38. Springer, Cham, 2017.
21. M. Svanadze, Fundamental solutions in the theory of elasticity for triple porosity materials. *Meccanica* **51** (2016), no. 8, 1825–1837.
22. M. Svanadze, *Potential Method in Mathematical Theories of Multi-Porosity Media*. Interdiscip. Appl. Math. 51, Springer, Cham, Switzerland, 2019.
23. M. Svanadze, Potential method in the coupled linear theory of elasticity for materials with triple porosity. *Trans. A. Razmadze Math. Inst.* **176** (2022), no. 1, 83–98.
24. M. Svanadze, On the coupled linear theory of thermoelasticity for nanomaterials which triple porosity. *Mechanics Research Communications* **132** (2023), 104161.
25. M. Svanadze, Potential method in the coupled theory of thermoelastic triple-porosity nanomaterials. *J. Therm. Stresses* **47** (2024), no. 10, 1277–1296.
26. M. Svanadze, A. Scalia, Mathematical problems in the coupled linear theory of bone poroelasticity. *Comput. Math. Appl.* **66** (2013), no. 9, 1554–1566.

27. I. Tsagareli, L. Bitsadze, Explicit solution of one boundary value problem in the full coupled theory of elasticity for solids with double porosity. *Acta Mech.* **226** (2015), no. 5, 1409–1418.
28. I. Tsagareli, M. M. Svanadze, Explicit solutions of the problems of elastostatics for an elastic circle with double porosity. *Mechanics Research Communications* **46** (2012), 76–80.
29. I. Vekua, On metaharmonic functions. *Proc. Tbilisi Math. Inst. Acad. Sci. Georgian SSR.* **12** (1943), 105–174. (Russian) *Lect. Notes TICMI* **14** (2013), 62 pp.
30. I. N. Vekua, *New Methods for Solving Elliptic Equations.* Translated from the Russian by D. E. Brown. Translation edited by A. B. Tayler. North-Holland Series in Applied Mathematics and Mechanics, vol. 1. North-Holland Publishing Co., Amsterdam; Interscience Publishers John Wiley & Sons, Inc., New York, 1967.
31. YS. Wu, HH. Liu, GS. Bodavarsson, A triple-continuum approach for modeling flow and transport processes in fractured rock. *J. Contam. Hydrol.* **73** (2004), 145–179.

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