

ON THE CONVOLUTION PRODUCT OF AN INTEGRABLE FUNCTION WITH A FUNCTION OF GENERALIZED BOUNDED VARIATION

HARDEEPBHAI J. KHACHAR^{1*} AND RAJENDRA G. VYAS²

Abstract. The convolution property of the class of integrable functions with the class of functions of generalized bounded variation is studied for functions of one variable and the result is extended to functions of several variables.

1. INTRODUCTION

In 2002, Akhobadze [1] studied various properties of the $B\Lambda(p(n) \uparrow p, \varphi)$ class, a generalization of the class of functions of bounded variation. In 2006, it was shown that various generalized classes of bounded variation can be regarded as non-unital modules over the ring of integrable functions under convolution [3]. Here, we generalize this property for the Akhobadze class and extend the result to functions of several variables.

In the sequel, we will assume that:

- φ is an increasing function defined on \mathbb{N} such that $\varphi \geq 2$ and $\lim_{n \rightarrow \infty} \varphi(n) = +\infty$.
- $\mathbb{T} := [0, 2\pi)$ and $\overline{\mathbb{T}}^N = \overline{\mathbb{T}} \times \overline{\mathbb{T}} \times \dots \times \overline{\mathbb{T}}$ (N times), $N \geq 2$.
- If f is a function of one variable then

$$\|f\|_1 = \int_{\overline{\mathbb{T}}} |f(x)| dx.$$

- If f is a function of two variables then

$$\|f\|_1 = \iint_{\overline{\mathbb{T}}^2} |f(x, y)| dx dy.$$

2. RESULT FOR FUNCTIONS OF ONE VARIABLE

Definition 1. [1, Definition 1] Let f be a 2π periodic measurable function. Let $p(n)$ be an increasing sequence such that $1 \leq p(n) \uparrow p$ for $1 \leq p \leq \infty$. Then $f \in B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$ if

$$\Lambda(f, p(n) \uparrow p, \varphi, \overline{\mathbb{T}}) := \sup_{m \geq 1} \sup_{h \geq \frac{1}{\varphi(m)}} \left\{ \frac{1}{h} \int_{\overline{\mathbb{T}}} |f(x+h) - f(x)|^{p(m)} dx \right\}^{\frac{1}{p(m)}} < \infty.$$

Note that, if $f \in B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$ then f is an essentially bounded function over $\overline{\mathbb{T}}$ [1, Corollary 1].

Definition 2. [4, p. 36] If f and g are two integral functions on $\overline{\mathbb{T}}$ then convolution product of f and g , denoted by $f * g$ is given by

$$f * g(x) = \frac{1}{2\pi} \int_{\overline{\mathbb{T}}} f(t)g(x-t)dt; \quad \forall x \in \overline{\mathbb{T}}.$$

2020 *Mathematics Subject Classification.* 42A85, 26A45, 26B30.

Key words and phrases. convolution product, generalized bounded variation, Akhobadze variation, module over ring under convolution product, functions of several variables.

*Corresponding author.

Theorem 2.1. *If $f \in L^1(\overline{\mathbb{T}})$ and $g \in B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$ then $f * g \in B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$.*

Proof. Let $m \in \mathbb{N}$ and $h > 0$ be given. Also, let $p(n)$ be an increasing sequence such that $1 \leq p(n) \uparrow p$ for $1 \leq p \leq \infty$, then by applying Hölder's inequality, we have

$$\begin{aligned} & \frac{1}{h} \int_{\overline{\mathbb{T}}} |f * g(x+h) - f * g(x)|^{p(m)} dx \\ &= \frac{1}{h} \int_{\overline{\mathbb{T}}} \left| \frac{1}{2\pi} \int_{\overline{\mathbb{T}}} f(y)(g(x+h-y) - g(x-y)) dy \right|^{p(m)} dx \\ &\leq \frac{1}{(2\pi)^{p(m)} h} \int_{\overline{\mathbb{T}}} \left(\int_{\overline{\mathbb{T}}} |f(y)|^{1-\frac{1}{p(m)}} |f(y)|^{\frac{1}{p(m)}} |g(x+h-y) - g(x-y)| dy \right)^{p(m)} dx \\ &\leq \frac{\|f\|_1^{p(m)-1}}{(2\pi)^{p(m)} h} \int_{\overline{\mathbb{T}}} \left(\int_{\overline{\mathbb{T}}} |g(x+h-y) - g(x-y)|^{p(m)} dx \right) dy. \end{aligned}$$

Now, taking supremum over $h \geq \frac{1}{\varphi(m)}$, we have,

$$\sup_{h \geq \frac{1}{\varphi(m)}} \frac{1}{h} \int_{\overline{\mathbb{T}}} |f * g(x+h) - f * g(x)|^{p(m)} dx \leq \left(\frac{\|f\|_1}{2\pi} \Lambda(g, p(n) \uparrow p, \varphi, \overline{\mathbb{T}}) \right)^{p(m)}.$$

Thus, by taking $p(m)$ th root and taking supremum over $m \geq 1$, we get $f * g \in B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$. \square

Remark 1. Since $L^1(\overline{\mathbb{T}})$ is a ring under the convolution product and in view of the previous result, $B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$ can be regarded as a non-unital module over the ring $L^1(\overline{\mathbb{T}})$.

3. RESULT FOR FUNCTIONS OF TWO VARIABLES

Definition 3. [2, Definition 3.1] Let f be a measurable function and 2π periodic in both variables. Let $p(n)$ and $q(n)$ be increasing sequences such that $p(n) \leq q(n)$, $1 \leq p(n) \uparrow p$ for $1 \leq p \leq \infty$ and $1 \leq q(n) \uparrow q$ for $1 \leq q \leq \infty$. Then $f \in B\Lambda(p(n) \uparrow p, q(n) \uparrow q, \varphi, \overline{\mathbb{T}}^2)$ if $\Lambda(f, p(n) \uparrow p, q(n) \uparrow q, \varphi, \overline{\mathbb{T}}^2)$

$$:= \sup_{m \geq 1} \sup_{hk \geq \frac{1}{\varphi(m)^2}} \left\{ \frac{1}{k} \int_{\overline{\mathbb{T}}} \left(\frac{1}{h} \int_{\overline{\mathbb{T}}} |\Delta f(x, y; h, k)|^{p(m)} dx \right)^{\frac{q(m)}{p(m)}} dy \right\}^{\frac{1}{q(m)}} < \infty,$$

where

$$\Delta f(x, y; h, k) = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y).$$

Definition 4. If f and g are integrable functions on $\overline{\mathbb{T}}^2$ then the convolution product of f and g , denoted by $f * g(x, y)$, is defined by

$$f * g(x, y) = \frac{1}{4\pi^2} \iint_{\overline{\mathbb{T}}^2} f(x, y) g(x-u, y-v) dudv.$$

Theorem 3.1. *If $f \in L^1(\overline{\mathbb{T}}^2)$ and $g \in B\Lambda(p(n) \uparrow p, q(n) \uparrow q, \varphi, \overline{\mathbb{T}}^2) \cap L^1(\overline{\mathbb{T}}^2)$ then*

$$f * g \in B\Lambda(p(n) \uparrow p, q(n) \uparrow q, \varphi, \overline{\mathbb{T}}^2) \cap L^1(\overline{\mathbb{T}}^2).$$

Proof. Clearly, $f * g \in L^1(\overline{\mathbb{T}}^2)$. Let $m \in \mathbb{N}$ and $h, k > 0$ be given. Let $p(n)$ and $q(n)$ be increasing sequences such that $1 \leq p(n) \uparrow p$ for $1 \leq p \leq \infty$, $1 \leq q(n) \uparrow q$ for $1 \leq q \leq \infty$ and $p(n) \leq q(n)$. Also, denote $\Delta f(x, y; h, k) = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y)$. Then by applying Hölder's inequality twice, we have

$$\begin{aligned} & \frac{1}{k} \int_{\overline{\mathbb{T}}} \left(\frac{1}{h} \int_{\overline{\mathbb{T}}} |\Delta(f * g(x, y; h, k))|^{p(m)} dx \right)^{\frac{q(m)}{p(m)}} dy \\ &= \frac{1}{k} \int_{\overline{\mathbb{T}}} \left(\frac{1}{h} \int_{\overline{\mathbb{T}}} \left| \frac{1}{4\pi^2} \iint_{\overline{\mathbb{T}}^2} f(u, v) \Delta g(x-u, y-v; h, k) dudv \right|^{p(m)} dx \right)^{\frac{q(m)}{p(m)}} dy \\ &\leq \frac{\|f\|_1^{q(m)(1-\frac{1}{p(m)})}}{(4\pi^2)^{q(m)}k} \int_{\overline{\mathbb{T}}} \left\{ \iint_{\overline{\mathbb{T}}^2} (|f(u, v)|) \left(\frac{1}{h} \int_{\overline{\mathbb{T}}} |\Delta g(x-u, y-v; h, k)|^{p(m)} dx \right) dudv \right\}^{\frac{q(m)}{p(m)}} dy \\ &\leq \frac{\|f\|_1^{q(m)-1}}{(4\pi^2)^{q(m)}} \iint_{\overline{\mathbb{T}}^2} \left\{ |f(u, v)| \frac{1}{k} \int_{\overline{\mathbb{T}}} \left(\frac{1}{h} \int_{\overline{\mathbb{T}}} |\Delta g(x-u, y-v; h, k)|^{p(m)} dx \right)^{\frac{q(m)}{p(m)}} dy \right\} dudv. \end{aligned}$$

Now, taking supremum over $hk \geq \frac{1}{\varphi(m)^2}$, we have,

$$\begin{aligned} & \sup_{hk \geq \frac{1}{\varphi(m)^2}} \frac{1}{k} \int_{\overline{\mathbb{T}}} \left(\frac{1}{h} \int_{\overline{\mathbb{T}}} |\Delta(f * g(x, y; h, k))|^{p(m)} dx \right)^{\frac{q(m)}{p(m)}} dy \\ & \leq \left(\frac{\|f\|_1}{4\pi^2} \Lambda(f, p(n) \uparrow p, q(n) \uparrow q, \varphi, \overline{\mathbb{T}}^2) \right)^{q(m)}. \end{aligned}$$

Thus, by taking $q(m)$ th root and taking supremum over $m \geq 1$, we get

$$f * g \in B\Lambda(p(n) \uparrow p, q(n) \uparrow q, \varphi, \overline{\mathbb{T}}^2).$$

□

Remark 2. Since $L^1(\overline{\mathbb{T}}^2)$ is a ring under the convolution product and in view of the previous result, the class $B\Lambda(p(n) \uparrow p, q(n) \uparrow q, \varphi, \overline{\mathbb{T}}^2) \cap L^1(\overline{\mathbb{T}}^2)$ can be regarded as a non-unital module over the ring $L^1(\overline{\mathbb{T}}^2)$.

4. RESULT FOR FUNCTIONS OF SEVERAL VARIABLES

Definition 5. [2, Definition 4.1] Let a measurable function f be 2π periodic in all the variables. Let $p_j(n)$ be increasing sequences such that $p_j(n) \leq p_{j-1}(n)$, $1 \leq p_j(n) \uparrow q_j$ and $1 \leq q_j \leq \infty$ for $j = 1, 2, \dots, N$, where $N \in \mathbb{N}$. Then $f \in B\Lambda(p_1(n) \uparrow q_1, \dots, p_N(n) \uparrow q_N, \varphi, \overline{\mathbb{T}}^N)$ if

$$\begin{aligned} & \Lambda(p_1(n) \uparrow q_1, \dots, p_N(n) \uparrow q_N, \varphi, \overline{\mathbb{T}}^N) \\ & := \sup_{m \geq 1} \sup_{\prod_1^N h_j \geq \frac{1}{\varphi(m)^N}} \left\{ \frac{1}{h_N} \int_{\overline{\mathbb{T}}} \dots \left(\frac{1}{h_1} \int_{\overline{\mathbb{T}}} |\Delta f(x_1, \dots, x_N; h_1, \dots, h_N)|^{p_1(m)} dx_1 \right)^{\frac{p_2(m)}{p_1(m)}} \dots dx_N \right\}^{\frac{1}{p_N(m)}} \\ & < \infty, \end{aligned}$$

$$\begin{aligned} \text{where } \Delta f(x_1; h_1) &= f(x_1 + h_1) - f(x_1), \\ \Delta f(x_1, x_2; h_1, h_2) &= f(x_1 + h_1, x_2 + h_2) - f(x_1, x_2 + h_2) - f(x_1 + h_1, x_2) + f(x_1, x_2), \\ &\vdots \\ \Delta f(x_1, \dots, x_N; h_1, \dots, h_N) &= \sum_{u_1=0}^1 \dots \sum_{u_N=0}^1 (-1)^{u_1+u_2+\dots+u_N} f(x_1 + u_1 h_1, \dots, x_N + u_N h_N). \end{aligned}$$

Definition 6. If f and g are integrable functions on $\overline{\mathbb{T}}^N$ then the convolution product of f and g , denoted by $f * g(x_1, x_2 \dots x_N)$, is defined by

$$f * g(x_1, x_2, \dots, x_N) = \frac{1}{(2\pi)^N} \int \dots \int_{\overline{\mathbb{T}}^N} f(x_1, \dots, x_N) g(x_1 - u_1, \dots, x_N - u_N) du_1 \dots du_N.$$

The result of functions of two variables can be extended for functions of N variables as below.

Theorem 4.1. If $f \in L^1(\overline{\mathbb{T}}^N)$ and $g \in B\Lambda(p_1(n) \uparrow q_1, \dots, p_N(n) \uparrow q_N, \varphi, \overline{\mathbb{T}}^N) \cap L^1(\overline{\mathbb{T}}^N)$ then

$$f * g \in B\Lambda(p_1(n) \uparrow q_1, \dots, p_N(n) \uparrow q_N, \varphi, \overline{\mathbb{T}}^N) \cap L^1(\overline{\mathbb{T}}^N).$$

Remark 3. Since $L^1(\overline{\mathbb{T}}^N)$ is a ring under the convolution product and in view of the previous result, the class $B\Lambda(p_1(n) \uparrow q_1, \dots, p_N(n) \uparrow q_N, \varphi, \overline{\mathbb{T}}^N) \cap L^1(\overline{\mathbb{T}}^N)$ can be regarded as a non-unital module over the ring $L^1(\overline{\mathbb{T}}^N)$.

REFERENCES

1. T. Akhobadze, A generalization of bounded variation, *Acta Math. Hungar.* **97** (2002), no. 3, 223–256
2. H. J. Khachar and R. G. Vyas, Properties of rational Fourier series and generalized Wiener class, *Georgian Math. J.* **30** (2023), no. 2, 247–253
3. R. G. Vyas, Convolution functions of several variables with generalized bounded variation, *Anal. Math.* **39** (2013), no. 2, 153–161
4. A. S. Zygmund, *Trigonometric series. Vol. I, II*, third edition, Cambridge Mathematical Library, Cambridge Univ. Press, Cambridge, 2002

(Received ???.???.20??)

¹DEPARTMENT OF MATHEMATICS, GOVERNMENT SCIENCE COLLEGE, BHILAD 396105, GUJARAT, INDIA

²DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, THE MAHARAJA SAYAJIRAO UNIVERSITY OF BARODA, VADODARA 390002, GUJARAT, INDIA

Email address: hardeep1996k@gmail.com

Email address: drrgvyas@yahoo.com