Transactions of A. Razmadze
Mathematical Institute
Vol. **1??** (202?), ?–??

ON ARTINIANNESS OF LOCAL COHOMOLOGY WITH SUPPORT OF DIMENSION $\leq d$

MIRSADEGH SAYEDSADEGHI[1*], MIRYOUSEF SADEGHI[2] AND FARZANEH RAFIEI[3]

Abstract. Let R be a commutative Noetherian ring and let M, N be two R-modules. Let d be a non-negative integer. Several results regarding the Artinian property of the modules $T_d^i(M,N)$ and $H_d^i(M,N)$ will be given. In the case that $(R,\mathfrak{m})$ is a local ring, we see that the Artinian property of $H_d^1(M)$ gives the same property for $H_d^{\dim_d(M)}(M)$. Then, among other things, in particular, we prove that the top non-vanishing d-transform of two modules M and N is Artinian.

1. Introduction

Throughout this paper, we assume that R is a commutative Noetherian ring with non-zero identity and M and N are two R-modules. Our aim is to study the Artinian properties of some local cohomology and transform functors supported in dimension $\leq d$, where d is a non-negative integer. To be precise, put $\Sigma = \{\mathfrak{a} \in \mathscr{I}(R) : \dim(R/\mathfrak{a}) \leq d\}$, where $\mathscr{I}(R)$ is the set of ideals of R. This Σ equipped with reverse inclusion is a directed set and we can define the functors $T_d(-) = \varinjlim_{\mathfrak{a}\in\Sigma} \operatorname{Hom}_R(\mathfrak{a}, -)$ and $L_d(-) = \varinjlim_{\mathfrak{a}\in\Sigma} \operatorname{Hom}_R(R/\mathfrak{a}, -)$. For an R-module M, it is easy to see that $L_d(M) = \{m \in M \mid Im = 0 \text{ for some } I \in \Sigma\}$, as stated in [1]. We denote the ith right derived functors of these two functors with $T_d^i(-)$ and $H_d^i(-)$ for $i \geq 0$. Then for each R-module M, $T_d^i(M)$ and $H_d^i(M)$ have natural R-module structures, which are referred to as d-transform ideal and d-local cohomology module respectively. Also, as a generalization of these last two modules, for two modules M and N, we define $T_d^i(M,N) = \varinjlim_{\mathfrak{a}\in\Sigma} \operatorname{Ext}_R^i(\mathfrak{a}M, N)$ and $H_d^i(M,N) = \varinjlim_{\mathfrak{a}\in\Sigma} \operatorname{Ext}_R^i(M/\mathfrak{a}M, N)$, where $i \geq 0$. Hence $H_d^i(M) = H_d^i(R,M)$ and $T_d^i(M) = T_d^i(R,M)$ for all $i \geq 0$. In [1, 11, 12], the authors have conducted investigations on d-local cohomology modules and their generalized modules. In [11, Lemma 2.1(6)], the authors, proved that $H_d^i(M,N) = \varinjlim_{\mathfrak{a}\in\Sigma} H_{\mathfrak{a}}^i(M,N)$ for all $i \geq 0$, where $H_{\mathfrak{a}}^i(M,N)$ is the usual generalized local cohomology of M and N supported in $\mathfrak{a}$ (see [10]). In [3] and [4], the authors have conducted studies on the Artinianness of $H_{\mathfrak{a}}^i(M,N)$, which has encouraged us to study d-local cohomology and d-transform ideal. In the following, we consider $\dim_d(M) = \sup\{\dim(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \Sigma \cap \operatorname{Spec}(R)\}$ (see [1, p. 12]) and $pd(M) = \min\{n \in \mathbb{N}_0 \mid M$ has a projective resolusion of length $n\}$ (see [9, p. 454]). In this paper, we want to study about the Artinian properties of these modules. In section 2, we will provide some auxiliary facts on these functors and modules. Then we will collect the main results in section 3. Among several results, we will see that when $(R,\mathfrak{m})$ is local ring and $0 \neq M$ is a finitely generated R-module, the fact that $H_d^1(M)$is Artinian, gives $H_d^{\dim_d(M)}(M)$ to be Artinian. Next, we see that the top d-transform module of two finitely generated R-modules M and N is Artinian.

2. Preliminaries

The study of Artinian properties in local cohomology and related functors presents fundamental challenges in homological algebra. This section establishes key technical tools and preliminary results that underpin our main theorems. We develop sufficient conditions for the Artinianness of modules

2020 *Mathematics Subject Classification.* 13C12, 13D45, 16P20.
Key words and phrases. Artinian; Local Cohomology; Transform ideal.
*Corresponding author: E-mail: m_sayedsadeghi@pnu.ac.ir.

$\mathrm{Hom}_R(R/\mathfrak{a}, H^1_d(M))$ and $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_d(M)/X)$, where $\mathfrak{a} \in \Sigma$, M and N are R-modules, t is a non-negative integer, and X is an R-submodule of $H^t_d(M)$. These foundational results create the necessary framework for the applications explored in subsequent sections.

Lemma 1. *Let M be a finitely generated R-module and N be an Artinian R-module. Then*
(1) $\mathrm{Ext}^i_R(M, N)$ is Artinian for all $i \geq 0$.
(2) $\mathrm{Tor}^R_i(M, N)$ is Artinian for all $i \geq 0$.

Proof. First, note that since M is a finitely generated R-module, then by [9, Lemma 8.49], there exists a chain of the submodules of M

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = M$$

such that for each $1 \leq i \leq k$, $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Supp}(M)$.
Now we proceed to the proof of parts (1) and (2).
(1) We use induction on i. For $i = 0$, let $\mathfrak{a}$ be an ideal of R. Since $\mathrm{Hom}_R(R/\mathfrak{a}, N) \cong (0 \underset{N}{:} \mathfrak{a})$ and $(0 \underset{N}{:} \mathfrak{a})$ is Artinian (as $(0 \underset{N}{:} \mathfrak{a}) \subseteq N$), then $\mathrm{Hom}_R(R/\mathfrak{a}, N)$ is Artinian. We have $\mathrm{Hom}_R(M_1, N) \cong \mathrm{Hom}_R(R/\mathfrak{p}_1, N)$ and so $\mathrm{Hom}_R(M_1, N)$ is Artinian. Also, from the short exact sequence

$$0 \to M_1 \to M_2 \to R/\mathfrak{p}_2 \to 0,$$

we get the long exact sequence

$$0 \to \mathrm{Hom}_R(R/\mathfrak{p}_2, N) \to \mathrm{Hom}_R(M_2, N) \to \mathrm{Hom}_R(M_1, N) \to \cdots.$$

Hence $\mathrm{Hom}_R(M_2, N)$ is Artinian, because $\mathrm{Hom}_R(M_1, N)$ and $\mathrm{Hom}_R(R/\mathfrak{p}_2, N)$ are Artinian. Now, by repeating this process, $\mathrm{Hom}_R(M, N)$ is also Artinian. In the following, we assume that $\mathrm{Ext}^i_R(M, N)$ is Artinian and prove that $\mathrm{Ext}^{i+1}_R(M, N)$ is an Artinian R-module. Let $E(N)$ be the injective hull of N. Since N is Artinian, then $E(N)$ is Artinian and so $E(N)/N$ is also Artinian. From the short exact sequence

$$0 \to N \to E(N) \to E(N)/N \to 0,$$

we get the exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, E(N)) \to \mathrm{Hom}_R(M, E(N)/N) \to \mathrm{Ext}^1_R(M, N) \to 0$$

and R-isomorphism $\mathrm{Ext}^i_R(M, E(N)/N) \cong \mathrm{Ext}^{i+1}_R(M, N)$ for all $i \geq 1$. Then $\mathrm{Ext}^1_R(M, N)$ is Artinian. Now, since by induction hypothesis, $\mathrm{Ext}^i_R(M, E(N)/N)$ is Artinian, hence $\mathrm{Ext}^{i+1}_R(M, N)$ is Artinian.
(2) Let $\mathfrak{a} \in \mathscr{I}(R)$. Since $R/\mathfrak{a} \otimes_R N \cong N/\mathfrak{a}N$ and N is Artinian, then $R/\mathfrak{a} \otimes_R N$ is also Artinian. Now, from the short exact sequence

$$0 \to M_{i-1} \to M_i \to R/\mathfrak{p}_i \to 0 \qquad (*)$$

and similar to the method used in part (1), it is easy to show that $M \otimes_R N$ is Artinian $(1 \leq i \leq k)$. From the short exact sequence

$$0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$$

we conclude that

$$0 \to \mathrm{Tor}^R_1(R/\mathfrak{a}, N) \to \mathfrak{a} \otimes_R N \to R \otimes_R N \to R/\mathfrak{a} \otimes_R N \to 0$$

is an exact sequence and $\mathrm{Tor}^R_n(\mathfrak{a}, N) \cong \mathrm{Tor}^R_{n+1}(R/\mathfrak{a}, N)$ for all $n \in \mathbb{N}$. Then since $\mathfrak{a}$ is finitely generated, thus $\mathfrak{a} \otimes_R N$ is Artinian and so $\mathrm{Tor}^R_1(R/\mathfrak{a}, N)$ is also Artinian. From the short exact sequence (*), we get the long exact sequence

$$\mathrm{Tor}^R_{n+1}(R/\mathfrak{p}_i, N) \to \mathrm{Tor}^R_n(M_{i-1}, N) \to \mathrm{Tor}^R_n(M_i, N) \to \mathrm{Tor}^R_n(R/\mathfrak{p}_i, N) \qquad (\eth)$$

for each $n \in \mathbb{N}_0$ and each $0 \leq i \leq k$. For $n = i = 1$, since $M_0 = 0$ and $\mathrm{Tor}^R_1(R/\mathfrak{p}_1, N)$ is Artinian, then $\mathrm{Tor}^R_1(M_1, N)$ is Artinian. Again, for $n = 1$ and $i = 2$, since $\mathrm{Tor}^R_1(M_1, N)$ and $\mathrm{Tor}^R_1(R/\mathfrak{p}_2, N)$ are Artinian, then $\mathrm{Tor}^R_1(M_2, N)$ is Artinian. Now, by repeating this process, $\mathrm{Tor}^R_1(M, N)$ is Artinian. Hence $\mathrm{Tor}^R_1(\mathfrak{a}, N)$ is Artinian and so $\mathrm{Tor}^R_2(R/\mathfrak{a}, N)$ is Artinian for all $\mathfrak{a} \in \mathscr{I}(R)$. Again, with the same method and using ($\eth$), we conclude that $\mathrm{Tor}_2^R(M,N)$ is also artinian. Now, continuing the above method and by induction, it is easy to show that $\mathrm{Tor}_n^R(M,N)$ is Artinian for all $n \geq 0$. □

Theorem 2.1. *Let M be an R-module. If $L_d(M)$ is an Artinian R-module, then $\mathrm{Hom}_R(R/\mathfrak{a}, M)$ is Artinian for all $\mathfrak{a} \in \Sigma$. Moreover, If $L_d(M)$ is a finitely generated R-module, then the converse is also true.*

Proof. First, let $L_d(M)$ be an Artinian module. By Lemma 2.1, $\mathrm{Hom}_R(R/\mathfrak{a}, L_d(M))$ is Artinian. Let $\mathfrak{a} \in \Sigma$. We have

$$\begin{aligned}
\mathrm{Hom}_R(R/\mathfrak{a}, L_d(M)) &= \mathrm{Hom}_R(R/\mathfrak{a}, \varinjlim_{\mathfrak{b}\in\Sigma} \mathrm{Hom}_R(R/\mathfrak{b}, M)) \\
&\cong \varinjlim_{\mathfrak{b}\in\Sigma} \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Hom}_R(R/\mathfrak{b}, M)) \\
&\cong \varinjlim_{\mathfrak{b}\in\Sigma} \mathrm{Hom}_R(R/\mathfrak{a} \otimes_R R/\mathfrak{b}, M) \\
&\cong \varinjlim_{\mathfrak{b}\in\Sigma} \mathrm{Hom}_R(R/\mathfrak{b} \otimes_R R/\mathfrak{a}, M) \\
&\cong \varinjlim_{\mathfrak{b}\in\Sigma} \mathrm{Hom}_R(R/\mathfrak{b}, \mathrm{Hom}_R(R/\mathfrak{a}, M)) \\
&= L_d(\mathrm{Hom}_R(R/\mathfrak{a}, M)).
\end{aligned}$$

Since $\mathfrak{a} \in \Sigma$ and $\mathfrak{a}.\mathrm{Hom}_R(R/\mathfrak{a}, M) = 0$, then

$$L_d(\mathrm{Hom}_R(R/\mathfrak{a}, M)) = \mathrm{Hom}_R(R/\mathfrak{a}, M).$$

Hence $\mathrm{Hom}_R(R/\mathfrak{a}, M)$ is Artinian.

For the next part, assume that $L_d(M) =< m_1, m_2, \cdots, m_t >$. Then for each $1 \leq i \leq t$, there exists $\mathfrak{a}_i \in \Sigma$ such that $\mathfrak{a}_i.m_i = 0$. Put $\mathfrak{b} = \prod_{i=1}^t \mathfrak{a}_i$. Clearly $\mathfrak{b} \in \Sigma$ and $\mathfrak{b}L_d(M) = 0$. Now, let $\mathrm{Hom}_R(R/\mathfrak{a}, M)$ be an Artinian R-module for all $\mathfrak{a} \in \Sigma$. Hence $(0 :_M \mathfrak{a})$ is an Artinian R-module for all $\mathfrak{a} \in \Sigma$. Assume that

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_i \supseteq M_{i+1} \supseteq \cdots$$

is a chain of the submodules of $L_d(M)$. Thus

$$(0 :_{M_1} \mathfrak{b}) \supseteq (0 :_{M_2} \mathfrak{b}) \supseteq \cdots \supseteq (0 :_{M_i} \mathfrak{b}) \supseteq (0 :_{M_{i+1}} \mathfrak{b}) \supseteq \cdots$$

is a chain of the submodules of $(0 :_M \mathfrak{b})$. Hence there exists $l \in \mathbb{N}$ such that $(0 :_{M_l} \mathfrak{b}) = (0 :_{M_{l+i}} \mathfrak{b})$ for all $i \geq 0$. We show that $M_l = M_{l+i}$ for all $i \geq 0$. For this, it is enough to prove that $M_l \subseteq M_{l+i}$. Let $x \in M_l$. Since M_l is a submodule of $L_d(M)$, then $\mathfrak{b}x = 0$ and so $x \in (0 :_{M_l} \mathfrak{b}) = (0 :_{M_{l+i}} \mathfrak{b}) \subseteq M_{l+i}$. Thus $x \in M_{l+i}$ and so $L_d(M)$ is Artinian. □

Theorem 2.2. *Let M be an R-module and $t \in \mathbb{N}_0$. Let $\mathfrak{a} \in \Sigma$ such that $\mathrm{Ext}_R^t(R/\mathfrak{a}, M)$ and $\mathrm{Ext}_R^j(R/\mathfrak{a}, H_d^i(M))$ be two Artinian modules for all $i < t$ and all $j \geq 0$. Also, let for any submodule N of $H_d^t(M)$, $\mathrm{Ext}_R^1(R/\mathfrak{a}, N)$ be an Artinian. Then $\mathrm{Hom}_R(R/\mathfrak{a}, H_d^t(M)/N)$ is Artinian.*

Proof. From the short exact sequence

$$0 \to N \to H_d^t(M) \to H_d^t(M)/N \to 0$$

we get the long exact sequence

$$\cdots \to \mathrm{Hom}_R(R/\mathfrak{a}, H_d^t(M)) \to \mathrm{Hom}_R(R/\mathfrak{a}, H_d^t(M)/N) \to \mathrm{Ext}_R^1(R/\mathfrak{a}, N) \to \cdots .$$

Since $\mathrm{Ext}_R^1(R/\mathfrak{a}, N)$ is Artinian, it suffices to prove that $\mathrm{Hom}_R(R/\mathfrak{a}, H_d^t(M))$ is Artinian. We do this by induction on t. Let $f \in \mathrm{Hom}_R(R/\mathfrak{a}, M/L_d(M))$. Then there exists $m \in M$ such that $f(1+\mathfrak{a}) = m + L_d(M)$. Since $\mathfrak{a}.f(1+\mathfrak{a}) = 0$, then $\mathfrak{a}m \subseteq L_d(M)$ and so there exists $\mathfrak{b} \in \Sigma$ such that $\mathfrak{b}\mathfrak{a}m = 0$. Hence $m \in L_d(M)$ and so $f(1+\mathfrak{a}) = 0$. Then $\mathrm{Hom}_R(R/\mathfrak{a}, M/L_d(M)) = 0$. Then from the exact sequence

$$0 \to \mathrm{Hom}_R(R/\mathfrak{a}, L_d(M)) \to \mathrm{Hom}_R(R/\mathfrak{a}, M) \to \mathrm{Hom}_R(R/\mathfrak{a}, M/L_d(M))$$

deduces $\operatorname{Hom}_R(R/\mathfrak{a}, L_d(M)) \cong \operatorname{Hom}_R(R/\mathfrak{a}, M)$. Now the result is obtained for $t = 0$. In the following, assume that $t > 0$ and the claim is true for $t-1$. From the long exact sequence

$$\cdots \to \operatorname{Ext}^t_R(R/\mathfrak{a}, M) \to \operatorname{Ext}^t_R(R/\mathfrak{a}, M/L_d(M)) \to \operatorname{Ext}^{t+1}_R(R/\mathfrak{a}, L_d(M)) \to \cdots, \quad (\sharp)$$

we conclude that $\operatorname{Ext}^t_R(R/\mathfrak{a}, M/L_d(M))$ is Artinian, as by assumption,

$$\operatorname{Ext}^t_R(R/\mathfrak{a}, M) \text{ and } \mathrm{Ext}^{t+1}_{\mathrm{R}}(\mathrm{R}/\mathfrak{a}, \mathrm{L_d(M)})$$

are Artinian. Assume that $E := E(M/L_d(M))$ is the injective hull of $M/L_d(M)$. Since $L_d(M/L_d(M)) = 0$ then $L_d(E) = 0$ and so $\operatorname{Hom}_R(R/\mathfrak{a}, E) = 0$. Let $K := E/(M/L_d(M))$. Thus from the short exact sequence

$$0 \to M/L_d(M) \to E \to K \to 0$$

we get

$$\operatorname{Ext}^i_R(R/\mathfrak{a}, K) \cong \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, M/L_d(M))$$

for all $\mathfrak{a} \in \Sigma$ and all $i \geq 0$. Passing to direct limit, we see that $H^i_d(K) \cong H^{i+1}_d(M/L_d(M))$. Hence by [11, Lemma 2.1(4)], $H^i_d(K) \cong H^{i+1}_d(M)$ for all $i \geq 0$. Then $\operatorname{Ext}^{t-1}_R(R/\mathfrak{a}, K)$ and also $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_d(K))$ are Artinian for all $i < t-1$ and $j \geq 0$. Hence K satisfies the inductive hypothesis and so $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_d(K))$ is Artinian. Therefore $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_d(M))$ is also Artinian. □

Theorem 2.3. *Let M be an R-module and $\mathfrak{a} \in \Sigma$ such that $\operatorname{Ext}^1_R(R/\mathfrak{a}, M)$ and $\operatorname{Ext}^2_R(R/\mathfrak{a}, L_d(M))$ be two Artinian modules. Then $\operatorname{Hom}_R(R/\mathfrak{a}, H^1_d(M))$ is Artinian.*

Proof. According to the assumption and by placing $t = 1$ in the long exact sequence ($\sharp$) in the proof of Theorem 2.3, $\operatorname{Ext}^1_R(R/\mathfrak{a}, M/L_d(M))$ is Artinian. Now, by [12, Remark 1(B)]

$$0 \to M/L_d(M) \to T_d(M) \to H^1_d(M) \to 0$$

is an exact sequence and so

$$0 \to \operatorname{Hom}_R(R/\mathfrak{a}, T_d(M)) \to \operatorname{Hom}_R(R/\mathfrak{a}, H^1_d(M)) \to \operatorname{Ext}^1_R(R/\mathfrak{a}, M/L_d(M)) \to \cdots$$

is a long exact sequence which deduces $\operatorname{Hom}_R(R/\mathfrak{a}, H^1_d(M))$ is Artinian, because

$$\begin{aligned}
\operatorname{Hom}_R(R/\mathfrak{a}, T_d(M)) &= \operatorname{Hom}_R(R/\mathfrak{a}, \varinjlim_{\mathfrak{b}\in\Sigma} \operatorname{Hom}_R(\mathfrak{b}, M)) \\
&\cong \varinjlim_{\mathfrak{b}\in\Sigma} \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Hom}_R(\mathfrak{b}, M)) \\
&\cong \varinjlim_{\mathfrak{b}\in\Sigma} \operatorname{Hom}_R(R/\mathfrak{a} \otimes_R \mathfrak{b}, M) \\
&\cong \varinjlim_{\mathfrak{b}\in\Sigma} \operatorname{Hom}_R(\mathfrak{b}, \operatorname{Hom}_R(R/\mathfrak{a}, M)) \\
&\cong T_d(\operatorname{Hom}_R(R/\mathfrak{a}, M)) = 0.
\end{aligned}$$

Note that $\mathfrak{a}.\operatorname{Hom}_R(R/\mathfrak{a}, M) = 0$ and so $L_d(\operatorname{Hom}_R(R/\mathfrak{a}, M)) = \operatorname{Hom}_R(R/\mathfrak{a}, M)$ which gives results $T_d(\operatorname{Hom}_R(R/\mathfrak{a}, M)) = 0$ by [12, Proposition 1]. □

3. Main Results

Let M and N be two R-modules and i be a non-negative integer. This section presents our central contributions concerning Artinian properties of d-transform and d-local cohomology modules. We establish new criteria for Artinianness in $T^i_d(M, N)$ and $H^i_d(M, N)$, extending classical finiteness results to the d-dimensional setting. Our theorems unify and generalize previous work, revealing deeper connections between homological conditions and Artinian behavior. The applications span various contexts, including flat modules and specific dimension bounds, offering fresh perspectives in commutative algebra.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a local ring. Assume that $0 \neq M$ is a finitely generated R-module such that $H^1_d(M)$ is Artinian. Then $H^{\dim_d(M)}_d(M)$ is Artinian R-module.*

Proof. We use induction on $n := \dim_d(M)$. If $n = 0$, then $\dim(M_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \Sigma \cap \operatorname{Spec}(R)$. Hence $\dim(M_{\mathfrak{m}}) = 0$ and so

$$\sqrt{(0 :_{R_{\mathfrak{m}}} M_{\mathfrak{m}})} = \mathfrak{m}R_{\mathfrak{m}} \Rightarrow \exists t \in \mathbb{N};\ (\mathfrak{m}R_{\mathfrak{m}})^t.M_{\mathfrak{m}} = 0.$$

Since R is a Noetherian ring and M is a finitely generated R-module, then $M_{\mathfrak{m}}$ is Noetherian $R_{\mathfrak{m}}$-module and so $M_{\mathfrak{m}}$ is an Artinian $R_{\mathfrak{m}}$-module. Hence M is an Artinian R-module and so $L_d(M)$ is Artinian too. Now suppose that $n \geq 2$ (by assumption $H^1_d(M)$ is Artinian) and the result is true for any non-zero finitely generated R-module X with $\dim_d(X) < n$. By virtue of [11, Lemma 2.1(4)], $H^i_d(M) \cong H^i_d(M/L_d(M))$ for all $i \geq 1$. Also $\dim_d(M/L_d(M)) \leq \dim_d(M)$. Now, if $\dim_d(M/L_d(M)) < n$, then by [1, p. 12], $H^n_d(M) \cong H^n_d(M/L_d(M)) = 0$ and the result is clear. Let $\dim_d(M/L_d(M)) = n$. Therefore, it can be assumed that $L_d(M) = 0$. Then $\mathfrak{a} \nsubseteq Z_R(M)$ for all $\mathfrak{a} \in \Sigma$. Hence $\mathfrak{m} \nsubseteq Z_R(M)$. Let $x \in \mathfrak{m} \setminus Z_R(M)$. From the short exact sequence

$$0 \to M \xrightarrow{x.} M \to M/xM \to 0$$

we get a long exact sequence

$$H^{n-1}_d(M/xM) \xrightarrow{f} H^n_d(M) \xrightarrow{x.} H^n_d(M) \to H^n_d(M/xM).$$

Now, we have

$$H^{n-1}_d(M/xM)/\mathrm{Ker} f \cong \mathrm{Im} f = \mathrm{Ker}(.x) = (0 \underset{H^n_d(M)}{:} x).$$

Since $\dim_d(M/xM) = \dim_d(M) - 1 = n-1$, then $H^{n-1}_d(M/xM)$ is Artinian by inductive hypothesis. Therefore $(0 \underset{H^n_d(M)}{:} x)$ is also Artinian. By [5, Corollary 3.11], $H^n_d(M)$ is a $\mathfrak{m}$-torsion and so it is also an Rx-torsion. Hence by [8, Theorem 1.3], $H^n_d(M)$ is Artinian. □

Theorem 3.2. *Let M be a finitely generated R-module and N be an R-module. Let $t \in \mathbb{N}$ and $H^i_d(N)$ is Artinian for all $i < t$. Then*
(1) $H^i_d(M, N)$ is Artinian for all $i < t$.
(2) $\mathrm{Ext}^i_R(R/\mathfrak{a}, N)$ is Artinian for all $i < t$ and for all $\mathfrak{a} \in \Sigma$.
(3) If $t \geq 2$, then for each $i < t-1$, $\mathrm{Ext}^i_R(M, N)$ is Artinian if and only if $T^i_d(M, N)$ is Artinian.
(4) If $t \geq 2$, then for each $i < t-1$, $T^i(R/\mathfrak{a}, N)$ is Artinian for all $\mathfrak{a} \in \Sigma$.

Proof. (1) We prove by induction on t. Let $t = 1$. By [11, Lemma 2.1(1)] we have

$$L_d(M, N) \cong \mathrm{Hom}_R(M, L_d(N)).$$

Since $L_d(N)$ is Artinian, then by Lemma 2.1, $L_d(M, N)$ is also Artinian. Let $t > 1$ and the result holds for $t-1$ and for any R-module X. Assume that $E(N)$ is the injective hull of N. From the short exact sequence

$$0 \to N \to E(N) \to E(N)/N \to 0, \qquad (\dagger)$$

we get the long exact sequences

$$\cdots \to H^i_d(E(N)) \to H^i_d(E(N)/N) \to H^{i+1}_d(N) \to H^{i+1}_d(E(N)) \to \cdots$$

and

$$\cdots \to H^i_d(M, E(N)) \to H^i_d(M, E(N)/N) \to H^{i+1}_d(M, N) \to H^{i+1}_d(M, E(N)) \to \cdots.$$

Since $E(N)$ is an injective R-module, then $H^i_d(E(N)) = \varinjlim_{\mathfrak{a} \in \Sigma} \mathrm{Ext}^i_R(R/\mathfrak{a}, E(N)) = 0$ and $H^i_d(M, E(N)) = \varinjlim_{\mathfrak{a} \in \Sigma} \mathrm{Ext}^i_R(M/\mathfrak{a}M, E(N)) = 0$ for all $i \geq 1$ and so $H^i_d(E(N)/N)$
$\cong H^{i+1}_d(N)$ and $H^i_d(M, E(N)/N) \cong H^{i+1}_d(M, N)$ for all $i \in \mathbb{N}$. Since $H^i_d(N)$ is Artinian for all $i < t$, then $H^i_d(E(N)/N)$ is also Artinian for all $i < t-1$ and so by induction hypothesis, $H^i_d(M, E(N)/N)$ is Artinian for all $i < t-1$. Therefore $H^i_d(M, N)$ is also Artinian for all $i < t$.
(2) We prove by induction on t. Let $t = 1$ and $\mathfrak{a} \in \Sigma$. Then by the assumption, $L_d(N)$ is Artinian and so by Theorem 2.2, $\mathrm{Hom}_R(R/\mathfrak{a}, N)$ is Artinian. The proof for $t > 1$ is similar to that of (1). Note

that $\mathrm{Ext}^i_R(R/\mathfrak{a}, E(N)/N) \cong \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, N)$ for all $i \geq 1$, which is obtained from (†).
(3) Let $\mathfrak{a} \in \Sigma$. Consider the short exact sequence

$$0 \to \mathfrak{a}M \to M \to M/\mathfrak{a}M \to 0.$$

Then the sequence

$$\mathrm{Ext}^i_R(M/\mathfrak{a}M, N) \to \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(\mathfrak{a}M, N) \to \mathrm{Ext}^{i+1}_R(M/\mathfrak{a}M, N)$$

is exact. Hence passing the direct limit we get the long exact sequence

$$H^i_d(M, N) \to \mathrm{Ext}^i_R(M, N) \to T^i_d(M, N) \to H^{i+1}_d(M, N). \qquad (\sharp\sharp)$$

Now, by part (1), the result is obtained immediately.
(4) This part is the result of parts (2) and (3). □

Theorem 3.3. *Let M be a finitely generated R-module and N be an R-module. Let $t \in \mathbb{N}$ and $T^i_d(N)$ is Artinian for all $i < t$. Then $T^i_d(M, N)$ is Artinian for all $i < t$.*

Proof. We prove by induction on t. If $t = 1$ then by [6, Theorem 3.1(2)] we have

$$T_d(M, N) \cong \mathrm{Hom}_R(M, T_d(N)).$$

By the assumption, $T_d(N)$ is Artinian, then by Lemma 2.1, $T_d(M, N)$ is also Artinian. Let $t > 1$ and the result holds for $t-1$ and for any R-module X. Assume that $E(N)$ is the injective hull of N. Then from the short exact sequence (†), we get the long exact sequences

$$T^i_d(E(N)) \to T^i_d(E(N)/N) \to T^{i+1}_d(N) \to T^{i+1}_d(E(N))$$

and

$$T^i_d(M, E(N)) \to T^i_d(M, E(N)/N) \to T^{i+1}_d(M, N) \to T^{i+1}_d(M, E(N)).$$

Using [12, Remark(B)], $T^i_d(L) \cong H^{i+1}_d(L)$ for any R-module L and all $i \geq 1$. Then for each $i \geq 1$, $T^i_d(E(N)) \cong H^{i+1}_d(E(N)) = 0$ and so $T^i_d(E(N)/N) \cong T^{i+1}_d(N)$. Similarly $T^i_d(M, E(N)) = 0$ for all $i \geq 1$ and so $T^i_d(M, E(N)/N) \cong T^{i+1}_d(M, N)$ for all $i \in \mathbb{N}$. Since $T^i_d(N)$ is Artinian for all $i < t$, then $T^i_d(E(N)/N)$ is also Artinian for all $i < t-1$ and so by induction hypothesis, $T^i_d(M, E(N)/N)$ is Artinian for all $i < t-1$. Therefore $T^i_d(M, N)$ is also Artinian for all $i < t$. □

Theorem 3.4. *Let M and N be two finitely generated R-modules. Assume that $t \in \mathbb{N}$ such that $H^t_d(M, R/\mathfrak{p})$ is Artinian for all $\mathfrak{p} \in \mathrm{Supp}(N)$. Then $H^t_d(M, N)$ is also Artinian.*

Proof. Since N is finitely generated, then by [9, Lemma 8.49], there exists a chain of the submodules of N

$$0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_k = N$$

such that for each $1 \leq i \leq k$, $N_i/N_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Supp}(N)$. For each $1 \leq i \leq k$, from the short exact sequence

$$0 \to N_{i-1} \to N_i \to R/\mathfrak{p}_i \to 0,$$

we get the long exact sequence

$$H^t_d(M, N_{i-1}) \to H^t_d(M, N_i) \to H^t_d(M, R/\mathfrak{p}_i) \to H^{t+1}_d(M, N_{i-1}). \qquad (**)$$

We have $H^t_d(M, N_1) \cong H^t_d(M, R/\mathfrak{p}_1)$, then $H^t_d(M, N_1)$ is Artinian. Also, the long exact sequence

$$\cdots \to H^t_d(M, N_1) \to H^t_d(M, N_2) \to H^t_d(M, R/\mathfrak{p}_2) \to \cdots,$$

deduces $H^t_d(M, N_2)$ is Artinian, because $H^t_d(M, N_1)$ and $H^t_d(M, R/\mathfrak{p}_2)$ are Artinian. Now by continuing this process and using the long exact sequence (**), it follows that $H^t_d(M, N)$ is Artinian. □

Corollary. *Let M and N be two finitely generated R-modules. Assume that $t \in \mathbb{N}$ such that $H^t_d(M, R/\mathfrak{p})$ is Artinian for all $\mathfrak{p} \in \mathrm{Supp}(N)$.*
(1) If K is finitely generated R-module such that $\mathrm{Supp}(K) \subseteq \mathrm{Supp}(N)$, then $H^t_d(M, K)$ is Artinian.
(2) If $\mathfrak{a}$ is an ideal of R and $\mathrm{V}(\mathfrak{a}) \subseteq \mathrm{Supp}(N)$, then $H^t_d(M, R/\mathfrak{a})$ is Artinian.

Proof. (1) Since K is finitely generated, then there exists a chain of the submodules of K

$$0 = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_l = K$$

such that for each $1 \leq i \leq l$, $K_i/K_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Supp}(K)$. For each $1 \leq i \leq l$, the sequence

$$0 \to K_{i-1} \to K_i \to R/\mathfrak{p}_i \to 0,$$

is exact. From $K_1 \cong R/\mathfrak{p}_1$, we have $H^t_d(M, K_1) \cong H^t_d(M, R/\mathfrak{p}_1)$. Since $\mathfrak{p}_1 \in \operatorname{Supp}(N)$, then $H^t_d(M, R/\mathfrak{p}_1)$ is Artinian and so $H^t_d(M, K_1)$ is also Artinian. Now, continuing the process of the Theorem 3.4, $H^t_d(M, K)$ is Artinian.
(2) Let $\mathfrak{p} \in \operatorname{Supp}(R/\mathfrak{a})$. we have

$$\begin{aligned}
&(R/\mathfrak{a})_{\mathfrak{p}} \neq 0 \Rightarrow R_{\mathfrak{p}} \neq \mathfrak{a}R_{\mathfrak{p}} \\
&\Rightarrow \mathfrak{a} \subseteq \mathfrak{p} \Rightarrow \mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \\
&\Rightarrow \operatorname{Supp}(R/\mathfrak{a}) \subseteq \mathrm{V}(\mathfrak{a}) \subseteq \operatorname{Supp}(N).
\end{aligned}$$

Now the result is obtained from part (1). □

Corollary. *Let M be a finitely generated R-module. Assume that $t \in \mathbb{N}$ such that $H^t_d(M, R/\mathfrak{p})$ is Artinian for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $H^t_d(M, K)$ is also Artinian for any finitely generated R-module K.*

Proof. Put $N := R$ in Corollary 3.5(1). □

Theorem 3.5. *Let M and N be two finitely generated R-modules. Assume that $t \in \mathbb{N}$ such that $T^t_d(M, R/\mathfrak{p})$ is Artinian for all $\mathfrak{p} \in \operatorname{Supp}(N)$. Then $T^t_d(M, N)$ is also Artinian.*

Proof. The proof is similar to Theorem 3.4. □

Theorem 3.6. *Let M and N be two finitely generated R-modules. Let M be a flat R-module and $n \geq 1$ be an integer. Suppose that $H^n_d(M, R/\mathfrak{p})$ is Artinian for all $\mathfrak{p} \in \operatorname{Supp}(N)$. Then $H^i_d(M, N)$ is Artinian for all $i \geq n$.*

Proof. First, we show that $H^n_d(M, N)$ is Artinian. Similar to the process of proving Theorem 3.4 and by keeping the signs and relations mentioned in it, we get the long exact sequence

$$\cdots \to H^{n-1}_d(M, R/\mathfrak{p}_i) \to H^n_d(M, N_{i-1}) \to H^n_d(M, N_i) \to H^n_d(M, R/\mathfrak{p}_i) \to \cdots$$

for all $1 \leq i \leq k$. By the assumption and induction on i, $H^n_d(M, N)$ is an Artinian module. Now, we show that $H^i_d(M, N)$ is Artinian for all $i > n$, it is enough to prove that $H^{n+1}_d(M, R/\mathfrak{p})$ is an Artinian module for all $\mathfrak{p} \in \operatorname{Supp}(N)$. Assume that $\mathfrak{p} \in \operatorname{Supp}(N)$ and $H^{n+1}_d(M, R/\mathfrak{p}) \neq 0$. Let $\mathfrak{a} \in \Sigma$ and $\mathfrak{a} \subseteq \mathfrak{p}$. Then $L_d(R/\mathfrak{p}) = R/\mathfrak{p}$ and so by [11, Lemma 2.1(3)], $H^{n+1}_d(M, R/\mathfrak{p}) = \operatorname{Ext}^{n+1}_R(M, R/\mathfrak{p})$. Now, since M is flat, then by [9, Corollary 3.57], M is projective too and so $H^{n+1}_d(M, R/\mathfrak{p}) = 0$, which is a contradiction. Hence $\mathfrak{a} \nsubseteq \mathfrak{p}$. Then there exists $x \in \mathfrak{a} \setminus \mathfrak{p}$. From the short exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{x.} R/\mathfrak{p} \to R/(\mathfrak{p} + Rx) \to 0,$$

we get the long exact sequence

$$H^n_d(M, R/(\mathfrak{p} + Rx)) \xrightarrow{\alpha} H^{n+1}_d(M, R/\mathfrak{p}) \xrightarrow{\beta} H^{n+1}_d(M, R/\mathfrak{p}).$$

Sinse $(0 :_R N) \subseteq \mathfrak{p} \subseteq \mathfrak{p} + Rx$, then $\operatorname{Supp}(R/(\mathfrak{p} + Rx)) \subseteq \operatorname{Supp}(N)$ and so by the first paragraph and Corollary 3.5(1), $H^n_d(M, R/(\mathfrak{p} + Rx))$ is Artinian. From the exact sequence of the latter, we conclude that

$$\begin{aligned}
H^n_d(M, R/(\mathfrak{p} + Rx))/\operatorname{Ker}\alpha &\cong \operatorname{Im}\alpha \\
&\cong \operatorname{Ker}\beta \\
&= (0 :_{H^{n+1}_d(M, R/\mathfrak{p})} x)
\end{aligned}$$

and so $(0 :_{H^{n+1}_d(M, R/\mathfrak{p})} x)$ is also Artinian. On the other hand, by [11, Lemma 2.1(1)], $L_d(M, N) \cong L_d(\operatorname{Hom}_R(M, N))$. Assume that E is an injective module. Clearly $\mathscr{R}^i L_d(M, E) = H^i_d(M, E) = 0$ for all $i \geq 1$. Also, since M is flat, then by [9, Corollary 3.57], it is projective and so by [9, Corollary 10.65]

$$\mathrm{Ext}_R^n(A, \mathrm{Hom}_R(M, E)) \cong \mathrm{Ext}_R^n(M \otimes_R A, E) = 0$$

for all R-modules A and all $n \geq 1$. Hence $\mathrm{Hom}_R(M, E)$ is an injective R-module and so

$$\mathscr{R}^i L_d(\mathrm{Hom}_R(M, E)) = H_d^i(\mathrm{Hom}_R(M, E)) = 0$$

for all $i \geq 1$. Then by an standard argument in homology (see [2, Exercise 1.3.4]), we have $H_d^i(M, N) \cong H_d^i(\mathrm{Hom}_R(M, N))$ for all $i \geq 0$. Hence

$$H_d^{n+1}(M, R/\mathfrak{p}) \cong H_d^{n+1}(\mathrm{Hom}_R(M, R/\mathfrak{p})).$$

Now, since $H_d^{n+1}(\mathrm{Hom}_R(M, R/\mathfrak{p}))$ is $\mathfrak{m}$-torsion by [5, Corollary 3.11] and so it is also Rx-torsion. Then $H_d^{n+1}(M, R/\mathfrak{p})$ is Rx-torsion and therefore by [8, Theorem 1.3], it is Artinian. □

Corollary. *Let N be a finitely generated R-module. Assume that $n \geq 1$ is an integer and $H_d^n(M, R/\mathfrak{p})$ is Artinian for all $\mathfrak{p} \in \mathrm{Supp}(N)$. Then $H_d^i(N)$ is Artinian for all $i \geq n$.*

Proof. In Theorem 3.8, by putting R instead of M, the result is obtained. Note that R is projective R-module and so it is flat. □

Theorem 3.7. *Let M and N be two finitely generated R-modules. Let M be a flat R-module and $n \geq 1$ be a positive integer.*
(1) If $T_d^n(M, R/\mathfrak{p})$ is Artinian for all $\mathfrak{p} \in \mathrm{Supp}(N)$, then $T_d^i(M, N)$ is also Artinian for all $i \geq n$.
(2) If $T_d^n(M, R/\mathfrak{p})$ and $H_d^n(M, R/\mathfrak{p})$ are Artinian modules for all $\mathfrak{p} \in \mathrm{Supp}(N)$, then $\mathrm{Ext}_R^i(M, N)$ is also Artinian for all $i \geq n$.

Proof. (1) The proof process is similar to the proof of Theorem 3.8. Here, by [6, Theorem 3.1(2)], $T_d(M, N) \cong \mathrm{Hom}_R(M, T_d(N))$ and if E is an injective R-module, then by [12, Remark 1(B)], $E/L_d(E) \cong T_d(E)$ and so by [11, Lemma 2.1(2)], $T(E)$ is injective. Hence the same argument as in Theorem 3.8, $\mathrm{Hom}_R(M, T_d(E))$ is injective.
(2) From the long exact sequence (♯♯) in proof of Theorem 3.2(3), Theorem 3.9 and part (1), the conclusion follows. □

Theorem 3.8. *Let M be a finitely generated R-module and N be an Artinian R-module. Then $H_d^i(M, N)$ is Artinian for all $i \geq 0$.*

Proof. We use induction on i. If $i = 0$, then by [11, Lemma 2.1(1)], $H_d^0(M, N) = \mathrm{Hom}_R(M, L_d(N))$. Since N is Artinian, then by Lemma 2.1, $\mathrm{Hom}_R(M, L_d(N))$ is Artinian and so $H_d^0(M, N)$ is also Artinian. Assume that $i > 0$ and $E(N)$ is the injective hull of N. Then $E(N)$ is Artinian, as N is Artinian. Now, by the same process of proving Theorem 3.2

$$H_d^{i-1}(M, E(N)/N) \cong H_d^i(M, N)$$

for all $i > 1$, as $H_d^i(M, E(N)) = 0$ for all $i \geq 1$. By inductive hypothesis, $H_d^{i-1}(M, E(N)/N)$ is Artinian, therfore $H_d^i(M, N)$ is also Artinian. □

Corollary. *Let M be a finitely generated R-module.*
(1) If N is an Artinian R-module, then $T_d^i(M, N)$ is Artinian for all $i \geq 0$.
(2) If N is a finitely generated R-module such that $\mathrm{pd}(M) + \dim(N) < \infty$, then $T_d^{\mathrm{pd}(M)+\dim(N)}(M, N)$ is Artinian.

Proof. (1) By the long exact sequence (♯♯) in proof of Theorem 3.2(3), Theorem 3.11 and Lemma 2.1 the result is complete.
(2) First, let $\dim(N) = 0$. Then N is Artinian and so by Lemma 2.1, $\mathrm{Ext}_d^i(M, N)$ is Artinian for all $i \geq 0$. Since $\mathrm{Ext}_d^i(M, N) = 0$ for all $i \geq \mathrm{pd}(M) + 1$, then by the exact sequence (♯♯),

$$\mathrm{Ext}_R^{\mathrm{pd}(M)}(M, N) \to T_d^{\mathrm{pd}(M)}(M, N) \to H_d^{\mathrm{pd}(M)+1}(M, N) \to 0$$

is an exact sequence and so by Theorem 3.11, $T_d^{\mathrm{pd}(M)}(M,N)$ is Artinian. Now, if $\dim(N) > 0$, hence by [12, Remark 1(B)]

$$T_d^{\mathrm{pd}(M)+\dim(N)}(M,N) \cong H_d^{\mathrm{pd}(M)+\dim(N)+1}(M,N)$$

and this finishes the proof by Theorem 3.11. □

Corollary. *Let N be an Artinian R-module. Then $H_d^i(N)$ and $T_d^i(N)$ are Artinian for all $i \geq 0$.*

Proof. In Theorem 3.11 and Corollary 3.12, use R instead of M. □

Theorem 3.9. *Let M, N and K be three finitely generated R-modules such that $pd(M) < \infty$ and $\mathrm{Supp}(K) \subseteq \mathrm{Supp}(N)$. If $H_d^i(M,N)$ is Artinian for all $i \geq t$, then $H_d^i(M,K)$ is Artinian for all $i \geq t$.*

Proof. By [11, Theorem 4.1], $H_d^i(M,N) = 0$ for all $i > pd(M) + \dim_d(N)$. Clearly $\dim_d(K) \subseteq \dim_d(N)$. We prove the result by using the descending induction on $i = t, t+1, \cdots, pd(M)+\dim_d(N)+1$. If $i = pd(M) + \dim_d(N) + 1$, then the result is complete. Hence we may assume that $t \leq i \leq pd(M)+\dim_d(N)$. Now assume that the result is true for $i+1$ and we prove it for i. By [13, Theorem 4.1], there is chain

$$0 = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_s = K$$

such that the factors K_j/K_{j-1} are homomorphic images of direct sum of finitely many copies of N. By using short exact sequences

$$0 \to K_{j-1} \to K_j \to K_j/K_{j-1} \to 0,$$

we can conclude that there is a short exact sequence

$$0 \to L \to N^m \to K \to 0$$

for some positive integer m and some finitely generated R-module L. Then we obtain a long exact sequence

$$H_d^i(M,N^m) \to H_d^i(M,K) \to H_d^{i+1}(M,L).$$

Since $\mathrm{Supp}(L) \subseteq \mathrm{Supp}(N)$, then by induction hypothesis, $H_d^{i+1}(M,L)$ is Artinian. On the other hand, $H_d^i(M,N^m)$ is Artinian, because $H_d^i(M,N^m) \cong H_d^i(M,N)^m$. Therefore $H_d^i(M,K)$ is also Artinian. □

Corollary. *Let M and N be two finitely generated R-modules and $pd(N) \leq t$. Then the following statements are equivalent:*
(1) $H_d^i(M,R/\mathfrak{p})$ is Artinian for all $\mathfrak{p} \in \mathrm{Supp}(N)$ and all $i \geq t$.
(2) $H_d^i(M,N)$ is Artinian for all $i \geq t$.

Proof. (1) ⇒ (2) This part is obtained from Theorem 3.4.
(2) ⇒ (1) Let $\mathfrak{p} \in \mathrm{Supp}(N)$. Then $\mathrm{Supp}(R/\mathfrak{p}) \subseteq \mathrm{Supp}(N)$ and so the result is deduced from Theorem 3.14. □

Theorem 3.10. *Let M and N be two R-modules and t be a non-negative integer such that*

$$\mathrm{Ext}_R^0(N,H_d^t(M)), \mathrm{Ext}_R^1(N,H_d^{t-1}(M)), \cdots, \mathrm{Ext}_R^t(N,H_d^0(M))$$

are artinan. Then, $H_d^t(N,M)$ is Artinian.

Proof. Let $G(-) = L_d(-)$ and $F(-) = \mathrm{Hom}_R(N,-)$ be two functors from the category of R-modules to itself. By [11, Lemma 2.1], $FG(-) = L_d(N,-)$ and F is the left exact functor. Also, for each injective R-module E, by [11, Lemma 2.1(2)], $L_d(E)$ is an injective R-module, then

$$\mathscr{R}^i F(G(E)) = \mathscr{R}^i \mathrm{Hom}_R(M, L_d(E)) = \mathrm{Ext}_R^i(M, L_d(E)) = 0$$

for all $i \geq 1$. Now, by [9, Theorem 10.47], there exists a Grothendieck spectral sequence

$$\mathrm{E}_2^{p,q} := \mathrm{Ext}_R^p(N, H_d^q(M)) \underset{p}{\Rightarrow} H_d^{p+q}(N,M).$$

For all i, $0 \leq i \leq t$, we have $E_\infty^{t-i,i} = E_{t+2}^{t-i,i}$ since $E_j^{t-i-j,i+j-1} = 0 = E_j^{t-i+j,i+1-j}$ for all $j \geq t+2$, so that $E_\infty^{t-i,i}$ is Artinian from the fact that $E_{t+2}^{t-i,i}$ is a subquotient of $E_2^{t-i,i}$, which is Artinian by assumption. There exists a finite filtration

$$0 = \phi^{t+1}H^t \subseteq \phi^t H^t \subseteq ... \subseteq \phi^1 H^t \subseteq \phi^0 H^t = H_d^t(N, M)$$

such that $E_\infty^{t-i,i} = \phi^{t-i}H^t/\phi^{t-i+1}H^t$, for all i, $0 \leq i \leq t$. Now, the exact sequences

$$0 \to \phi^{t-i+1}H^t \to \phi^{t-i}H^t \to E_\infty^{t-i,i} \to 0,$$

for all i, $0 \leq i \leq t$, yield the assertion. □

References

1. C. Banica and M. Soia, *Singular sets of a module on local cohomology,* Boll. Un. Mat. Ital. B **16** (1976), 923-934.
2. M. P. Brodmann and R. Y. Sharp, *Local Cohomology- An Algebraic Introduction with Geometric Applications,* Cambridge University Press, 1998.
3. L. Chu and Q. Wang, *Some results on local cohomology modules defined by a pair of ideals,* J. Math. Kyoto Univ. **49** (2009) 193-200.
4. L. Z. Chu and Z. M. Tang, *On the Artinianness of generalized local cohomology,* Comm. Algebra **35** (2007), 3821-3827.
5. A. ghafari, N. Zamani and M. S. Sayedsadeghi, $(d, \mathfrak{b})$*-injectivity and some ideal transforms,* submitted.
6. A. Makhsoos, N. Zamani and M. S. Sayedsadeghi, *some d-transform functors based on an ideal,* submitted.
7. H. Matsumura, *Commutative ring theory,* Cambridge University Press, 1986.
8. L. Melkersson, *On asymptotic stability for sets of prime ideals connected with the powers of an ideal,* Math. Proc. Cambridge Philos. Soc. **107** (1990) 267–271.
9. J. Rotman, *Introduction to homological algebra,* Academic Press, 1979.
10. N. Zamani, *Generalized local cohomology relative to (I,J),* Southeast Asian Bull. Math. **35** (2011) 1045-1050.
11. N. Zamani, M. H. Bijan-zadeh and M. S. Sayedsadeghi, *cohomology with supports of dimension* $\leq d$, Journal of Algebra and Its Applications, **15** (2016), 1650042(1)-1650051(10).
12. N. Zamani, M. S. Sayedsadeghi, M. H. Bijan-Zadeh, *d-Transform Functor and Some Finiteness and Isomorphism Results.* Vie. J. Math. **41** (2013), 179-186.
13. W. Vasconcelos, *Divisor theory in module categories.* Amsterdam: North-Holand, 1974.

(Received ??.??.20??)

[1]Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-4697, Tehran, Iran.

[2]Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-4697, Tehran, Iran.

[3]Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-4697, Tehran, Iran.
Email address: m_sayedsadeghi@pnu.ac.ir
Email address: my.sadeghi@pnu.ac.ir
Email address: f.rafiei@student.pnu.ac.ir