SOLVABILITY AND WELL-POSEDNESS OF THE CAUCHY–NICOLETTI WEIGHTED PROBLEM FOR NONLINEAR SINGULAR FUNCTIONAL DIFFERENTIAL SYSTEMS

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Abstract. For nonlinear singular functional differential systems, the unimprovable sufficient conditions for the solvability and well-posedness of the Cauchy–Nicoletti weighted problem are established.

Let $-\infty < a < b < +\infty$, $t_i \in [a, b]$ $(i = 1, \ldots, n)$, $\varphi = (\varphi_i)_{i=1}^n : [a, b] \to \mathbb{R}^n$ be a continuous vector function such that

$$\varphi_i(t_i) = 0, \quad \varphi_i(t) > 0 \ \text{for} \ t \neq t_i \ (i = 1, \ldots, n),$$

and let $C_\varphi([a, b]; \mathbb{R}^n)$ be the space of continuous vector functions $x = (x_i)_{i=1}^n : [a, b] \to \mathbb{R}^n$, satisfying the condition

$$\|x\|_{C_\varphi} = \sum_{i=1}^n \|x_i\|_{C_{\varphi_i}} < +\infty,$$

where

$$\|x_i\|_{C_{\varphi_i}} = \sup\left\{ \frac{|x_i(t)|}{\varphi_i(t)} : \ a \leq t \leq b, \ t \neq t_i \right\} \ (i = 1, \ldots, n).$$

Let, moreover, $I_k = [a, b] \setminus \{t_k\}$ $(k = 1, \ldots, n)$ and let $L_{\mathrm{loc}}(I_k; \mathbb{R})$ be the space of functions $u : I_k \to \mathbb{R}$, Lebesgue integrable on every closed interval, contained in I_k.

We consider the nonlinear functional differential system

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n)(t) \ (i = 1, \ldots, n) \tag{1}$$

with the weighted boundary conditions

$$\limsup_{t \to t_i} \frac{|x_i(t)|}{\varphi_i(t)} < +\infty \ (i = 1, \ldots, n). \tag{2}$$

Here, the operators $f_k : C_\varphi([a, b]; \mathbb{R}^n) \to L_{\mathrm{loc}}(I_k; \mathbb{R})$ $(k = 1, \ldots, n)$ are such that:

(i) for any $\rho > 0$, the condition

$$f_k^*(\varphi, \rho) \in L_{\mathrm{loc}}(I_k; \mathbb{R}) \ (k = 1, \ldots, n)$$

is satisfied, where

$$f_k^*(\varphi, \rho)(t) = \sup\left\{ |f_k(x_1, \ldots, x_n)(t)| : \ \|(x_i)_{i=1}^n\|_{C_\varphi} \leq \rho \right\};$$

(ii) for any $\rho > 0$, $k \in \{1, \ldots, n\}$, $[a_0, b_0] \subset I_k$, and for uniformly converging sequence of continuous vector functions $(x_{im})_{i=1}^n : [a, b] \to \mathbb{R}^n$ $(m = 1, 2, \ldots)$ satisfying the conditions

$$\|(x_{im})_{i=1}^n\|_{C_\varphi} \leq \rho \ (m = 1, 2, \ldots), \tag{3}$$
$$\lim_{m \to \infty} (x_{im}(t))_{i=1}^n = (x_i(t))_{i=1}^n \ \text{for} \ t \in [a, b],$$

we have

$$\lim_{m \to \infty} \int_{a_0}^t f_k(x_{1m}, \ldots, x_{nm})(s) \, ds = \int_{a_0}^t f_k(x_1, \ldots, x_n)(s) \, ds \ \text{uniformly on} \ [a_0, b_0].$$

From (2), we have the conditions

$$x_i(t_i) = 0 \quad (i = 1, \ldots, n).$$

Problem (1), (2) is known as the Cauchy–Nicoletti problem (see, e.g., [1, 2, 4, 6, 7]).

Thus we call (1), (2) the Cauchy–Nicoletti weighted problem.

Everywhere below, in addition to the above-introduced notation, we use the following notation:

$X = (x_{ik})_{i,k=1}^n$ is the $n \times n$ matrix with components x_{ik} $(i, k = 1, \ldots, n)$;

$r(x)$ is the spectral radius of an $n \times n$ matrix;

$[x]_+$ is the positive part of a real number x, i.e.,

$$[x]_+ = \frac{x + |x|}{2}.$$

Theorem 1. *Let there exist a positive number ρ such that for any $\lambda \in [0, 1]$ and $\delta > 0$ every solution of problem (1), (2) admits estimate (3), then problem (1), (2) has at least one solution.*

Theorem 2. *Let there exist nonnegative numbers p_{ik} $(i, k = 1, \ldots, n)$ and a function $q : [0, +\infty[\to [0, +\infty[$ such that*

$$r(p) < 1, \quad \text{where } p = (p_{ik})_{i,k=1}^n, \tag{4}$$

$$\lim_{s \to +\infty} \frac{q(s)}{s} = 0,$$

and for any $(x_i)_{i=1}^n \in C_\varphi([a, b]; \mathbb{R}^n)$, the inequalities

$$\left| \int_{t_i}^t \left[f_i(x_1, \ldots, x_n)(s) \operatorname{sgn}((s - t_i) x_i(s)) \right]_+ ds \right|$$

$$\leq \varphi_i(t) \Big(\sum_{k=1}^n p_{ik} \|x_k\|_{G_{\varphi_k}} + q\Big(\sum_{k=1}^n \|x_k\|_{C_{\varphi_k}} \Big) \Big) \quad (i = 1, \ldots, n) \tag{5}$$

are satisfied in the interval $[a, b]$. Then problem (1), (2) has at least one solution.

Definition. *Problem (1), (2) is said to be well-posed if:*

(i) *it has a unique solution $(x_i)_{i=1}^n$;*

(ii) *there exits a positive constant ρ such that for any integrable functions $h_i :]a, b[\to \mathbb{R}$ $(i = 1, \ldots, n)$, satisfying the conditions*

$$\nu_k(h_k) = \sup \left\{ \frac{1}{\varphi_k(t)} \left| \int_{t_k}^t |h_k(s)| \, ds \right| : \ a \leq t \leq b, \ t \neq t_k \right\} < +\infty,$$

the problem

$$\frac{dy_i(t)}{dt} = f_i(y_1, \ldots, y_n)(t) + h_i(t) \quad (i = 1, \ldots, n),$$

$$\limsup_{t \to t_i} \frac{|y_i(t)|}{\varphi_i(t)} < +\infty \quad (i = 1, \ldots, n)$$

has at least one solution and every such solution admits the estimate

$$\|(y_i - x_i)_{i=1}^n\|_{C_\varphi} \leq \rho \sum_{k=1}^n \nu_k(h_k).$$

Theorem 3. *Let there exist nonnegative numbers p_{ik} $(i, k = 1, \ldots, n)$ satisfying conditions (4) such that for any $(x_i)_{i=1}^n \in C_\varphi([a, b]; \mathbb{R}^n)$, in the interval $[a, b]$, the inequalities*

$$\left| \int_{t_i}^t \left[f_i(x_1, \ldots, x_n)(s) \operatorname{sgn}((s - t_i) x_i(s)) \right]_+ ds \right| \leq \varphi_i(t) \sum_{k=1}^n p_{ik} \|x_k\|_{G_{\varphi_k}} \quad (i = 1, \ldots, n)$$

are fulfilled. Then problem (1), (2) is well-posed.

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