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GROWTH OF LEBESGUE CONSTANTS

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Abstract. Let

$$\Big\{k \in \mathbb{Z}^n : 1 \le \prod_{j=1}^{n} |k_j|^{\gamma_j} \le R^{\gamma_1+\dots+\gamma_n}\Big\},$$

where $\gamma_1, ..., \gamma_n > 0$, be a "hyperbolic cross" dilated homothetically as $R \to +\infty$. In our study, their Lebesgue constants are, as expected, always of power growth R^p, $p > 0$, sometimes times a logarithmic factor. Surprising is that contrary to the expected $p = \frac{n-1}{2}$ in any case, p may become, for an appropriate choice of $\gamma_1, ..., \gamma_n$, arbitrarily larger than that fraction. In many cases, the estimates of the Lebesgue constants are sharp in the sense that those from above and from below differ from one another only by coefficients.

1. Introduction

Behavior of the partial sums of Fourier expansions is an important ingredient of research in many problems. On the other hand, it is an independent topic of harmonic analysis. The norms of corresponding operators are called the Lebesgue constants. They are analytic characteristics in the studies of convergence and summability of Fourier series, approximation and interpolation, and many other problems; see [1, 3, 5, 13, 15, 25] to refer to some sources. Frequently, for instance, in the study of uniform convergence, such operators are considered in C (or in L^1, the Lebesgue constants are the same in these spaces). Everything is clear in dimension one: for the R-th partial sum the norm differs from $\frac{4}{\pi^2}\ln R$ by a bounded value (see, e.g., [27, Sect. 2.12]). In the multivariate setting things become (much) more complicated. The main reason is obvious: unlike the one-dimensional case, there are different ways of ordering the partial sums in higher dimension. These lead to different types of convergence or divergence, for example, in cubes, spheres or polyhedra, and the difference may be drastic. Very often a partial sum of the Fourier series is generated by a finite set $B \cap \mathbb{Z}^n$:

$$S_B(f) = S_B(f;x) := \sum_{k\in B\cap\mathbb{Z}^n} \widehat{f}_k e^{i\langle k,x\rangle},$$

where $f \in L^1(\mathbb{T}^n)$, $\mathbb{T}^n = [-\pi,\pi)^n$, $\langle k,x\rangle = k_1x_1 + ... + k_nx_n$, and $\widehat{f}_k$ is the k-th Fourier coefficient of f. The norm of the operator $f \mapsto S_B(f)$ (in the space of continuous or integrable functions on $\mathbb{T}^n$) is called the B-th Lebesgue constant. It can be represented as

$$\mathscr{L}(B) = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{T}^n}\Big|\sum_{k\in B\cap\mathbb{Z}^n} e^{i\langle k,x\rangle}\Big|\,dx. \tag{1.1}$$

The sum of harmonics on the right-hand side is called the B-th Dirichlet kernel.

If the intersection $B \cap \mathbb{Z}^n$ is infinite, then the operator is defined only for good enough functions f, say for polynomials. Considering it in the space of all the polynomials, one may define the Lebesgue constant $\mathscr{L}(B)$ in a usual way as the least number C which satisfies the inequality $\|S_B(f)\| \le C\|f\|$ for any polynomial f. This does not exclude the case $\mathscr{L}(B) = +\infty$. The works [20, Ch.IX § 4] or [4], for example, give an impression on such situations.

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In order to get information of full value, a family of the sets B is usually used such that the integer lattice is exhausted by them. One of the most natural ways of ordering and exhausting is the R-dilation of a fixed set $B \subset \mathbb{R}^n$. By this, B is replaced with the set $RB = \{Ru : u \in B\}$. Of course, the generating set B is assumed to be known. The main feature of the one-dimensional case is the fact that the Dirichlet kernel can be expressed in a relatively simple manner ("rolled up"), while in the multivariate case this is possible only in certain rare and trivial situations. This leads to the need of dealing with trigonometric sums as a whole.

A very detailed survey on the Lebesgue constants is given in [18]; it deals equally with both linear methods of summability of the Fourier series and partial sums. There are other survey type texts but they (see, e.g., [8]) have the Lebesgue constants as a section only and in addition, they are outdated now. As for the Lebesgue constants for just partial sums, a concise though almost comprehensive overview of the basic results can be found in [11]. Of course, the very recent publications, like [12], could be referred to in none of these surveys.

Let us outline briefly the picture of behavior of different Lebesgue constants arisen prior to our study. A trivial L^2 estimate leads to the bound $\sqrt{\mathrm{card}(B\cap\mathbb{Z}^n)}$ for $\mathscr{L}(B)$. However rough it seems, it is non-refinable in a sense that for any B, there is a subset $B_0 \subset B$ such that $\mathscr{L}(B_0) \ge C_1\sqrt{\mathrm{card}(B)}$, where C_1 is a positive absolute constant; see [20, Problem IX.4.54]. This reveals the fact that not only the number of lattice points involved is of importance but also arithmetic and geometric properties of B. By this, one of the main tasks is to find natural assumptions on B such that the above bound can be improved. It has long ago been known that the Lebesgue constants, for bounded B with nonempty interior, tend to infinity as R grows. There are two main types of the rate of growth of the Lebesgue constants $\mathscr{L}(RB)$: power and logarithmic. On the one hand, it was shown in [19] (see also [17]) that if the smooth enough boundary of B contains at least one point with non-vanishing Gaussian curvature, the lower bound $R^{\frac{n-1}{2}}$ is guaranteed. The spherical Lebesgue constants as well as those similar to them but generated by a more general set satisfy the same estimate from above. On the other hand, lack of the points with non-vanishing curvature on the boundary of the generating set B may lead to a lower growth of the Lebesgue constants than $R^{\frac{n-1}{2}}$. For example, if B is a polyhedron (cube say), than the rate of the growth is $\ln^n R$. We mention that it had not been known for some time whether the sets B exist that generate an intermediate growth (see, e.g., [22]). Thus, for "normal" situations, that is, for bounded B, the rate $R^{\frac{n-1}{2}}$ seemed to be a natural maximal limit of the growth of the Lebesgue constants for the R-dilated set B.

In this work, we are mainly interested in the hyperbolic Lebesgue constants. Since the appearance of Babenko's paper [2] interest has continued in various questions of Approximation Theory and Fourier Analysis in $\mathbb{R}^n$ connected with the study of linear means with harmonics in the "hyperbolic crosses". It is meaningful to immediately describe our basic object in detail. Given a vector $\gamma \in \mathbb{R}^n_+$, we associate with it the hyperbolic cross $H_\gamma = \big\{t \in \mathbb{R}^n : |t_1|^{\gamma_1}\cdots|t_n|^{\gamma_n} \le 1\big\}$. Our further reasoning will mostly be concerned with its positive part

$$H_\gamma^+ = \Big\{t \in \mathbb{R}^n_+ : t_1^{\gamma_1}\cdots t_n^{\gamma_n} \le 1\Big\}.$$

Without loss of generality, we shall assume $\gamma_1 + \ldots + \gamma_n = 1$. We are interested in how large the Lebesgue constants corresponding to $RH_\gamma^+ = \Big\{t \in \mathbb{R}^n_+ : t_1^{\gamma_1}\cdots t_n^{\gamma_n} \le R\Big\}$ are, as $R \gtrsim 1$, that is, the values, with $k \in \mathbb{N}^n$,

$$\mathscr{L}(RH_\gamma^+) = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{T}^n}\Bigg|\sum_{k\in RH_\gamma^+} e^{i\langle k,x\rangle}\Bigg|\,dx.$$

When studying the Lebesgue constants one may consider the crosses with $k_j = 0$ as well, moreover, the set H_γ is dilated. However, this case differs a little from the one considered, since they distinct from one another by a bounded operator $f(\cdot) \mapsto \int_{\mathbb{T}} f(\cdot + te_j)\,dt$, where e_j is the j-th basis vector in $\mathbb{R}^n$. Secondly, we then would be deprived of advantages the representation (1.1) gives, since the intersection $B \cap \mathbb{Z}^n$ is infinite for $B = RH_\gamma$.

What has been known at present in this problem and what motivated us to return to it? The exact degree of growth for them $\mathscr{L}(RH_\gamma) \asymp R^{\frac{1}{2}}$ was established in the two-dimensional case independently by Belinsky [3] and by A. and V. Yudins [26], and afterwards was generalized to the case of arbitrary dimension in [16], where the bilateral bound $R^{\frac{n-1}{2}}$ was asserted. It is worth noting that the results obtained were applied to the problem of approximation on the classes of functions with continuous partial and mixed derivatives in [3] and [16] and in the study of the uniform convergence in [10]. However, the works by Temlyakov are apparently better adjoin to our problem; see [23] and [24, Ch.4] or [6] to mention some.

The above makes an impression that all goes well and the hyperbolic case does not slip out of the general picture, though possesses certain technical difficulties. It turned out that such an impression is erroneous, and just these Lebesgue constants destroy the picture drawn above for certain parameters of hyperbolic crosses. In a sense, $R^{\frac{n-1}{2}}$ can no more be considered as "Everest" among the "Lebesgue-constants-mountains". Well, the two-dimensional case is the most studied and brings no discomfort but a larger and sharp rate of growth is possible already in the three-dimensional case. It turned out that not all the cases were considered in [16] but only those subject to the formed stereotype. It is not difficult to figure out that the latter is violated already in dimension three. Indeed,

$$\int\limits_{\mathbb{T}^3}\Bigg|\sum_{1\le|k_1|^{\gamma_1}|k_2|^{\gamma_2}|k_3|^{\gamma_3}\le R} e^{i\langle k,x\rangle}\Bigg|\,dx \ge 2\pi\int\limits_{\mathbb{T}^2}\Bigg|\sum_{1\le|k_1|^{\gamma_1}|k_2|^{\gamma_2}\le R} e^{i(k_1x_1+k_2x_2)}\Bigg|\,dx_1\,dx_2.$$

Applying now the known two-dimensional estimate and recalling that $\gamma_1+\gamma_2+\gamma_3=1$, we immediately arrive at the lower bound $\mathscr{L}(RH_\gamma^+) \gtrsim R^{\frac{1}{2(\gamma_1+\gamma_2)}}$ (of course, any other pair may be in the denominator). It is not a big deal to choose γ_1, γ_2, and γ_3 such that the power be arbitrarily greater than one. Similar arguments for any dimension are not much more complicated.

Of course, these new bounds have to be supplemented with the corresponding upper ones. This is our main goal to establish all possible rates of growth for the Lebesgue constants, whereas $R^{\frac{n-1}{2}}$ will be the least one. All these will be attained according to the parameters $\gamma_1, ..., \gamma_n$ of the hyperbolic cross. Besides sharp power estimates, certain crosses are subject to additional logarithmic factors. It is an open problem whether such factors can be removed or not. Moreover, the value of these factors is questionable. For instance, we can prove in dimension three that the upper bound can be reduced to $R\ln^{\frac{1}{2}} R$ rather than $R\ln R$ that Theorem 2.2 gives. A corresponding picture in higher dimensions is still open, and we shall search for it elsewhere.

Also, $A \asymp B$ denotes $A \lesssim B \lesssim A$, where here and below we use the notation "$\lesssim$" as abbreviation for "$\le C$" with C being a positive constant, maybe different in different occurrences (and of course different in $A \lesssim B \lesssim A$). Similarly, "$\gtrsim$" substitutes for "$\ge C$". The constants C in this work never depend on the coefficient of dilation R but may depend on $\gamma_1, ..., \gamma_n$.

2. Results

We start with general estimates for the Lebesgue constants. Being well inclined for the constants generated by convex domains, they will nevertheless be actively used in the hyperbolic case.

2.1. **General estimates.**

For the set $V \subset \mathbb{R}^n$ of finite volume, we assume

$$I_V = \int\limits_{\mathbb{T}^n}\big|\widehat{\chi}(x)\big|\,dx,$$

where the function χ is 1 on V and 0 otherwise (the characteristic function of V) and $\widehat{\chi}$ is its Fourier transform:

$$\widehat{\chi}(x) = \int\limits_{\mathbb{R}^n}\chi(y)\,e^{i\langle x,y\rangle}dy = \int\limits_{V} e^{i\langle x,y\rangle}dy.$$

Our goal is to make the trivial estimate $I_V \lesssim \sqrt{mes(V)}$ more precise by means of certain restrictions on geometry of V.

Theorem 2.1. *Let the set V satisfy the conditions:*

1) Its projection E on the space of the first $n-2$ variables is of finite volume (in $\mathbb{R}^{n-2}$);

2) Intersection of V with any line parallel to the $(n-1)$-st or n-th coordinate axis is an interval, maybe empty;

3) There exist numbers Δ and Δ', $1 \le \Delta \le \Delta'$, such that for each point $(x_1 \dots, x_{n-2})$ in E the corresponding section of V

$$\{(x_{n-1}, x_n) \in \mathbb{R}^2 : (x_1, \dots, x_{n-1}, x_n) \in V\}$$

is located in the rectangle with the sides Δ and Δ'.
Then there exist a coefficient C, depending only on dimension n, such that

$$I_V \le C\sqrt{\Delta\, mes(E)}\,\Big(1 + \ln\frac{\Delta'}{\Delta}\Big).$$

In particular, $I_V \le CR^{\frac{n-1}{2}}$ if V is contained in a cube with side R and satisfies 2). Moreover, any convex subset of this cube admits this estimate.

Remark 1. An intent reader may observe that an assumption $n > 2$ is hidden in the formulation of the theorem. Nevertheless, if $n = 2$ the situation becomes even simpler. The changes needed are minimal: remove 1), simplify 3) to "the set V is located in the rectangle with the sides Δ and Δ'", and reduce the last inequality to $I_V \le C\sqrt{\Delta}\,\big(1 + \ln\frac{\Delta'}{\Delta}\big)$.

There is an alternative way to formally reduce the plane problem to a volumetric one by applying the theorem with $n = 3$ to the product $[0, 1] \times V$.

The following less general corollary is proved in [21] for two dimensions regardless of Theorem 2.1. Just this result will be used in the proof of the main result in an indirect and non-immediate way.

Corollary. *Let $B \subset [0, A_1] \times [0, A_2] \times ... \times [0, A_n]$, where $1 \le A_1 \le A_2 \le ... \le A_n$, so that the intersection of B with an arbitrary line parallel to a coordinate axis is an interval, maybe empty. Then,*

$$\mathscr{L}(B) \lesssim \sqrt{A_1\, A_2 \cdots A_{n-1}}\,\Big(1 + \ln\frac{A_n}{A_{n-1}}\Big). \tag{2.1}$$

This estimate is simpler in dimension two: $\mathscr{L}(B) \lesssim \sqrt{A_1}\,\big(1 + \ln\frac{A_2}{A_1}\big)$.

It is worth mentioning that a similar result was independently obtained in [26, Cor.2]. On the other hand, it was proved in [7] that if the finite set $B \subset \mathbb{Z}^n_+$ is such that with every point $k = (k_1, ..., k_n) \in B$, there holds

$$\prod_{j=1}^{n} [1, k_j] \cap \mathbb{Z}^n_+ \subseteq B, \tag{2.2}$$

then

$$\mathscr{L}(B) \le 50n^3 |\mathrm{card}(B \cap \mathbb{Z}^n)|^{\frac{n-1}{2n}}. \tag{2.3}$$

All these results are not applicable to hyperbolic cross directly; for example, (2.3) gives additional and unnecessary factor $\ln R$ even for $n = 2$ and the simplest cross $B = \{(x_1, x_2) : 1 \le |x_1 x_2| \le R\}$.

2.2. **Hyperbolic case.**

We are going to formulate our main result. All the formulations will be given for the case $\gamma_1 \le \dots \le \gamma_n$, whereas any other case reduces to that merely by renumeration of the variables.

Theorem 2.2. *Let* $0 < \gamma_1 \leqslant \ldots \leqslant \gamma_n$ *and* $\gamma_1 + \ldots + \gamma_n = 1$. *Then,*

$$R^{\frac{1}{2}\theta} \lesssim \mathscr{L}(RH_\gamma) \lesssim R^{\frac{1}{2}\theta} \ln^{k-1} R, \tag{2.4}$$

where

$$\theta = \max_{1\le \ell < n} \frac{\ell}{\gamma_1 + \ldots + \gamma_{\ell+1}},$$

and k *is the number of* ℓ *for which* $\theta = \frac{\ell}{\gamma_1+\ldots+\gamma_{\ell+1}}$.

In particular, in a "generic" situation where all the fractions $\frac{\ell}{\gamma_1+\ldots+\gamma_{\ell+1}}$ *are different* ($k = 1$), *a bilateral estimate*

$$\mathscr{L}(RH_\gamma) \asymp R^{\frac{1}{2}\theta} \tag{2.5}$$

holds.

Observe that here and in what follows we use notation $\mathscr{L}(RH_\gamma)$ with a certain abuse of rigidity, since it cannot be identified with (1.1). However, the difference between them is a constant, which is described in the introduction. We hope that such a licence will result in no confusion. In connection with this theorem, let us mention a different result due to Dyachenko. In [9], under assumption (2.2) a more precise result than (2.3) is obtained. It gives a correct upper estimate in the case $B = \{(x_1, ..., x_n) : 1 \le |x_1 \cdots x_n| \le R\}$ but not in "anisotropic" cases.

3. Specifications and discussion

It is desirable to have direct estimates of the constants $\mathscr{L}(RH_\gamma)$ via the coordinates of the vector γ rather than via the related numbers $\theta_1, \ldots, \theta_{n-1}$. We cannot say that this goal is achieved in full, since there is no formula for calculating θ and k immediately through $\gamma_1, ..., \gamma_n$. It is quite possible that such a formula is not achievable.

For $n = 2$, all is simple: $\theta = \theta_1 = 1$ and $k = 1$, which results in the bilateral estimate $\mathscr{L}(RH_\gamma) \asymp R^{\frac{1}{2}}$ known earlier. For higher dimensions, the situation becomes more complicated.

Let us discuss the case of three dimensions. Then either $k = 1$ or $k = 2$, which yields

$$\theta = \max\Big\{\frac{1}{\gamma_1+\gamma_2}, \frac{2}{\gamma_1+\gamma_2+\gamma_3}\Big\} = \max\Big\{\frac{1}{1-\gamma_3}, 2\Big\}.$$

Therefore, $k = 1$ provided $\gamma_3 \neq \frac{1}{2}$; hence, a bilateral estimate holds:

$$\mathscr{L}(RH_\gamma) \asymp \begin{cases} R, & \text{if } \gamma_3 < \frac{1}{2}; \\ R^{\frac{1}{2(1-\gamma_3)}}, & \text{if } \gamma_3 > \frac{1}{2}. \end{cases} \tag{3.1}$$

Observe that for γ_3 close to 1, the power $\frac{1}{2(1-\gamma_3)}$ becomes arbitrarily large. In particular, greater than 1 if $\gamma_3 > \frac{1}{2}$.

If $\gamma_3 = \frac{1}{2}$, then $k = 2$ and a logarithmic factor appears in our upper estimate:

$$R \lesssim \mathscr{L}(RH_\gamma) \lesssim R \ln R. \tag{3.2}$$

There is a geometric interpretation for conditions on γ_1, γ_2 and γ_3: if these are the lengths of the sides of a triangle, for instance, $\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{3}$, then the first relation in (3.1) is valid. If the triangle degenerates into an interval, then (3.2) holds. Finally, if the sum of the two first numbers is smaller than the third one, then the second relation in (3.1) is valid.

Let now $n = 4$, then $k \in \{1, 2, 3\}$ and

$$\theta = \max\Big\{\frac{1}{\gamma_1+\gamma_2}, \frac{2}{\gamma_1+\gamma_2+\gamma_3}, 3\Big\} = \max\Big\{\frac{1}{1-\gamma_3-\gamma_4}, \frac{2}{1-\gamma_4}, 3\Big\}. \tag{3.3}$$

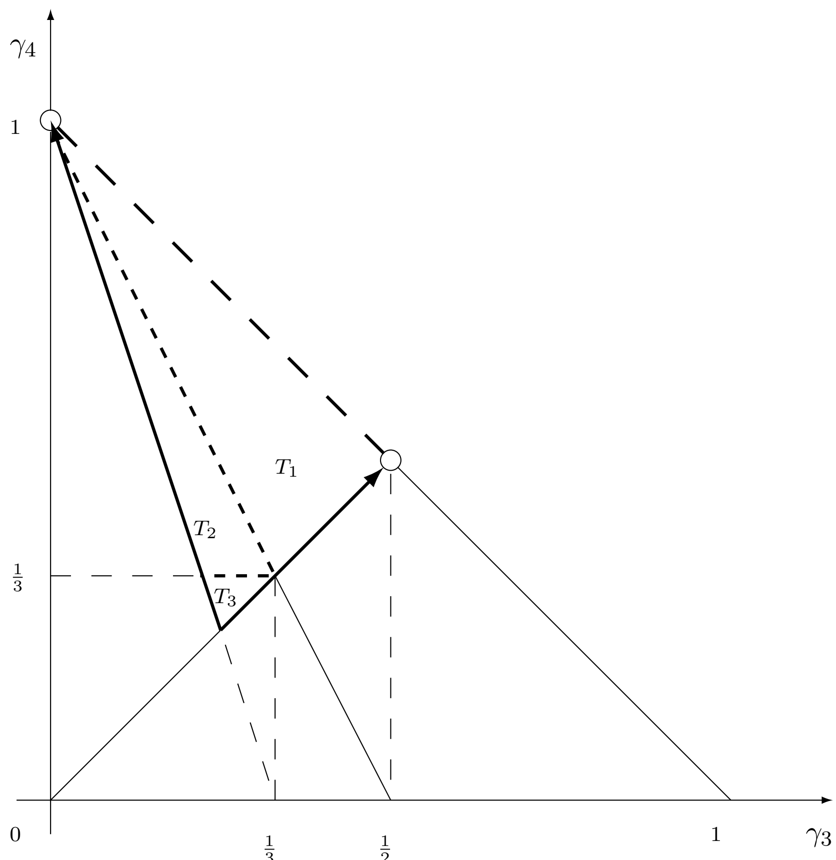
Thus, θ is a function of the variables γ_3 and γ_4. Where is it defined? Besides the obvious conditions $0 < \gamma_3 \le \gamma_4$ and $\gamma_3 + \gamma_4 < 1$, the inequality $1 - \gamma_3 - \gamma_4 = \gamma_1 + \gamma_2 \le 2\gamma_3$ should be taken into account. Under all these restrictions, we see that the point (γ_3, γ_4) belongs to a triangle

$$T = \big\{(\gamma_3, \gamma_4) : 0 < \gamma_3 \le \gamma_4,\ \gamma_3 + \gamma_4 < 1,\ 3\gamma_3 + \gamma_4 \ge 1\big\},$$

with vertices $(0,1)$, $\left(\frac{1}{4},\frac{1}{4}\right)$ and $\left(\frac{1}{2},\frac{1}{2}\right)$. Since $\gamma_3+\gamma_4<1$ for any pair (γ_3,γ_4) in T, there exists a pair (γ_1,γ_2) such that $\gamma=(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ satisfies all the restrictions for the parameters of the cross H_γ:

$$0<\gamma_1\le\gamma_2\le\gamma_3\le\gamma_4 \quad\text{and}\quad \gamma_1+\gamma_2+\gamma_3+\gamma_4=1.$$

In other words, the function θ is defined everywhere on T.



Omitting elementary calculations, we point out the parts of T on which

- all the fractions on the right-hand side (3.3) are equal ($k=3$),
- only two of them are equal ($k=2$),
- all the fractions are different ($k=1$).

Equality $k=3$ takes place only at the point $(\frac{1}{3},\frac{1}{3})$, so

$$R^{\frac{3}{2}}\lesssim\mathscr{L}(RH_\gamma)\lesssim R^{\frac{3}{2}}\ln^2R \quad\text{if}\quad \gamma=\Big(\gamma_1,\frac{1}{3}-\gamma_1,\frac{1}{3},\frac{1}{3}\Big),$$

where $0<\gamma_1\le\frac{1}{6}$.

Equality $k=2$ is valid on two intervals connecting the point $\left(\frac{1}{3},\frac{1}{3}\right)$ with the points $(0,1)$ and $\left(\frac{2}{9},\frac{1}{3}\right)$, where only the last one $\left(\frac{2}{9},\frac{1}{3}\right)$ among those three belongs to the corresponding (horizontal) interval. In these cases, the estimates hold, respectively,

$$R^{\frac{1}{2(\gamma_1+\gamma_2)}}\lesssim\mathscr{L}(RH_\gamma)\lesssim R^{\frac{1}{2(\gamma_1+\gamma_2)}}\ln R$$

and

$$R^{\frac{3}{2}}\lesssim\mathscr{L}(RH_\gamma)\lesssim R^{\frac{3}{2}}\ln R.$$

Equality $k=1$ holds true at the rest of the points of T, which form three smaller triangles (see figure):

$T_1=\{(\gamma_3,\gamma_4):\,\gamma_3\le\gamma_4,\,\gamma_3+\gamma_4<1,\,2\gamma_3+\gamma_4>1\}$, where $\theta=\theta_1=\frac{1}{1-\gamma_3-\gamma_4}$;

$T_2 = \{(\gamma_3, \gamma_4) : \gamma_4 > \frac{1}{3}, 2\gamma_3 + \gamma_4 < 1 \le 3\gamma_3 + \gamma_4\}$, where $\theta = \theta_2 = \frac{2}{1-\gamma_4}$
and

$T_3 = \{(\gamma_3, \gamma_4) : \gamma_3 \le \gamma_4 < \frac{1}{3}, 3\gamma_3 + \gamma_4 \ge 1\}$, where $\theta = \theta_3 = 3$.

In each of them its own power growth of the Lebesgue constants hold:

$$\mathscr{L}(RH_\gamma) \asymp \begin{cases} R^{\frac{1}{2(\gamma_1+\gamma_2)}}, & (\gamma_3, \gamma_4) \in T_1, \\ R^{\frac{1}{\gamma_1+\gamma_2+\gamma_3}}, & (\gamma_3, \gamma_4) \in T_2, \\ R^{\frac{3}{2}}, & (\gamma_3, \gamma_4) \in T_3. \end{cases}$$

As for arbitrary dimensions, searching of appropriate values of parameters θ and k basic in Theorem 2.2 in low dimensions has shown that even there the only modus operandi is brute force. Thus, in every concrete situation, that is, for given $\gamma_1, ..., \gamma_n$, it is clear how to find θ and k. However, there are simple general situations where it is quite easy to indicate θ omitting lengthy calculations. For example, if $\gamma_1 + \gamma_2 \le \gamma_3$, then the numbers $\{\theta_\ell\}_{1\le\ell<n}$ do not increase, and hence $\theta = \theta_1 = \frac{1}{\gamma_1+\gamma_2}$. By this, the estimates are almost obvious.

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References

1. L. Aizenberg and N. Tarkhanov, *Stable expansions in homogeneous polynomials*, Complex analysis and dynamical systems III, 1–22, Contemp. Math., **455**, Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI, 2008.
2. K. I. Babenko, *Approximation by trigonometric polynomials in a certain class of periodic functions of several variables*, Dokl. Akad.Nauk SSSR **132** (1960), 982–985 (Russian). – English transl. in Soviet Math. Dokl. **1** (1960), 672–675.
3. E. Belinsky, *Behavior of the Lebesgue constants of certain methods of summation of multiple Fourier series*, Metric Questions of the Theory of Functions and Mappings, Naukova Dumka, Kiev, 1977, 19–39 (Russian).
4. E. Belinsky and E. Liflyand, *Behavior of the Lebesgue constants of hyperbolic partial sums*, Mat. Zametki **43** (1988), 192–196 (Russian). – English transl. in Math. Notes **43** (1988), 107–109.
5. P. Dencker, W. Erb, Yu. Kolomoitsev and T. Lomako, *Lebesgue constants for polyhedral sets and polynomial interpolation on Lissajous-Chebyshev nodes*, J. Complexity **43** (2017), 1–27.
6. D. Dung, V. Temlyakov, and T. Ullrich, *Hyperbolic Cross Approximation*, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, Cham, 2018.
7. M. Dyachenko, *The order of growth of Lebesgue constants of Dirichlet kernels of monotone type*, Vestnik Moskov. Univ. Ser. I Mat. Mekh., no. 6 (1989), 33–37, 98 (Russian). – English transl. in Moscow Univ. Math. Bull. **44:6** (1989), 41–46.
8. M. Dyachenko, *Some problems in the theory of multiple trigonometric series*, Uspekhi Mat. Nauk **47:5** (1992), 97–162 (Russian). – English transl. in Russian Math. Surveys **47:5** (1992), 103–171.
9. M. Dyachenko, *Norms of Dirichlet kernels and some other trigonometric polynomials in L_p-spaces*, Matem. sb. **184:3** (1993), 3–20 (Russian). – English transl. in Russian Acad. Sci. Sb. Math. **78:2** (1994), 267–282.
10. M. I. Dyachenko, *Uniform Convergence of Hyperbolic Partial Sums of Multiple Fourier Series*, Mat. Zametki **76** (2004), 723–731 (Russian). – English transl. in Math. Notes **76** (2004), 673–681.
11. M. Ganzburg and E. Liflyand, *The Lebesgue constants of Fourier partial sums*, Topics in classical and modern analysis-in memory of Yingkang Hu, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, Cham, 2019, 147–158.
12. Yu. Kolomoitsev and E. Liflyand, *Asymptotics of the Lebesgue constants for a d-dimensional simplex*, Proc. Amer. Math. Soc. **149** (2021), 2911–2926.
13. S. Konyagin and M. Skopina, *Comparison of the L^1-norms of total and truncated exponential sums*, Mat. Zametki **69** (2001), 699–707 (Russian). – English transl. in Math. Notes **69** (2001), 644–651.
14. S. Konyagin, *On Pringsheim Convergence of a Subsequence of Partial Sums of a Multiple Trigonometric Fourier Series*, Tr. Mat. Inst. Steklova **323** (2023), 167–180 (Russian). – English transl. in Proc. Steklov Inst. Math. **323** (2023), 159–172.

15. S. Konyagin, H. Queffélec, E. Saksman, and K. Seip, *Riesz projection and bounded mean oscillation for Dirichlet series*, Stud. Math. **262** (2022), 121–149.
16. E. Liflyand, *Exact order of the Lebesgue constants of hyperbolic partial sums of multiple Fourier series*, Mat. Zametki **39** (1986), 674–683 (Russian). – English transl. in Math. Notes **39** (1986), 369–374.
17. E. Liflyand, *Sharp estimates of the Lebesgue constants of partial sums of multiple Fourier series*, Trudy Mat. Inst. im. V. A. Steklova **180** (1987), 151–152 (Russian). – English transl. in Proc. Steklov Math. Inst. **180** (1989), 176–177.
18. E. Liflyand, *Lebesgue constants of multiple Fourier series*, Online J. Anal. Combin. **1** (2006), 112 p.
19. E. Liflyand, A.G. Ramm, A.I. Zaslavsky, *Estimates from below for Lebesgue constants*, J. Fourier Anal. Appl. **2** (1996), 287–301.
20. B.M. Makarov, M.G. Goluzina, A.A. Lodkin, and A.N. Podkorytov, *Problèmes d'analyse reéelle*, Cassini, Paris, 2010. – French transl. of 2nd Russian edition, Nevskii Dialekt & BHV-Peterburg, St. Petersburg, 2004.
21. A. N. Podkorytov, *On the Lebesgue constants for double Fourier series*, Vestnik Leningrad. Univ. Matem. **7** (1977), 79–84 (Russian). – English transl. in Vestnik Leningrad Univ. Math. **10** (1982), 187–192.
22. A. N. Podkorytov, *Intermediate rates of growth of Lebesgue constants in the two-dimensional case*, Numerical Methods and Questions of the Organization of Calculations, Part 7 Notes of Scientific Seminars of the V.A.Steklov Math. Inst. Leningrad. Branch, Acad. Sci. USSR, Nauka, Leningrad **139** (1984), 148–155 (Russian). – English transl. in J. Soviet Math. **36** (1987), 276–282.
23. V. Temlyakov, *Approximation of Functions with Bounded Mixed Derivative*, Trudy Mat. Inst. im. V. A. Steklova **178** (1986) (Russian). – English transl. in Proc. Steklov Math. Inst. **178** (1989).
24. V. Temlyakov, *Multivariate Approximation*, Cambridge Univ. Press, Cambridge, 2018.
25. R.M.Trigub and E.S. Belinsky, *Fourier Analysis and Appoximation of Functions*, Kluwer, 2004.
26. A.A. Yudin and V.A. Yudin, *Discrete Imbedding Theorems and Lebesgue Constants*, Mat. Zametki **22** 1977, 381–394 (Russian). – English transl. in Math. Notes **22** 1977, 702–711.
27. A. Zygmund, *Trigonometric series: Vols. I, II*, Second edition, reprinted with corrections and some additions, Cambridge University Press, London, 1968.

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