Involutions and associated partitions of a measure space into two congruent nonmeasurable sets

Alexander Kharazishvili

Abstract. Under certain conditions, it is shown that if a ground set E is equipped with an involution s and with a measure μ, then there exists a partition of E into two s-congruent μ-nonmeasurable subsets. On the other hand, no such partition consists of sets which are absolutely nonmeasurable with respect to the class $\mathcal{M}_\mu$.

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Let E be a ground set and let f be a mapping of E into itself such that $f(x) \neq x$ for all $x \in E$. It is easy to check that the following two assertions are equivalent:

(*) f is an involution (i.e., $f^2 = \mathrm{Id}_E$);

(**) the family of sets $\{\{x, f(x)\} : x \in E\}$ forms a partition of E into two-element subsets.

Conversely, if one has a partition $\mathcal{P}$ of E into two-element subsets of E, then $\mathcal{P}$ uniquely determines an involution $f : E \to E$ without fixed elements such that $\mathcal{P} = \{\{x, f(x)\} : x \in E\}$. This fact is a theorem of **ZF** set theory, i.e., it does not need the Axiom of Choice (**AC**).

Example 1. Let $(\mathbf{R}, +)$ denote the additive group of reals, let $(\mathbf{Q}, +)$ denote the subgroup of all rationals, and let $\mathbf{R}/\mathbf{Q}$ be the quotient group. In [6] Sierpiński considered the ground set $E = \mathbf{R}/\mathbf{Q} \setminus \{\mathbf{Q}\}$ and a canonically associated with E involution $s : E \to E$ defined by

$$s(z + \mathbf{Q}) = -z + \mathbf{Q} \qquad (z \in \mathbf{R} \setminus \mathbf{Q}).$$

Using this involution, it was shown in [6] that the existence of a selector of the family of two-element sets

$$\{\{z + \mathbf{Q}, -z + \mathbf{Q}\} : z \in \mathbf{R} \setminus \mathbf{Q}\}$$

implies (in **ZF** & **DC** theory) the existence of a non-Lebesgue measurable subset of $\mathbf{R}$. In particular, it is impossible to define (in the same theory) a linear ordering of the family $\{z + \mathbf{Q} : z \in \mathbf{R}\}$.

Usual examples of involutions in geometry of the n-dimensional Euclidean space $\mathbf{R}^n$ are various kinds of symmetries. In particular, the central symmetry $s : \mathbf{R}^n \to \mathbf{R}^n$ is given by the formula $s(x) = -x$. The analogous formula describes the cental symmetry of the n-dimensional unit sphere $\mathbf{S}^n$ in $\mathbf{R}^{n+1}$. The symmetry of $\mathbf{R}^n$ has a unique fixed point 0 and the symmetry of $\mathbf{S}^n$ does not possess fixed points in $\mathbf{S}^n$.

A. Kolmogorov conjectured that the circle group $\mathbf{S}_1$ does not admit a partition into two congruent nonmeasurable subsets, where nonmeasurability is meant with respect to the standard Lebesgue measure λ_1 on $\mathbf{S}_1$. V. Uspensky disproved Kolmogorov's conjecture by giving an example of such a partition of $\mathbf{S}_1$ (for more details about this fact, see e.g. [8]; as far as we know, Uspensky's example was never published). Since one of the elements of $\mathbf{S}_1$ is the symmetry (corresponding to angle π), it makes sense to consider a more general situation when there are given a ground set E instead of $\mathbf{S}_1$, an involution s of E and a nonzero σ-finite measure μ on E.

It turns out that if the triple (E, s, μ) satisfies certain conditions, then Uspensky's example admits a substantial generalization. A few preliminary notions are necessary to formulate the generalized result.

A family $\{X_i : i \in I\}$ of μ-measurable subsets of E is called a pseudo-base of μ if $\mu(X_i) > 0$ for each index $i \in I$ and, for every μ-measurable set Y with $\mu(Y) > 0$, there exists an index $j = j(Y) \in I$ such that $X_j \subset Y$.

Accordingly, the pseudo-weight of μ is defined as the minimum of the cardinalities of all pseudo-bases of μ.

A subset Z of E is called μ-thick (or μ-massive) if the equality $\mu_*(E \setminus Z) = 0$ holds true, where μ_* denotes the inner measure on E produced by μ.

It is not hard to see that if $\{P, Q\}$ is a partition of E into two μ-thick subsets, then both members P and Q of this partition are nonmeasurable with respect to μ. The converse assertion is not true in general.

Theorem 1. *Suppose that (E, s, μ) satisfies the following conditions:*

(1) E is an infinite ground set and s is an involution of E without fixed elements;

(2) μ is a nonzero σ-finite s-quasi-invariant measure on E whose pseudo-weight does not exceed $\mathrm{card}(E)$*;*

(3) for every μ-measurable set Z with $\mu(Z) > 0$, one has $\mathrm{card}(Z) = \mathrm{card}(E)$.

Then there exists a partition $\{P_1, P_2\}$ of E such that:

(a) $s(P_1) = P_2$ (hence $s(P_2) = P_1$);

(b) both sets P_1 and P_2 are μ-thick in E (so none of them is μ-measurable).

The proof of this theorem uses a Bernstein type argument based on the method of transfinite induction.

Example 2. Fix a nonzero natural number n, consider the unit sphere $\mathbf{S}_n$ in the space $\mathbf{R}^{n+1}$ and equip $\mathbf{S}_n$ with the standard Lebesgue measure λ_n. For this sphere, we have a canonical involution $s : \mathbf{S}_n \to \mathbf{S}_n$ given by the formula $s(x) = -x$. Obviously, s is an isometric transformation of $\mathbf{S}_n$ onto itself, without fixed points. The conditions of Theorem 1 are trivially fulfilled for

$$E = \mathbf{S}_n, \quad s : \mathbf{S}_n \to \mathbf{S}_n, \quad \mu = \lambda_n,$$

so we come to a partition of $\mathbf{S}_n$ into two s-congruent λ_n-thick subsets.

Example 3. Following the argument of [7], denote by $\mathbf{Z}$ the set of all integers and let $\mathbf{Q}$ stand again for the set of all rationals. Consider a selector K of the quotient group $\mathbf{Q}/\mathbf{Z}$ and define

$$A_1 = \cup\{K + 2n : n \in \mathbf{Z}\}, \quad A_2 = \cup\{K + 2n + 1 : n \in \mathbf{Z}\}.$$

Clearly, we have

$$A_1 \cup A_2 = \mathbf{Q}, \quad A_1 \cap A_2 = \emptyset, \quad A_1 + 1 = A_2.$$

Further, since $(\mathbf{Q}, +)$ is a divisible subgroup of $(\mathbf{R}, +)$, it follows that $(\mathbf{Q}, +)$ is a direct summand in $(\mathbf{R}, +)$. Therefore, we come to a representation

$$\mathbf{R} = \mathbf{Q} + V \qquad (\mathbf{Q} \cap V = \{0\}),$$

where V is a vector subspace (over $\mathbf{Q}$) of $\mathbf{R}$. In fact, V is a Vitali subset of $\mathbf{R}$ (see, e.g., [1], [2], [4], [5], [9], [10] for more detailed information about properties of Vitali sets). It is not difficult to verify that the two sets

$$P_1 = A_1 + V, \quad P_2 = A_2 + V$$

satisfy the following relations:

(1) $P_1 \cup P_2 = \mathbf{R}$ and $P_1 \cap P_2 = \emptyset$;

(2) $P_1 + 1 = P_2$;

(3) both sets P_1 and P_2 are thick with respect to the standard Lebesgue measure on $\mathbf{R}$ (hence none of them is measurable with respect to this measure).

Using a similar argument, it can be proved that, for every natural number $k > 1$, there exists a partition $\{P_1, P_2, ..., P_k\}$ of $\mathbf{R}$ such that all sets P_i $(i = 1, 2, ..., k)$ are pairwise translation-congruent and thick with respect to the Lebesgue measure on $\mathbf{R}$. Moreover, arguing analogously one obtains a partition of $\mathbf{R}$ of the form $\{P_i : i \in I\}$, where $1 < \text{card}(I) \leq \text{card}(\mathbf{R})$, all sets P_i are pairwise translation-congruent and thick with respect to the same Lebesgue measure on **R**. In [11] closely related questions are discussed for the more general case of uncountable non-discrete locally compact commutative topological groups (cf. also [3]).

Theorem 2. *Let E be an infinite ground set equipped with a bijection $f : E \to E$ and with a σ-finite measure μ. Suppose also that a partition $\{P_1, P_2\}$ of E satisfies the equality $f(P_1) = P_2$ (hence $f(P_2) = P_1$ and $\{P_1, P_2\}$ is an invariant partition with respect to the group G_f of transformations of E, generated by $\{f\}$).*

If one of the sets P_1 and P_2 is μ-nonmeasurable in E, then there exists a measure μ' on E such that:

(a) μ' extends μ;

(b) $\{P_1, P_2\} \subset \mathrm{dom}(\mu')$;

(c) if the initial measure μ is G_f-invariant (respectively G_f-quasi-invariant), then μ' is also G_f-invariant (respectively, G_f-quasi-invariant).

Let E be a ground set and let $\mathcal{M}$ be a class of measures on E (the domains of members of $\mathcal{M}$ may be various σ-algebras of subsets of E). By definition, a subset X of E is relatively measurable with respect to $\mathcal{M}$ if there exists at least one measure $\nu \in \mathcal{M}$ such that $X \in \mathrm{dom}(\nu)$. Otherwise, X is called absolutely nonmeasurable with respect to $\mathcal{M}$.

Example 4. Let E be a ground set equipped with a transformation group G and let $\mathcal{M}$ be the class of all nonzero σ-finite G-invariant (G-quasi-invariant) measures on E. Any subset of E absolutely nonmeasurable with respect to this $\mathcal{M}$ is called G-absolutely nonmeasurable. If E itself is an uncountable commutative group identified with the group of all its translations, then there exist E-absolutely nonmeasurable subsets of E (see [2]).

As a consequence of Theorem 2, one obtains

Theorem 3. *Let E be an infinite ground set equipped with a bijection f, let G_f denote the group of transformations of E generated by $\{f\}$, and let μ be a σ-finite G_f-invariant (G_f-quasi-invariant) measure on E.*

Further, let $\{P_1, P_2\}$ be a partition of E such that $f(P_1) = P_2$ and let $\mathcal{M}_\mu$ denote the class of all those G_f-invariant (G_f-quasi-invariant) measures on E which extend μ.

Then both sets P_1 and P_2 are relatively measurable with respect to $\mathcal{M}_\mu$.

In terms of absolute nonmeasurability, Kolmogorov's conjecture may be reformulated as follows: there does not exist a partition $\{P_1, P_2\}$ of E which satisfies $f(P_1) = P_2$ and both members of which are G_f-absolutely nonmeasurable.

In fact, if θ is a nonzero finite finitely additive G_f-invariant measure on E and $\{P_1, P_2\}$ is a G_f-invariant partition of E, then there exists a finitely additive

G_f-invariant measure θ' on E extending θ and satisfying $\{P_1, P_2\} \subset \mathrm{dom}(\theta')$. Moreover, θ' can be defined for all subsets of E (see [10]).

Example 5. Let E be an uncountable ground set, G be a group of transformations of E generated by two distinct bijections of E onto itself, and let $\{P_1, P_2, P_3\}$ be a partition of E into three G-congruent subsets. From the classical result of Hausdorff (widely known as Hausdorff paradox; see e.g. [10]) one can conclude that a situation is realizable where each of the sets P_1, P_2 and P_3 is absolutely nonmeasurable with respect to the class of all nonzero finite finitely additive G-invariant measures on E.

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Author's Address:
A. Razmadze Mathematical Institute,
2 Merab Aleksidze II Lane,

Tbilisi 0193,
Georgia
I. Vekua Institute of Applied Mathematics,
University Street, 2,
Tbilisi 0186,
Georgia
e-mail: kharaz2@yahoo.com