

## CAPUTO DYNAMICAL SYSTEMS

PETER E. KLOEDEN<sup>1,\*</sup>

**Abstract.** Recent developments in the theory of dynamical systems and their attractors for Caputo fractional differential equations are reviewed.

### 1. INTRODUCTION

The theory of fractional analysis and fractional differential equations is vast and has deep historical roots. The introduction of Caputo fractional differential equations (FDEs) in the 1960s allowed initial value problems to be handled more naturally and the asymptotic behaviour of models based on them to be investigated. More recently, mathematically defined dynamical systems generated by Caputo FDEs and their attractors have been introduced.

Dissipative Caputo FDEs have vector fields which satisfy a dissipativity property. For ordinary differential equations (ODEs) it follows from such a property that an absorbing set exists which contains all of the long-term dynamical behaviour of the system such as the existence of an attractor. The situation is more complicated for Caputo FDEs, since these are essentially integral equations and the dissipative inequalities cannot be so easily exploited.

These developments are presented in the new SpringerBriefs monograph *Attractors of Caputo Fractional Differential Equations* by Doan Thai Son, Hoang The Tuan and the author [9]. The ideas and results there will be outlined here.

### 2. CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

Consider an autonomous Caputo fractional differential equation (FDE) of order  $\alpha \in (0, 1)$  on  $\mathbb{R}^d$ ,

$${}^C D_{0+}^{\alpha} x(t) = g(x(t)) \quad (2.1)$$

where  $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$  satisfies a global Lipschitz property. (This can be weakened later). The Caputo FDE (2.1) with the initial condition  $x(0) = x_0$  is essentially the integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(x(s)) ds =: (\mathcal{T}x)(t), \quad (2.2)$$

where  $\Gamma(\alpha) := \int_0^{\infty} t^{\alpha-1} e^{-t} dt$  is the Gamma function. Here  $\mathcal{T}$  is an operator from  $C([0, T], \mathbb{R}^d)$  into itself.

The proof of existence of a unique solution of (2.2) is based on a contraction mapping fixed point argument for the operator  $\mathcal{T}$ . This gives only local existence when the usual supremum norm is used [5, Theorem 6.2]. Instead, the proof in [9, Theorem 3.1] uses a weighted norm

$$\|\xi\|_{\gamma} := \sup_{t \in [0, T]} \frac{\|\xi(t)\|}{E_{\alpha}(\gamma t^{\alpha})} \quad \text{for all } \xi \in C([0, T], \mathbb{R}^d), \quad (2.3)$$

---

*Date:* 10 October 2025.

*2020 Mathematics Subject Classification.* 26A33, 42B20.

*Key words and phrases.* Caputo differential equations; translation semigroup; trajectory attractors.

\*Corresponding author.

where  $\gamma > 0$  is a suitable constant which is chosen to give a contraction. The weight function  $E_\alpha(\cdot)$  is a Mittag-Leffler function [5, 9]. This is essentially a generalisation of the exponential function and is defined with one or two parameters by

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + 1)}, \quad E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + \beta)}, \quad t \in \mathbb{R},$$

with  $\alpha, \beta > 0$ , not necessarily integers, and  $t \in \mathbb{R}$ . Clearly,  $E_\alpha(t) = E_{\alpha,1}(t)$ .

The solutions of Caputo FDEs depend on their history and so cannot be concatenated, i.e., patched together, as for ODEs. They can even intersect in finite time in two dimensions or more [1, 2, 9]. In particular, they do not generate a semi-group, i.e., a semi-dynamical system, on  $\mathbb{R}^d$ .

### 3. SCALAR CAPUTO FDEs

In the scalar case, Cong & Tuan [1, 2] showed that any two solutions of (2.1) satisfy the lower bound

$$|x_2(t) - x_1(t)| \geq |x_2(0) - x_1(0)| E_\alpha(-Lt^\alpha).$$

Hence two solutions with different initial values cannot intersect in finite time. This is also true for higher dimensional systems with a triangular vector field, i.e., with

$$g(x_1, x_2, \dots, x_d) = (g_1(x_1), g_2(x_1, x_2), \dots, g_d(x_1, x_2, \dots, x_d)).$$

Otherwise, the solutions may intersect, see [1] for a two dimensional counter example.

Cong & Tuan [1] showed that the evolution mappings

$$\Phi_{s,t}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}, \quad s, t \geq 0,$$

generated by the solutions of a scalar Caputo FDE with  $\Phi_{0,t}x_0 = x(t, x_0)$  are a two-parameter family of bijections satisfying

$$\Phi_{s,t} := \Phi_{0,t} \circ \Phi_{0,s}^{-1} \quad \text{for all, } s, t \geq 0. \quad (3.1)$$

This says that to continue a solution at  $x(s)$  beyond a time  $s > 0$  one needs to find its starting point  $x_0 = \Phi_{0,s}^{-1}x(s)$  and then to map forward from there to obtain

$$x(t) = \Phi_{0,t}x_0 = \Phi_{0,t} \circ \Phi_{0,s}^{-1}x(s), \quad t \geq s \geq 0.$$

Cong & Tuan called this a *nonlocal dynamical system*. It differs profoundly from the two-parameter families generated by nonsunonomous ODEs via the concatenation of solutions, see [11].

### 4. CAPUTO SEMI-DYNAMICAL SYSTEMS

The integral equation (2.2) representation of a Caputo FDE (2.1) is a special case of a Volterra integral equation

$$x_f(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(x_f(s)) ds \quad (4.1)$$

where  $f \in \mathfrak{C} := C(\mathbb{R}^+, \mathbb{R}^d)$ .

Sell [12] showed that the family of Volterra operators  $\{\mathfrak{T}_t\}_{t \geq 0}$  on  $\mathfrak{C}$  defined by

$$(\mathfrak{T}_t f)(\theta) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t+\theta-s)^{\alpha-1} g(x(s)) ds, \quad \theta \in \mathbb{R}^+, \quad (4.2)$$

with  $x(t)$  given by (7.3). so  $(\mathfrak{T}_t f)(0) = x(t)$ ,  $t \geq 0$ , is a semi-group on  $\mathfrak{C}$  with the topology uniform convergence on compact subsets. This topology is induced by the metric

$$\rho(f, h) := \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(f, h), \quad \rho_n(f, h) := \frac{\sup_{t \in [0, n]} \|f(t) - h(t)\|}{1 + \sup_{t \in [0, n]} \|f(t) - h(t)\|}.$$

With the restriction to  $f(t) \equiv id_{x_0}$ , Doan & Kloeden [6] called this the *Caputo semi-group*. This semi-group represents the Caputo FDE (2.1) as an autonomous semi-dynamical system on the space  $\mathfrak{C}$ .

Of course, the  $\mathfrak{T}_t f$  do not remain equal to  $id_{x_0}$  unless  $x_0$  is a zero of the vector field  $g$ , i.e., a steady state solution of the Caputo FDE. Two solutions of the Caputo FDE for different initial values may intersect in finite time here, but in the above semi-group formulation they are distinguished by the "flag"  $(\mathfrak{T}_t id_{x_0})(\theta)$ ,  $\theta > 0$ , that they "carry".

## 5. DISSIPATIVE CAPUTO FDES

By a fractional calculus counterpart of the chain rule of Tuan & Trinh [13], the solutions of the Caputo FDE (2.1) satisfy

$${}^C D_{0+}^\alpha \|x(t)\|^2 \leq 2 \langle x(t), {}^C D_{0+}^\alpha x(t) \rangle.$$

Hence, if the vector field  $g$  of (2.1) satisfies the dissipativity condition

$$\langle x, g(x) \rangle \leq a - b\|x\|^2 \quad (5.1)$$

for  $a, b > 0$ , then along the solutions of (2.1)

$${}^C D_{0+}^\alpha \|x(t)\|^2 \leq 2 \langle x(t), g(x(t)) \rangle \leq 2a - 2b\|x(t)\|^2.$$

This can be integrated, see Kloeden [10], to give

$$\|x(t)\|^2 \leq \|x_0\|^2 E_\alpha(-2bt^\alpha) + \frac{a}{b} (1 - E_\alpha(-2bt^\alpha)). \quad (5.2)$$

The global existence and uniqueness of solutions of a Caputo FDE can be shown using this inequality when the vector field satisfies a local Lipschitz condition and the above dissipativity condition [10].

It follows from (5.2) that  $\|x(t, x_0)\| \leq R$  for all  $t \geq 0$  when  $\|x_0\| \leq R$  and  $R^2 \geq 1 + \frac{a}{b} =: R_*^2$ . Thus the set

$$\mathcal{B} := \left\{ x \in \mathbb{R}^d : \|x\|^2 \leq 1 + \frac{a}{b} =: R_*^2 \right\}$$

is an absorbing set for the solutions of the Caputo FDE (2.1). In particular, there exists  $T_R \geq 0$  such that  $\|x(t, x_0)\| \in \mathcal{B}^*$ , i.e.,  $\|x(t, x_0)\| \leq R^*$  for all  $\|x_0\| \leq R$  and  $t \geq T_R$ . The set  $\mathcal{B}$  is also positive invariant, i.e., solutions starting in it stay in it for all future time.

This set  $\mathcal{B}$  is compact in  $\mathbb{R}^d$  and contains the corresponding omega limit set

$$\Omega := \left\{ y \in \mathbb{R}^d : \exists \text{ all bnd'd } x_{0,n} \in \mathbb{R}^d, t_n \rightarrow \infty \text{ such that } x(t_n, x_{0,n}) \rightarrow y \right\},$$

which is a nonempty compact subset of  $\mathcal{B}$  and contains the future dynamics of the autonomous Caputo FDE (2.1). However,  $\Omega$  cannot, strictly speaking, be called the attractor of the Caputo FDE, since this does not generate a semi-group in  $\mathbb{R}^d$ . It could, perhaps, be called an *attracting set*.

Note that  $\Omega$  contains the steady state solutions of the Caputo FDE (2.1). See [9].

## 6. ATTRACTORS OF CAPUTO SEMI-DYNAMICAL SYSTEMS

For a some time a major difficulty was how to apply the dissipativity condition (5.1) to the vector field  $g$  inside the integral equations (7.4) for the  $(\mathfrak{T}_t f)(\theta)$  terms in order to establish the existence of an absorbing set in the space  $\mathfrak{C}$ .

The chain rule inequality of Tuan & Trinh can be used for the case  $\theta = 0$ , i.e., for the Caputo FDE (2.1). As seen above this leads to a compact positively invariant absorbing set  $\mathcal{B}$  for its solutions  $\mathbb{R}^d$ . Let  $R^2 \geq 1 + \frac{a}{b} =: R_*^2$ . Then, the dissipativity condition (5.1) and solution estimates give

$$B_R := \sup_{t \geq 0, \|x_0\| \leq R} \|x(t, x_0)\| < \infty, \quad B_R^g := \sup_{\|x\| \leq R} \|g(x)\| < \infty.$$

where the continuity of the vector field  $g$  has been used in the second bound. Such bounds can then be used to estimate the  $(\mathfrak{T}_t id_{x_0})(\theta)$  terms for  $\theta > 0$ . However, due to the growth nature of the bounds obtained, Doan & Kloeden [6] found it necessary to restrict to a subspace  $\mathfrak{C}_\alpha$  of  $\mathfrak{C}$  with the weighted norm

$$\|f\|_\alpha := \|f(0)\| + \sum_{N=1}^{\infty} \frac{1}{2^N N^\alpha} \|f\|_N, \quad \|f\|_N := \sup_{t \in [1/N, N]} \|f(t)\|, \quad N = 1, 2, \dots$$

Then they showed that the closed and bounded subset

$$\mathfrak{B} := \left\{ \chi \in \mathfrak{C} : \|\chi\|_\alpha \leq 2R^* + \frac{B_{R^*}^g}{\alpha\Gamma(\alpha)} =: \widehat{R}^* \right\}$$

of  $\mathfrak{C}_\alpha$  is an absorbing set for the Caputo semi-group  $(\mathfrak{T}_t id_{x_0})(\cdot)$  and absorbs bounded sets of constant initial data  $id_{x_0}$  with  $\|x_0\| \leq R$  in time  $t \geq T_R$

Doan & Kloeden [6] also showed that the operators  $(\mathfrak{T}_t id_{x_0})(\cdot)$  are asymptotically compact, i.e., for every sequence  $t_n \rightarrow \infty$  there is a subsequence  $t_n \rightarrow \infty$  such that  $(\mathfrak{T}_{t_n} id_{x_0})(\cdot) \rightarrow \infty \chi^*(\cdot) \in \mathfrak{C}_\alpha$ .

The theory of autonomous semi-dynamical systems on Banach space [11] can then be applied to give the existence of a global attractor.

**Theorem 6.1.** [7, 9] *Suppose that the vector field  $g$  is locally Lipschitz and satisfies the dissipativity condition (5.1). Then, the semi-group  $\{\mathfrak{T}_t\}_{t \geq 0}$  on the space  $\mathfrak{C}_\alpha$  has a global attractor  $\mathfrak{A} \subset \mathfrak{C}_\alpha$ , which has the structure*

$$\mathfrak{A} = \bigcap_{t \geq 0} \mathfrak{T}_t \mathfrak{B}$$

and attracts all bounded subsets  $id_D$  of  $\mathfrak{C}_\alpha$  consisting of constant initial functions  $id_{x_0}$  for  $x_0 \in D$  for bounded subsets  $D$  of  $\mathbb{R}^d$  in the sense that

$$\lim_{t \rightarrow \infty} \text{dist}_{\mathfrak{C}_\alpha}(\mathfrak{T}_t id_D, \mathfrak{A}) = 0.$$

In particular,  $\mathfrak{A}$  is a compact subset of  $\mathfrak{C}_\alpha$  and is invariant in the sense that

$$\mathfrak{T}_t \mathfrak{A} = \mathfrak{A} \quad t \geq 0.$$

A distinctive feature of the attractor  $\mathfrak{A}$  here is that its universe of attraction consists of bounded subsets  $id_D$  of  $\mathfrak{C}_\alpha$  as described in the theorem. The solutions usually do not remain in this universe.

Note that the corresponding omega limit set  $\Omega$  in  $\mathbb{R}^d$  satisfies

$$\Omega = \{\chi(0) \in \mathbb{R}^d : \chi \in \mathfrak{A}\}$$

This set is determined solely by the limiting dynamics of the solution  $x(t, x_0) = (\mathfrak{T}_t id_{x_0})(0)$  of the Caputo FDE. The solution then determines the other parts of the  $\mathfrak{T}_t id_{x_0}$  and hence the attractor  $\mathfrak{A}$  itself.

**6.1. Scalar and triangular systems.** The above discussion also applies for Caputo FDEs with scalar and triangular vector fields. Their special structure allows more to be said about the dynamics inside the omega limit set  $\Omega$ .

In the scalar case a dissipative vector field must have at least one zero, which corresponds to a steady state solution. When these are hyperbolic, i.e., the eigenvalue of the linear part has non-zero real part, then the omega limit set consists of the steady state solutions and the heteroclinic trajectories joining them.

It has essentially the same geometric structure as the omega limit set (in this case is also the global attractor) of the corresponding ODE, with the major exception that the rates of attraction are not exponential, but rather subexponential of the form  $E_\alpha(-t^\alpha) \sim t^{-\alpha}$ .

A similar situation holds for the triangular vector fields with some restrictions. See [7, 9].

## 7. NON-AUTONOMOUS CAPUTO FDES

The non-autonomous situation with time dependent vector fields is considerably more difficult both for fractional and ordinary differential equations [11]. More can be said when the change of time in the vector field is given by a driving system.

Consider the non-autonomous Caputo ffDE of order  $\alpha \in (0, 1)$  on  $\mathbb{R}^d$  of the form

$${}^C D_{0+}^\alpha x(t) = g(x(t), \vartheta_t(p)) \quad \text{for } t \in [0, T[, \quad (7.1)$$

with a driving system  $\vartheta_t$  in the vector field  $g(x, p)$ . Specifically,  $\vartheta_t : P \rightarrow P$ ,  $t \in \mathbb{R}$ , is a group of operators, i.e., an autonomous dynamical system, and  $P$  is a compact metric space. The solution of the Caputo FDE (7.1) with initial condition  $x(0) = x_0$  and  $p_0 \in P$  satisfies the integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(x(s), \vartheta_s(p_0)) ds. \quad (7.2)$$

Define operators  $\mathfrak{T}_t : \mathfrak{C}_\alpha \times P \rightarrow \mathfrak{C}_\alpha$  by

$$(\mathfrak{T}_t(id_{x_0}, p_0))(0) := x(t, x_0, p_0) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(x(s, x_0, p_0), \vartheta_s(p_0)) ds \quad (7.3)$$

and

$$(\mathfrak{T}_t(id_{x_0}, p_0))(\theta) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t+\theta-s)^{\alpha-1} g(x(s, x_0, p_0), \vartheta_s(p_0)) ds \quad (7.4)$$

for  $\theta > 0$ . The resulting Caputo semi-group  $\{(\mathfrak{T}_t, \vartheta_t)\}_{t \in \mathbb{R}^+}$  on  $\mathfrak{C}_\alpha \times P$  is called a *Caputo skew-product flow*.

As in the autonomous case, the attractor in the skew-product flow case also attracts only a restricted class of initial values in  $\mathfrak{C}_\alpha$ , which is not invariant under the dynamics. The following theorem is due to Cui & Kloeden [3, Theorem 2].

**Theorem 7.1.** *Suppose that the vector field  $g$  is locally Lipschitz in both variables and satisfies the dissipativity condition (5.1) uniformly in  $p \in P$ . Then semi-group  $\{(T_t, \vartheta_t)\}_{t \in \mathbb{R}^+}$  on the space  $\mathfrak{C}_\alpha \times P$  corresponding to the integral equations (7.2) has an attractor  $\mathfrak{A} \subset \mathfrak{C}_\alpha \times P$ , which attracts bounded subsets of  $\mathfrak{C}_\alpha \times P$  consisting of constant initial functions in  $\mathfrak{C}_\alpha$  and has the structure*

$$\mathfrak{A} = \bigcup_{p \in P} \mathfrak{A}(p) \times \{p\},$$

where the  $\mathfrak{A}(p)$  are compact subsets of  $\mathfrak{C}_\alpha$ . Moreover, the sets  $\mathfrak{A}(p)$  are positively invariant in the sense that

$$T_t(\mathfrak{A}(p), p) = \mathfrak{A}(\vartheta_t(p)), \quad t \geq 0, p \in P,$$

and pullback attracting in the sense that

$$\lim_{t \rightarrow \infty} \text{dist}_{\mathfrak{C}_\alpha}(T_t(\mathfrak{D}, \vartheta_{-t}(p)), \mathfrak{A}(p)) = 0, \quad p \in P,$$

attracts all bounded subsets  $id_D$  of  $\mathfrak{C}_\alpha$  consisting of constant initial functions  $id_{x_0}$  for  $x_0 \in D$  for bounded subsets  $D$  of  $\mathbb{R}^d$ .

**7.1. An explicit example.** As a motivational example consider the scalar Caputo FDE

$${}^C D_{0+}^\alpha x(t) = -x(t) + \cos t. \quad (7.5)$$

Here  $P$  is the hull [11, 12] of the functions  $\cos(\cdot)$ , i.e.,

$$P = \bigcup_{0 \leq \tau \leq 2\pi} \cos(\tau + \cdot),$$

which is a compact metric space with the metric induced by the supremum norm  $d_P(p_1, p_2) = \sup_{t \in \mathbb{R}} |p_1(t) - p_2(t)|$ . In addition, let  $\vartheta_t : P \rightarrow P$  be the left shift operator  $\vartheta_t(\cos(\cdot)) = \cos(t + \cdot)$ . This shift operator is continuous in the above metric. Indeed, it is an isometry with  $d_P(\vartheta_t(p_1), \vartheta_t(p_2)) = d_P(p_1, p_2)$ ,  $p_1, p_2 \in P$ .

Note that the linear Caputo equation (7.5) is strictly contracting. Let  $x(t) = x(t, x_0, p_0)$  and  $y(t) = y(t, y_0, p_0)$ . Then

$${}^C D_{0+}^\alpha z(t) = -z(t), \quad z(t) := x(t) - y(t).$$

Hence  $z(t) = E_\alpha(-t^\alpha)z(0)$ , which gives the strictly contracting property

$$|z(t)| = E_\alpha(-t^\alpha)|z(0)|. \quad (7.6)$$

In particular,  $z(t) \rightarrow 0$  as  $t \rightarrow \infty$  at the non-exponential rate  $t^{-\alpha}$ , see [9].

To see what happens to individual solutions note that the linear Caputo FDE (7.5) has explicit solutions given by the variation of constants formula [9, Lemma 1.4]

$$x(t, x_0, p_0) = E_\alpha(-t^\alpha)x_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha) p(s, p_0) ds.$$

Let  $\tau > 0$  and let  $q_{-\tau} = p(-\tau, p_0)$ , so  $p_0 = p(\tau, q_{-\tau})$ . Then following [8]

$$x(\tau, x_0, q_{-\tau}) = E_\alpha(-\tau^\alpha)x_0 + \int_{-\tau}^0 (-\nu)^{\alpha-1} E_{\alpha,\alpha}(-(-\nu)^\alpha) p(\nu, p_0) d\nu,$$

since  $p(\nu + \tau, q_{-\tau}) = p(\nu, p_0)$ . Hence by the pullback limit (see [3, 11])

$$\lim_{\tau \rightarrow \infty} x(\tau, x_0, q_{-\tau}) = a(p_0) := \int_{-\infty}^0 (-\nu)^{\alpha-1} E_{\alpha,\alpha}(-(-\nu)^\alpha) p(\nu, p_0) d\nu.$$

The strictly contractive condition (7.6) gives

$$|x(\tau, x_0, q_{-\tau}) - y(\tau, y_0, q_{-\tau})| = E_\alpha(-\tau^\alpha)|x_0 - y_0|, \quad (7.7)$$

which means all such solutions converge in the pullback sense to the same limit, i.e.,  $a(p_0)$ . In particular, the pullback attractor sets  $\mathfrak{A}(p)$  in Theorem 7.1 are singleton sets  $\mathfrak{A}(p) = \{a(p)\}$ . In fact this corresponds to a single entire solution  $\chi^*(t) = a(p^*(t))$  in  $\mathbb{R}$  of the Caputo FDE (7.5).

In its original form the strictly contractive condition (7.6) also shows that all of the solutions of the Caputo FDE (7.5) converge forwards in time to this entire pullback solution.

#### REFERENCES

1. N.D. Cong and H.T. Tuan, Generation of nonlocal dynamical systems by fractional differential equations, *J. Integral Equations Appl.*, **29** (2017), 585–608.
2. N.D. Cong, H.T. Tuan and Hieu Trinh, On asymptotic properties of solutions to fractional differential equations. *J. Math. Anal. Appl.* **484** (2020), 123759.
3. Hongyong Cui and P.E. Kloeden, Skew-product attractors of non-autonomous Caputo fractional differential equations, *Chaos*, **34** 083125 (2024) <https://doi.org/10.1063/5.0214041>
4. K. Diethelm, On the separation of solutions of fractional differential equations, *Fract. Calc. Appl. Anal.* **11** (2008), no. 3, 259–268.
5. K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer Lecture Notes in Mathematics, Vol. 2004, Springer, Heidelberg, 2010.
6. T.S. Doan and P.E. Kloeden, Semi-dynamical systems generated by autonomous Caputo fractional differential equations, *Vietnamese J. Math.*, **49** (2021), 1305–1315.
7. T.S. Doan and P.E. Kloeden, Attractors of Caputo fractional differential equations with triangular vector field, *Fract. Calc. Appl. Anal.*, **25**(2), 720–734 (2022).
8. T.S. Doan and P.E. Kloeden, Asymptotic behaviour of non-autonomous Caputo fractional differential equations with a one-sided dissipative vector field, *Buletinul Academiei de Stinte a Republicii Moldova Matematica*, No. 1 (104)-2(105), (2024), 44-52. ISSN 1024-7696, E-ISSN 2587-4322
9. T.S. Doan, P.E. Kloeden and H.T. Tuan, *Attractors of Caputo Fractional Differential Equations*, SpringerBirefs, Chalm, (2025)
10. P.E. Kloeden, An elementary inequality for dissipative Caputo fractional differential equations, *Fract. Calc. Appl. Anal.* **26** (2023), 2166-2174.
11. P.E. Kloeden and M. Rasmussen, *Nonautonomous Dynamical Systems*, Amer. Math. Soc. Providence, 2011.
12. G.R. Sell, *Topological Dynamics and Ordinary Differential Equations*, Van Nostrand Reinhold Mathematical Studies, London, 1971.
13. H.T. Tuan and H. Trinh, Stability of fractional-order nonlinear systems by Lyapunov direct method, *IET Control Theory Appl.* **12**, 2417-2422 (2018)

<sup>1</sup> MATHEMATISCHES INSTITUT, UNIVERSITÄT TÜBINGEN, D-72076 TÜBINGEN, GERMANY.  
Email address: kloeden@math.uni-frankfurt.de