

Topological insulators and stable isomorphism versus isomorphism of vector bundles

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Abstract

This note gives an overview of the mathematical framework underlying topological insulators, highlighting the connection to K-theory and vector bundles. We see “real” and “quaternionic” vector bundles arise naturally in the presence of time-reversal symmetry. Our recent results about when stable isomorphism implies isomorphism are summarised, including some ongoing work for G -equivariant K-theory for finite groups. This clarifies when K-theory completely distinguishes topological phases.

1 Introduction

Topological insulators have become a central topic in condensed matter physics because they exhibit remarkable transport properties: they are insulating in the bulk, but they support robust conducting boundary states, which are protected by topological invariants of the system. The electronic properties of these materials are determined by global topological invariants, making them resilient to perturbations. Mathematically, they may be described in the language of operator algebras, K-theory, and vector bundles.

In this note, we briefly review the mathematical formalism that models topological insulators, emphasizing connections to C^* -algebras and vector bundles. We describe how time-reversal symmetry introduces “real” and “quaternionic” structures. We explain the physical motivation for the question whether stably isomorphic vector bundles are isomorphic, and we discuss our recent results on this in joint work with Malkhaz Bakuradze [2]. We conclude with some remarks on ongoing research in G -equivariant K-theory, which exhibits markedly different phenomena compared to the non-equivariant case.

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2 Lattice models and the Hilbert space formalism

We consider an electron moving in a d -dimensional crystal with k internal degrees of freedom per lattice site. The Hilbert space of the system is $\ell^2(\mathbb{Z}^d, \mathbb{C}^k)$. The dynamics on this space is governed by a bounded, self-adjoint Hamiltonian H . We assume translation invariance, that is, $S_n H = H S_n$ for all $n \in \mathbb{Z}^d$ for the translation operator S_n defined by $(S_n f)(x) = f(x - n)$. Under this assumption, there are matrices $H_a \in \mathbb{M}_k(\mathbb{C})$ with

$$(Hg)(m) = \sum_{a \in \mathbb{Z}^d} (H_a g)(m - a) \quad (1)$$

for all $g \in \ell^2(\mathbb{Z}^d, \mathbb{C}^k)$. We assume that H has finite range, that is, $H_a = 0$ for all but finitely many a . A Hamiltonian H describes an insulator if and only if it is invertible.

To describe a crystal with a boundary, we restrict H to a half-space, that is, we consider the operator $\hat{H} := I^* H I$ on the subspace $\ell^2(\mathbb{N} \times \mathbb{Z}^{d-1}, \mathbb{C}^k)$, where $I: \ell^2(\mathbb{N} \times \mathbb{Z}^{d-1}, \mathbb{C}^k) \hookrightarrow \ell^2(\mathbb{Z}^d, \mathbb{C}^k)$ denotes the inclusion map. Even if H is invertible, \hat{H} may fail to be invertible. This means physically that conducting states appear on the boundary of a finite-size chunk of the material. Next, we sketch how the presence of such conducting states may be topologically protected by a non-vanishing K-theory index.

3 C*-Algebraic formulation and the index map

The C*-algebra generated by the finite-range, translation invariant operators on $\ell^2(\mathbb{Z}^d, \mathbb{C}^k)$ is

$$C^*(\mathbb{Z}^d) \otimes \mathbb{M}_k(\mathbb{C}),$$

where $C^*(\mathbb{Z}^d) \cong C(\mathbb{T}^d)$ is the group C*-algebra of \mathbb{Z}^d . This is the smallest C*-algebra that contains all the Hamiltonians allowed above. On the half-space, one unitary generator is replaced by a unilateral shift on $\ell^2(\mathbb{N})$. So the allowable Hamiltonians for the half-space generate the C*-algebra

$$\mathcal{T} \otimes C^*(\mathbb{Z}^{d-1}) \otimes \mathbb{M}_k(\mathbb{C}),$$

where \mathcal{T} is the Toeplitz C*-algebra (see [5] for more details). These C*-algebras fit in an extension

$$\mathcal{K} \otimes C^*(\mathbb{Z}^{d-1}) \otimes \mathbb{M}_k(\mathbb{C}) \hookrightarrow \mathcal{T} \otimes C^*(\mathbb{Z}^{d-1}) \otimes \mathbb{M}_k(\mathbb{C}) \rightarrow C^*(\mathbb{Z}^d) \otimes \mathbb{M}_k(\mathbb{C}), \quad (2)$$

where \mathcal{K} is the C*-algebra of compact operators on $\ell^2(\mathbb{N})$. Roughly speaking, a half-space Hamiltonian in $\mathcal{T} \otimes C^*(\mathbb{Z}^{d-1}) \otimes \mathbb{M}_k(\mathbb{C})$ yields a bulk Hamiltonian in $C^*(\mathbb{Z}^d) \otimes \mathbb{M}_k(\mathbb{C})$ by identifying $\mathbb{N} \times \mathbb{Z}^{d-1}$ with $(\mathbb{N} - s) \times \mathbb{Z}^{d-1}$ for $s \in \mathbb{N}$ and letting $s \rightarrow \infty$.

Next we bring K-theory and the index map into play. The Hamiltonian of an insulator is a self-adjoint invertible element of the C*-algebra $C^*(\mathbb{Z}^d) \otimes \mathbb{M}_k(\mathbb{C})$. Functional calculus allows us to deform it among self-adjoint invertible operators to the self-adjoint involution $F = \text{sign}(H)$. This satisfies $F^2 = 1$ and $F = F^*$. It contains the same information as the associated projection

$$p = \frac{1 + F}{2} = \frac{1 + \text{sign}(H)}{2} \in C^*(\mathbb{Z}^d) \otimes \mathbb{M}_k(\mathbb{C}).$$

The latter represents a class $[H] := [p] \in K_0(C^*(\mathbb{Z}^d))$. The boundary map for the extension (2) maps this class to its index in $K_1(\mathcal{K} \otimes C^*(\mathbb{Z}^{d-1}))$. If the latter is nonzero, then H cannot lift to an invertible operator in $\mathcal{T} \otimes C^*(\mathbb{Z}^{d-1}) \otimes \mathbb{M}_k(\mathbb{C})$. This means that any Hamiltonian on the half-space that behaves like H in the bulk is a conductor. That is, there are conducting states on the boundary, and these are forced to exist by the nonvanishing index. Since the index is homotopy invariant, the existence of these boundary states is not affected by small perturbations of the Hamiltonian.

4 Vector bundles and the Bloch bundle

Via Fourier transform, $C^*(\mathbb{Z}^d)$ is identified with $C(\mathbb{T}^d)$, so that the K_0 -class $[H]$ defines a class in $K^0(\mathbb{T}^d)$. The associated vector bundle is called the *Bloch bundle*. Its fibre at $z \in \mathbb{T}^d$ is the image of the Fourier transform of p at z ; the latter is a projection in $\mathbb{M}_k(\mathbb{C})$.

It is physically interesting to know whether or not this vector bundle is trivial because this is equivalent to the existence of “exponentially localised Wannier functions” (see [4, Proposition 4.3]), which are a tool used for computations in physics. It is usually much easier to decide whether the Bloch bundle is stably trivial, that is, becomes trivial after adding a trivial bundle. This means that its class in reduced K-theory vanishes. If the reduced K-theory is torsion-free, this happens if and only if its Chern numbers vanish. Physicists have long studied the Chern numbers of the Bloch bundle as topological invariants related to conductivity phenomena. As a result, it is physically relevant to know whether the triviality of the Bloch bundle follows from its stable triviality. It is well known that vector bundles of sufficiently high rank that are stably trivial are automatically trivial; more generally, stably isomorphic bundles of sufficiently high rank are isomorphic (see [3]). Generalizations of these classical results are needed to treat physical systems with certain extra symmetries.

5 “Real” and “Quaternionic” bundles

In quantum mechanics, a time-reversal symmetry is represented by an anti-unitary operator that commutes with the Hamiltonian. The square of this anti-unitary operator is ± 1 , where $+1$ occurs for bosons, and -1 for fermions. In the Hilbert space $\ell^2(\mathbb{Z}^d, \mathbb{C}^k)$, we assume that time-reversal symmetry is given

by applying a certain anti-unitary operator on \mathbb{C}^k pointwise. Depending on the sign, this has the effect that the coefficients H_a in (1) now belong to $\mathbb{M}_k(\mathbb{R})$ or $\mathbb{M}_k(\mathbb{H})$ instead of $\mathbb{M}_k(\mathbb{C})$. The same happens for the matrix coefficients of the projection p . When we take the Fourier transform, however, this does not correspond to the Bloch bundle over the torus being a real or quaternionic bundle. Instead, the Bloch bundle is a complex vector bundle equipped with a conjugate-linear involution θ mapping E_z to $E_{\bar{z}}$ for all $z \in \mathbb{T}^d$. If $\theta^2 = 1$ (bosons), this is a “real” vector bundle; if $\theta^2 = -1$, this is a “quaternionic” vector bundle.

Such vector bundles may be considered over a space X with an involution such as the map $z \mapsto \bar{z}$ above. If the involution on X is the identity map, then “real” and “quaternionic” vector bundle become equivalent to vector bundles over the fields \mathbb{R} of real numbers and \mathbb{H} of quaternions, respectively. We need the case, however, where the involution is nontrivial. While “real” vector bundles have been known in index theory for a long time (see [1]), they have not received so much attention. In particular, it has not been shown that “real” or “quaternionic” bundles of sufficiently high rank are isomorphic once they are stably isomorphic. The recent article [2] fills this gap. A crucial step in the proof is showing that a bundle of sufficiently high rank contains a trivial vector bundle of rank 1 as a direct summand. Both results combine into the statement that the stabilisation map from bundles of rank k to bundles of rank $k + 1$ that adds a trivial bundle of rank 1 induces a bijection on isomorphism classes for sufficiently large k . The following theorems from [2] give the details of these statements:

Theorem 3. *Let $d_1, d_0, k \in \mathbb{N}$. Let X be a $\mathbb{Z}/2$ -CW-complex. Assume that the free cells in X have at most dimension d_1 and that the trivial cells have at most dimension d_0 . Let*

$$k_0 := \max \left\{ d_0, \left\lceil \frac{d_1 - 1}{2} \right\rceil \right\}, \quad k_1 := \max \left\{ d_0 + 1, \left\lceil \frac{d_1}{2} \right\rceil \right\}.$$

1. *Let E be a “real” vector bundle over X of rank $k \geq k_0$. There is an isomorphism $E \cong E_0 \oplus (X \times \mathbb{C}^{k-k_0})$ for some “real” vector bundle E_0 over X and the trivial “real” vector bundle $X \times \mathbb{C}^{k-k_0}$ of rank $k - k_0$.*
2. *Let E_1 and E_2 be two “real” vector bundles over X of rank $k \geq k_1$. If E_1 and E_2 are stably isomorphic, that is, $E_1 \oplus E_3 \cong E_2 \oplus E_3$ for some “real” vector bundle E_3 , then they are isomorphic.*

Theorem 4. *Let $d_1, d_0, k \in \mathbb{N}$. Let X be a $\mathbb{Z}/2$ -CW-complex. Assume that the free cells in X have at most dimension d_1 and that the trivial cells have at most dimension d_0 . Let*

$$k_0 := \max \left\{ \left\lceil \frac{d_0 - 3}{2} \right\rceil, \left\lceil \frac{d_1 - 1}{2} \right\rceil \right\}, \quad k_1 := \max \left\{ \left\lceil \frac{d_0 - 2}{2} \right\rceil, \left\lceil \frac{d_1}{2} \right\rceil \right\}.$$

1. *Let E be a “quaternionic” vector bundle over X of rank $k \geq k_0$. There is an isomorphism $E \cong E_0 \oplus \theta_X^{\oplus 2 \lfloor (k-k_0)/2 \rfloor}$ for some “quaternionic” vector bundle E_0 and the trivial “quaternionic” vector bundle $\theta_X^{\oplus 2 \lfloor (k-k_0)/2 \rfloor}$ of rank $2 \lfloor (k - k_0)/2 \rfloor$.*

2. Let E_1 and E_2 be two “quaternionic” vector bundles over X of rank $k \geq k_1$. If E_1 and E_2 are stably isomorphic, that is, $E_1 \oplus E_3 \cong E_2 \oplus E_3$ for some “quaternionic” vector bundle E_3 , then they are isomorphic.

6 Equivariant K-theory and ongoing work

When the system has classical crystallographic symmetries, then the Bloch bundle becomes an equivariant vector bundle for a certain group. In G -equivariant K-theory for, say, a finite group G , new phenomena may occur. The main new issue is that there may be several non-isomorphic trivial vector bundles, corresponding to inequivalent representations of G . This is not the case for “real” and “quaternionic” bundles, although they may at first sight seem more complicated than equivariant vector bundles because they involve the group $\mathbb{Z}/2$ acting on the vector bundle by anti-unitary maps.

For example, consider a $\mathbb{Z}/2$ -equivariant complex vector bundle over the circle $\mathbb{T}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ with $\mathbb{Z}/2$ acting by $z \mapsto \bar{z}$. Its fibres over ± 1 are complex representations of the group $\mathbb{Z}/2$, and both may be arbitrary. It may happen that at $+1$ we have the trivial representation and at -1 the nontrivial sign representation of some rank k . This vector bundle has arbitrarily high rank k , but has no trivial subbundles because the representations at ± 1 have no common subrepresentation. Analogues of the main theorems above exist, but their assumptions use multiplicities of representations of stabiliser subgroups instead of just ranks.

7 Conclusion

Topological insulators illustrate a rich interplay between condensed matter physics and K-theory. Their mathematical description via C^* -algebras, vector bundles, and index maps explains the topological protection of boundary states and motivates questions on when stable isomorphism implies isomorphism for “real”, “quaternionic”, and equivariant bundles. We stated two theorems from [2] about trivial direct summands and stable isomorphism and isomorphism of “real”, “quaternionic” vector bundles.

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